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Ringed Spaces of Valuation Rings and Projective Limits of Schemes

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Introduction.

Let K be a field and A a subring of K. We denote by $\mathrm{Zar}(K|A)$ the set of valuation rings of K which contain A. Introducing a topology and a sheaf of rings, $\mathrm{Zar}(K|A)$ becomes a local ringed space. In fact, the topology is defined by the open base: $\Sigma=\{\mathrm{Zar}(K|A[E]) \mid E \text{ is a finite subset of } K\}$, and the sheaf of rings $\mathcal{O}$ is defined by $\mathcal{O}(V)=\bigcap_{R\in V}R$ for any non empty open subsets V of $\mathrm{Zar}(K|A)$ (see also Lemma 1 in [3]).

In the case that A is a subfield of K, the set of closed points of $\mathrm{Zar}(K|A)$ was treated by O. Zariski as a topological space (see [4]). So our local ringed spaces $\mathrm{Zar}(K|A)$ are said to be Zariski ringed spaces in this paper (see also §5). We shall give a characterization of Zariski ringed spaces among local ringed spaces and prove that these are projective limits of schemes. Actually we prove,

MAIN THEOREM. *Let K be a field and A a subring of K. If A is noetherian and K is a finitely generated extension over the quotient field of A, then we obtain*

$$\mathrm{Zar}(K|A) \simeq \mathrm{proj.lim}\, X\,,$$

where X runs over all proper integral schemes over $\mathrm{Spec}\,A$ *with rational function field K. Note that X could be assumed to be projective over* $\mathrm{Spec}\,A$.

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§1. By (L.R.S.) we denote the category of local ringed spaces and (Rings) the category of commutative rings with unity. We introduce a natural transformation $\pi: I\to \mathrm{Spec}\circ\Gamma$, in the following way. Here I is the identity functor of (L.R.S.) and Γ is the contravariant functor from (L.R.S.) to (Rings) defined by global sections.

For a local ringed space $(X, \mathcal{O}_X)$, we define a mapping $\pi_X: X\to \mathrm{Spec}\,\mathcal{O}_X(X)$ by

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$$\pi_X : x \mapsto \rho_{X,x}^{-1}(m(\mathcal{O}_{X,x})) \tag{1}$$

for any $x \in X$. Here we denote by $\rho_{X,x} : \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ the canonical mapping. The mapping $\rho_{X,x}$ induces a homomorphism of local rings

$$\bar{\rho}_{X,x} : \mathcal{O}_X(X)_{\pi_X(x)} \longrightarrow \mathcal{O}_{X,x} . \tag{2}$$

Next we define a morphism $\pi_X^\sharp : \widetilde{\mathcal{O}_X(X)} \to \pi_{X*}\mathcal{O}_X$ of sheaves on $\operatorname{Spec}\mathcal{O}_X(X)$. For this purpose, we define a ring homomorphism $\pi_X^\sharp(U) : \widetilde{\mathcal{O}_X(X)}(U) \to \mathcal{O}_X(\pi_X^{-1}(U))$ for any open set U of $\operatorname{Spec}\mathcal{O}_X(X)$. Let $s \in \widetilde{\mathcal{O}_X(X)}(U)$. We may assume that s is a continuous mapping from U to $\coprod_{P \in U} \mathcal{O}_X(X)_P$. Then we put

$$t(x) = (\bar{\rho}_{X,x} \circ s \circ \pi_X)(x) \tag{3}$$

for $x \in \pi_X^{-1}(U)$. Hence t is a mapping from $\pi_X^{-1}(U)$ to $\coprod_{x \in X} \mathcal{O}_{X,x}$. We also put

$$\pi_X^\sharp(U) : s \mapsto t . \tag{4}$$

Then the following lemma is easy to prove.

LEMMA 1. *Let $(X, \mathcal{O}_X)$ be a local ringed space. Then*

(i) $(\pi_X, \pi_X^\sharp) : (X, \mathcal{O}_X) \to (\operatorname{Spec}\mathcal{O}_X(X), \widetilde{\mathcal{O}_X(X)})$ *is a morphism of* (L.R.S.), *that satisfies*

(5) $\pi_X^\sharp(\operatorname{Spec}\mathcal{O}_X(X))$ *is the identity mapping of* $\mathcal{O}_X(X)$,

(6) $(\pi_X^\sharp)_x = \bar{\rho}_{X,x}$ *for any* $x \in X$.

(ii) $\pi : I \to \operatorname{Spec} \circ \Gamma$ *is a natural transformation.*

§2. Here we introduce the condition (8) for local ringed spaces. Differentiable manifolds and schemes satisfy the condition (8).

In general, for a ring A, let $\operatorname{nil} A$ denote the nilradical of A that is the intersection of all the prime ideals of A. Let $(X, \mathcal{O}_X)$ be a local ringed space. Then we obtain

$$\operatorname{nil}\mathcal{O}_X(X) \subset \bigcap_{x \in X} \pi_X(x) , \tag{7}$$

and the following lemma:

LEMMA 2. *Let $(X, \mathcal{O}_X)$ be a local ringed space, F a closed set in* $\operatorname{Spec}\mathcal{O}_X(X)$ *and $\mathfrak{a}$ an ideal of $\mathcal{O}_X(X)$. If $F = V(\mathfrak{a})$, then the following two conditions are equivalent*:

(a) $\overline{\pi_X(\pi_X^{-1}(F))} = F$.

(b) $\displaystyle\bigcap_{P \in \operatorname{Im}\pi_X \cap F} P = \sqrt{\mathfrak{a}}$.

COROLLARY. *The following two conditions for a local ringed space $(X, \mathcal{O}_X)$ are* *equivalent*:

(a) π_X *is dominant*.

(b) $$\bigcap_{x\in X} \pi_X(x)=\operatorname{nil}\mathcal{O}_X(X).$$

We consider the following condition for a local ringed space $(X, \mathcal{O}_X)$:

(8) X has an open base consisting of open sets V such that π_V is dominant.

Note that any scheme satisfies the condition (8).

Certain properties and conditions of schemes are naturally generalized to those of local ringed spaces. In the following, we deal with reduced, integral or normal local ringed spaces, and also consider the ring of rational functions $\operatorname{Rat}(X, \mathcal{O}_X)$ of such a local ringed space $(X, \mathcal{O}_X)$, which is abbreviated as $\operatorname{Rat}X$.

Let U be a dense open subset of X. Then we denote by

(9) $$f\mapsto\langle U,f\rangle$$

the canonical mapping: $\mathcal{O}_X(U)\to\operatorname{Rat}X$. For $\alpha\in\operatorname{Rat}X$, we put

(10) $$\operatorname{dom}(\alpha)=\bigcup U,$$

where U runs over the set of all dense open subsets of X such that $\langle U,f\rangle=\alpha$ for some $f\in\mathcal{O}_X(U)$.

Here we consider reduced local ringed spaces satisfying the condition (8). The following two lemmas are easy.

LEMMA 3. *Let $(X, \mathcal{O}_X)$ be a local ringed space. Then $(X, \mathcal{O}_X)$ is reduced and satisfies the condition* (8), *if and only if $\bigcap_{x\in V}\pi_V(x)=0$ in $\mathcal{O}_X(V)$ holds for any open subset V of X.*

LEMMA 4. *Let $(X, \mathcal{O}_X)$ be a reduced local ringed space satisfying the condition* (8).

(i) *Let U, V be dense open subsets of X, and $f\in\mathcal{O}_X(U)$, $g\in\mathcal{O}_X(V)$. If $\langle U,f\rangle=\langle V,g\rangle$ in $\operatorname{Rat}X$, then $\rho_{U,U\cap V}(f)=\rho_{V,U\cap V}(g)$ in $\mathcal{O}_X(U\cap V)$ holds.*

(ii) *For any $\alpha\in\operatorname{Rat}X$, there exists an $f\in\mathcal{O}_X(\operatorname{dom}(\alpha))$ such that $\alpha=\langle\operatorname{dom}(\alpha),f\rangle$.*

(iii) *For a dense open subset U of X, we obtain*

(11) $$\mathcal{O}_X(U)=\{\alpha\in\operatorname{Rat}X \mid U\subset\operatorname{dom}(\alpha)\}\subset\operatorname{Rat}X.$$

Then we also have,

(12) $$\operatorname{Rat}X=\bigcup\mathcal{O}_X(U)$$

and $\operatorname{Rat}X$ *is reduced.*

(iv) $\operatorname{Rat}X$ *is a total quotient ring.*

Next we consider integral local ringed spaces satisfying the condition (8). The following three lemmas are easy to prove.

LEMMA 5. *Let* $(X, \mathcal{O}_X)$ *be a local ringed space. If* π_X *is dominant and* X *is irreducible, then* $\operatorname{Spec}\mathcal{O}_X(X)$ *is also irreducible. Hence* $\operatorname{nil}\mathcal{O}_X(X)$ *is a prime ideal of* $\mathcal{O}_X(X)$.

LEMMA 6. *Let* $(X, \mathcal{O}_X)$ *be a local ringed space satisfying the condition* (8). *Then* $(X, \mathcal{O}_X)$ *is integral if and only if* $(X, \mathcal{O}_X)$ *is reduced and irreducible.*

LEMMA 7. *Let* $(X, \mathcal{O}_X)$ *be an integral local ringed space satisfying the condition* (8). *Then*

(i) $\operatorname{Rat}X$ *is a field.*

(ii) *For* $x \in X$, *we have*

$$\mathcal{O}_{X,x}=\{\alpha\in\operatorname{Rat}X \mid x\in\operatorname{dom}(\alpha)\}\subset\operatorname{Rat}X\ . \tag{13}$$

Therefore,

$$\mathcal{O}_X(U)\subset\mathcal{O}_{X,x}\subset\operatorname{Rat}X\ , \tag{14}$$

for $x\in U$, *and*

$$\operatorname{dom}(\alpha)=\{x\in X \mid \alpha\in\mathcal{O}_{X,x}\}\ , \tag{15}$$

for any $\alpha\in\operatorname{Rat}X$. *We also obtain*

$$\mathcal{O}_{X,x}=\bigcup_{U\ni x}\mathcal{O}_X(U)\subset\operatorname{Rat}X\ , \tag{16}$$

$$\mathcal{O}_X(U)=\bigcap_{x\in U}\mathcal{O}_{X,x}\subset\operatorname{Rat}X\ , \tag{17}$$

$$\operatorname{Rat}X=\bigcup_{x\in X}\mathcal{O}_{X,x}=\bigcup_{U\neq\varnothing}\mathcal{O}_X(U)\ . \tag{18}$$

COROLLARY. Rat *is a contravariant functor from the category of integral local ringed spaces satisfying the condition* (8) *and dominant morphisms to the category of fields. Then*

$$\operatorname{Rat}\pi_X : Q\mathcal{O}_X(X)\hookrightarrow\operatorname{Rat}X\ . \tag{19}$$

As the restriction of (19), *we obtain the mappings* $\pi_X^\sharp(U)$ *and* $(\pi_X^\sharp)_x$. *Here we denote by* QA *the quotient field of an integral domain* A.

§3. For a ring A, we denote by $(\text{L.R.S}\mid A)$ the category of local ringed spaces over $\operatorname{Spec}A$. Then the following two propositions are easy to prove.

PROPOSITION 1. *The following two conditions for a local ringed space* $(X, \mathcal{O}_X)$ *and a field* k *are equivalent.*

(a) *The next diagram*:

$$\begin{tikzcd} & k \arrow[dl] \arrow[dr] & \\ \mathcal{O}_X(V) \arrow[rr, hook] \arrow[d, "\rho_{V,U}"'] & & \mathrm{Map}(V, k) \arrow[d, "\textit{restriction of mappings}"] \\ \mathcal{O}_X(U) \arrow[rr, hook] & & \mathrm{Map}(U, k) \end{tikzcd}$$

commutes for any open subsets U and V of X such that $U \subset V$.

(b) *$(X, \mathcal{O}_X)$ is a local ringed space over k, and is reduced satisfying the condition* (8). *Further*

$$\mathcal{O}_{X,x}/m(\mathcal{O}_{X,x}) \simeq k \qquad \textit{for any} \quad x \in X .$$

COROLLARY. *Differentiable manifolds and algebraic varieties consisting of closed points satisfy the condition* (8).

PROPOSITION 2. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be local ringed spaces satisfying all the conditions of Proposition* 1.

(i) *Let $f: X \to Y$ be a continuous mapping such that*

(20) *if $s \in \mathcal{O}_Y(V)$, then $s \circ (f|_{f^{-1}(V)}) \in \mathcal{O}_X(f^{-1}(V))$ for any open subset V of Y.*

Defining $f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$ by $f^{\#}(V): s \mapsto s \circ (f|_{f^{-1}(V)})$, we obtain a morphism $(f, f^{\#}): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of (L.R.S$|k$).

(ii) *Conversely, let $(f, \theta): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of* (L.R.S$|k$). *Then f satisfies the condition* (20) *and we obtain $\theta = f^{\#}$.*

COROLLARY. (i) *The category of real C^r-class manifolds* ($r = 0, 1, 2, \cdots, \infty, \omega$) *is a full subcategory of* (L.R.S$|\boldsymbol{R}$).

(ii) *The category of complex manifolds is a full subcategory of* (L.R.S$|\boldsymbol{C}$).

(iii) *Let k be an algebraically closed field. Then the category of algebraic varieties consisting of closed points over k is a full subcategory of* (L.R.S$|k$).

§4. Here we introduce the condition (21) for local ringed spaces. This is a local condition characterizing the generalizations of a point by ring theory.

We consider the following condition for a local ringed space $(X, \mathcal{O}_X)$:

(21) For any $x \in X$, there exists a morphism of local ringed spaces $(j_x, j_x^{\#}): \mathrm{Spec}\,\mathcal{O}_{X,x} \to X$ such that $\mathrm{Im}(j_x) = \{y \in X \mid x \in \overline{\{y\}}\}$ and $(j_x^{\#})_P: \mathcal{O}_{X, j_x(P)} = (\mathcal{O}_{X,x})_P$ for any $P \in \mathrm{Spec}\,\mathcal{O}_{X,x}$.

The following two lemmas are easy.

LEMMA 8. *Let $(X, \mathcal{O}_X)$ be a local ringed space satisfying the condition* (21). *Then*
(i) *j_x is injective.*

(ii) $j_x(m(\mathcal{O}_{X,x}))=x$ *for* $x\in X$.

(iii) (j_x, j_x^*) *is a morphism over* $\mathrm{Spec}\,\mathcal{O}_X(X)$. *That is, the next triangle commutes*:

$$\begin{tikzcd} \mathrm{Spec}\,\mathcal{O}_{X,x} \arrow[r, "j_x"] \arrow[dr, "\mathrm{Spec}\,\rho_{X,x}"'] & X \arrow[d, "\pi_X"] \\ & \mathrm{Spec}\,\mathcal{O}_X(X)\,. \end{tikzcd} \tag{22}$$

(iv) (j_x, j_x^*) *is uniquely determined by* $x\in X$.

(v) *Let* $(Y, \mathcal{O}_Y)$ *be a local ringed space satisfying the condition* (21), *and* (f, θ): $(Y, \mathcal{O}_Y)\to(X, \mathcal{O}_X)$ *a morphism of local ringed spaces. If* $x=f(y)$, *then we have the following commutative diagram*:

$$\begin{tikzcd} Y \arrow[r, "f"] & X \\ \mathrm{Spec}\,\mathcal{O}_{Y,y} \arrow[u, "j_y"] \arrow[r, "\mathrm{Spec}\,\theta_y"'] & \mathrm{Spec}\,\mathcal{O}_{X,x}\,. \arrow[u, "j_x"'] \end{tikzcd} \tag{23}$$

EXAMPLES. (i) Any scheme satisfies the condition (21). Especially if X is an affine scheme, then $j_x=\mathrm{Spec}\,\rho_{X,x}$ by (22).

(ii) Differentiable manifolds do not satisfy the condition (21). In general, a local ringed space treated in §3 does not satisfy the condition (21).

LEMMA 9. *Let X be a topological space with a generic point and satisfy the separable condition T_0. Assume that $(X,\mathcal{O}_X)$ is an integral local ringed space satisfying the conditions* (8) *and* (21). *Then j_x is dominant and* $\mathrm{Rat}\,j_x$ *is the identity mapping*:

$$\mathrm{Rat}\,j_x : \mathrm{Rat}\,X = Q(\mathcal{O}_{X,x})\,. \tag{24}$$

§5. Here we consider the Zariski ringed spaces (see [3]).

Let K be a field and $\mathrm{Zar}\,K$ the set of valuation rings of K. If X is an irreducible subset of $\mathrm{Zar}\,K$, then $(X, \mathcal{O}_X)$ is a local ringed space. We also call this a Zariski ringed space associated to the field K. Zariski ringed spaces are normal and integral.

For a Zariski ringed space $(X, \mathcal{O}_X)$ and $R\in X$, we obtain

$$\pi_X(R)=\mathcal{O}_X(X)\cap m(R)\,. \tag{25}$$

Therefore,

$$\bigcap_{R\in X}\pi_X(R)=\bigcap_{R\in X}m(R)\,. \tag{26}$$

LEMMA 10. *Let $(X, \mathcal{O}_X)$ be a Zariski ringed space associated to a field K. Then*

(i) $\mathrm{Rat}\,X\simeq\bigcup_{R\in X}R\subset K$ *holds*.

In what follows, we identify the above two rings. Then we obtain $K=Q(\mathrm{Rat}\,X)$ *and*

$$\mathrm{dom}(\alpha)=\{R\in X \mid \alpha\in R\} \tag{27}$$

for any $\alpha\in\mathrm{Rat}\,X\subset K$.

(ii) *The following five conditions for* $(X, \mathcal{O}_X)$ *and* K *are equivalent*:

(a) X *is dense in* $\mathrm{Zar}\,K$.

(b) $K=\mathrm{Rat}\,X$.

(c) $\bigcap_{R\in X}\pi_X(R)=0$.

(d) π_X *is dominant.*

(e) $(X, \mathcal{O}_X)$ *satisfies the condition* (8).

PROOF. (i) The mapping $\bigcup_{R\in X}R\to\mathrm{Rat}\,X$ defined by $f\mapsto\langle X\cap\mathrm{Zar}(K\mid\{f\}), f\rangle$ is a ring isomorphism. (ii) is easy to prove.

In what follows, let Z denote the Zariski ringed space $\mathrm{Zar}(K|A)$, where A is a subring of K. Then $\mathcal{O}_Z(Z)$ is the integral closure of A in K, and the next triangle

$$\begin{array}{ccc} Z & \xrightarrow{\pi_Z} & \mathrm{Spec}\,\mathcal{O}_Z(Z) \\ & \searrow^{\Phi_{K|A}} & \downarrow \\ & & \mathrm{Spec}\,A \end{array} \tag{28}$$

commutes. For the morphism $\Phi_{K|A}$, see (10), (11) and (12) in [3]. We also obtain

$$\mathrm{dom}(\alpha)=\mathrm{Zar}(K|A[\alpha]) \tag{27'}$$

for any $\alpha\in K=\mathrm{Rat}\,X$.

We denote by (P.Fields) the category of projective fields in which morphisms are places. Since the canonical functor: (Fields) $\to$ (P.Fields) exists, (Fields) can be regarded as a subcategory of (P.Fields), which is not a full subcategory.

For a place $\varphi: K_\infty\to L_\infty$, we define a morphism of local ringed spaces $\mathrm{Zar}\,\varphi$: $\mathrm{Zar}\,L\to\mathrm{Zar}\,K$ by $Q\mapsto\varphi^{-1}(Q)$. Thus we obtain a contravariant functor Zar : (P.Fields) $\to$ (L.R.S.).

LEMMA 11. *Let* K *and* L *be fields,* A *a subring of* K *and* B *an integrally closed subring of* L. *Let* $X=\mathrm{Zar}(K|A)$ *and* $Y=\mathrm{Zar}(L|B)$. *If we put*

$$M=\{\varphi\in\mathrm{Mor}_{(\mathrm{P.Fields})}(K_\infty, L_\infty)\mid \varphi(A)\subset B\},$$

then the mapping:

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Mor}_{(\mathrm{L.R.S.})}(Y, X) \\ \varphi & \longmapsto & (\mathrm{Zar}\,\varphi)|_Y \end{array}$$

is bijective. Moreover, we obtain that φ *is a morphism of* (Fields) *if and only if* $(\mathrm{Zar}\,\varphi)|_Y$ *is dominant.*

PROOF. Surjectivity: For any $(f, \theta): Y \to X$, we put $R_0 = f(L) \in X$. Then we obtain a ring homomorphism $\theta_L: R_0 \to L$. Letting $\varphi: K_\infty \to L_\infty$ be an extension of θ_L, we have $(f, \theta) = (\mathrm{Zar}\,\varphi)|_Y$. Injectivity: Let $(\mathrm{Zar}\,\varphi)|_Y = (\mathrm{Zar}\,\psi)|_Y = (f, \theta)$. Then we have $\varphi|_{R_0} = \theta_L = \psi|_{R_0}$ and hence $\varphi = \psi$. Q.E.D.

COROLLARY. *The contravariant functor* Zar : (P.Fields) $\to$ (L.R.S.) *is fully faithful.*

Topology $\boldsymbol{T}(0)$. Until the previous paragraph, we have been using the Zariski topology only. We denote by $\boldsymbol{T}(0)$ the Zariski topology on $\mathrm{Zar}(K|A)$. Therefore $\boldsymbol{T}(0)$ has an open base:

$$\Sigma = \{\mathrm{Zar}(K|A[E]) \mid E \text{ is a finite subset of } K\}.$$

Topology $\boldsymbol{T}(1)$. For $R \in Z = \mathrm{Zar}(K|A)$, the inclusion $i_R: \mathrm{Zar}(K|R) \hookrightarrow \mathrm{Zar}(K|A)$ is defined. Then we denote by $\boldsymbol{T}(1)$ the strongest topology among the topologies such that i_R is continuous for any $R \in \mathrm{Zar}(K|A)$. We can also write:

$$\boldsymbol{T}(1) = \{O \subset Z \mid O \cap \mathrm{Zar}(K|R) \text{ is open in } \mathrm{Zar}(K|R) \text{ for any } R \in Z\}.$$

It is clear that $\boldsymbol{T}(0) \subset \boldsymbol{T}(1)$.

LEMMA 12. *Let K be a field, A a subring of K, $Z = \mathrm{Zar}(K|A)$ and $R \in Z$. If a topology $\boldsymbol{T}$ on Z satisfies $\boldsymbol{T}(0) \subset \boldsymbol{T} \subset \boldsymbol{T}(1)$, then we obtain*

(i) $\overline{\{R\}} = \{R' \in Z \mid R' \subset R\}$.

(ii) *The relative topology of $\boldsymbol{T}$ to* $\mathrm{Zar}(K|R)$ *is the Zariski topology.*

PROOF. (i) Let $R' \in \overline{\{R\}}$. For any $\alpha \in R'$, we put $V = \mathrm{dom}(\alpha) \in \boldsymbol{T}(0) \subset \boldsymbol{T}$. Then $V \cap \{R\} \neq \varnothing$ implies $R \in V$ and hence $\alpha \in R$. Therefore we have $R' \subset R$. Conversely, let $R' \in Z$ satisfy $R' \subset R$. For any $O \in \boldsymbol{T} \subset \boldsymbol{T}(1)$, we can write $O = \bigcup_{R \in O} \mathrm{Zar}(K|R)$. Hence $R' \in O$ implies $R \in O$. Therefore we have $R' \in \overline{\{R\}}$. (ii) is easy to prove.

Presheaf of Rings. For any topology $\boldsymbol{T}$ on $Z = \mathrm{Zar}(K|A)$, we can define the presheaf $\mathcal{O}_Z$ by (8) in [3]. Especially if $\boldsymbol{T}$ satisfies $\boldsymbol{T}(0) \subset \boldsymbol{T} \subset \boldsymbol{T}(1)$, then we have $\mathcal{O}_{Z,R} = R$ for any $R \in Z$. Therefore $(Z, \mathcal{O}_Z)$ becomes a local ringed space. The proof is similar to that of Lemma 1 in [3]. We also write $(Z, \boldsymbol{T}, \mathcal{O}_Z)$ instead of $(Z, \mathcal{O}_Z)$, if necessary.

§6. Here we introduce the separatedness and the properness by means of valuation theory without using direct products. As an application, we consider the projective limit of proper (or projective) integral schemes.

DEFINITION. A morphism $X \to Y$ of local ringed spaces is called valuative-separated (resp. valuative-proper), if the mapping: $\mathrm{Mor}_Y(\mathrm{Spec}\,Q, X) \to \mathrm{Mor}_Y(\mathrm{Spec}\,L, X)$ defined by $(f, \theta) \mapsto (f, \theta) \circ \mathrm{Spec}\,i$ is injective (resp. bijective), for any valuation ring Q over Y. Here we denote by L the quotient field of Q and i the inclusion mapping: $Q \subset L$.

LEMMA 13. *Let K be a field, A a subring of K, and let $(X, \mathcal{O}_X)$ satisfy the following conditions*:

(29) *X is a topological space with a generic point and satisfies the separable condition T_0*.

(30) *$(X, \mathcal{O}_X)$ is an integral local ringed space satisfying the conditions* (8) *and* (21).

(31) $K = \mathrm{Rat}(X, \mathcal{O}_X)$.

Then the morphism $X \to \mathrm{Spec}\,A$ is valuative-separated (*resp. valuative-proper*), *if and only if the set $\{x \in X \mid R$ dominates $\mathcal{O}_{X,x}\}$ has at most one point* (*resp. only one point*), *for any $R \in \mathrm{Zar}(K \mid A)$.*

PROOF. We only prove the separatedness case. The "only if" part: Let M be the subset of $\mathrm{Mor}_A(\mathrm{Spec}\,R, X)$ consisting of dominant morphisms (f, θ) such that $\mathrm{Rat}(f, \theta)$ is the identity mapping of K. Then there exists a bijection between the sets M and $\{x \in X \mid R$ dominates $\mathcal{O}_{X,x}\}$ defined by

$$\begin{array}{ccc} (f, \theta) & \longmapsto & f(m(R)) \\ j_x \circ \mathrm{Spec}\,i' & \longleftarrow\!\shortmid & x\,. \end{array}$$

Here we denote by i' the inclusion mapping: $\mathcal{O}_{X,x} \subset R$. By the separatedness, we have $\mathrm{card}\,M \leq 1$.

The "if" part: Take $(f_k, \theta_k) : \mathrm{Spec}\,Q \to X$ $(k=1,2)$ that satisfy $(f_1, \theta_1) \circ \mathrm{Spec}\,i = (f_2, \theta_2) \circ \mathrm{Spec}\,i = (g, \eta)$. Putting $x_0 = g(0) \in X$, and we consider $\eta_0 : \mathcal{O}_{X,x_0} \to L$. Let L_{alg} be an algebraic closure of L. Then we define $\varphi' : \mathcal{O}_{X,x_0} \to L \hookrightarrow L_{\mathrm{alg}}$ by composition. φ' can be extended to a place $\varphi : K_\infty \to (L_{\mathrm{alg}})_\infty$. Then by $\varphi(A) \subset Q$ and Lemma 11, we obtain a morphism $\mathrm{Zar}\,\varphi : \mathrm{Zar}(L_{\mathrm{alg}} \mid Q) \to \mathrm{Zar}(K \mid A)$. For any $P \in \mathrm{Spec}\,Q$, we put $x_k = f_k(P)$ $(k=1,2)$. Since the mapping $\mathrm{Zar}(L_{\mathrm{alg}} \mid Q) \to \mathrm{Zar}(L \mid Q)$ is surjective, there exists $Q' \in \mathrm{Zar}(L_{\mathrm{alg}} \mid Q)$ such that $Q_P = L \cap Q'$. If we put $R = \varphi^{-1}(Q') \in \mathrm{Zar}(K \mid A)$, then R dominates $\mathcal{O}_{X,x_k}$ $(k=1,2)$. Therefore, we have $x_1 = x_2$ and hence $f_1 = f_2$. Since $(\theta_k)_P : \mathcal{O}_{X,x_k} \to Q_P$ is the restriction of φ, we have $\theta_1 = \theta_2$. Q.E.D.

COROLLARY. *If the morphism $X \to \mathrm{Spec}\,A$ is valuative-separated, then $\mathcal{O}_{X,x} = \mathcal{O}_{X,y}$ implies $x = y$ for any $x, y \in X$.*

DEFINITION. For an integral domain A, we define a subcategory $\mathscr{C}(A)$ of $(\mathrm{L.R.S} \mid \mathrm{A})$ as follows:
The objects $(X, \mathcal{O}_X)$ satisfy (29), (30) and the following condition:

(32) The structure morphism $X \to \mathrm{Spec}\,A$ is dominant and valuative-separated.

The morphisms of $\mathscr{C}(A)$ are dominant morphisms over $\mathrm{Spec}\,A$.

LEMMA 14. *The contravariant functor* Rat : $\mathscr{C}(A) \to (QA\text{-Fields})$ *is faithful.*

PROOF. Let $K = \mathrm{Rat}X$, $L = \mathrm{Rat}Y$ and take $(f_k, \theta_k) : Y \to X$ $(k=1, 2)$ which satisfy $\mathrm{Rat}(f_1, \theta_1) = \mathrm{Rat}(f_2, \theta_2) = \varphi$. For any $y \in Y$, there exists $Q \in \mathrm{Zar}(L|A)$ such that Q dominates $\mathcal{O}_{Y,y}$. If we put $R = \varphi^{-1}(Q) \in \mathrm{Zar}(K|A)$, then R dominates $\mathcal{O}_{X, f_k(y)}$ $(k=1,2)$. Therefore we have $f_1(y) = f_2(y)$ and hence $f_1 = f_2$. Since $(\theta_k)_y$ is the restriction of φ, we have $\theta_1 = \theta_2$. Q.E.D.

DEFINITION. Let K be a field and A a subring of K. Then we define a subcategory $\mathscr{C}(K|A)$ of (L.R.S$|A$) as follows:

The objects $(X, \mathcal{O}_X)$ satisfy (29), (30), (31) and the following condition:

(33) The structure morphism $X \to \mathrm{Spec}A$ is valuative-proper.

The morphisms (f, θ) of $\mathscr{C}(K|A)$ satisfy the following condition:

(34) (f, θ) is a dominant morphism over $\mathrm{Spec}A$ and $\mathrm{Rat}(f, \theta)$ is the identity mapping of K.

The category $\mathscr{C}(K|A)$ is a subcategory of $\mathscr{C}(A)$, which is not a full subcategory. In fact, for any objects X and Y of $\mathscr{C}(K|A)$, the set $\mathrm{Mor}_{\mathscr{C}(K|A)}(Y, X)$ has at most one point by Lemma 14.

LEMMA 15. *Let K be a field, A a subring of K and $Z = \mathrm{Zar}(K|A)$. If a topology $\boldsymbol{T}$ on Z satisfies $\boldsymbol{T}(0) \subset \boldsymbol{T} \subset \boldsymbol{T}(1)$, then $(Z, \boldsymbol{T}, \mathcal{O}_Z)$ is an object of $\mathscr{C}(K|A)$.*

The proof follows from Lemmas 10, 12 and 13.

For the topology $\boldsymbol{T}(1)$ on Z, the following lemma is important.

LEMMA 16. *Let K, A and Z be as in Lemma 15. Then,*

(i) $(Z, \boldsymbol{T}(1), \mathcal{O}_Z) \simeq \text{ind.lim}\ \mathrm{Spec}R$ *in* $\mathscr{C}(A)$, *where R runs over Z.*

(ii) $(Z, \boldsymbol{T}(1), \mathcal{O}_Z)$ *is the initial object of* $\mathscr{C}(K|A)$.

PROOF. We put $\boldsymbol{D} = (Z, \boldsymbol{T}(1), \mathcal{O}_Z)$.

(i) By Lemma 13, the morphism $\mathrm{Spec}R \to \mathrm{Spec}A$ is valuative-separated for any $R \in Z$. Therefore we obtain the following commutative diagram in $\mathscr{C}(A)$:

$$\begin{tikzcd} \mathrm{Spec}R_1 \arrow[r] \arrow[dr, "j_{R_1}"'] & \mathrm{Spec}R_2 \arrow[d, "j_{R_2}"] \\ & \boldsymbol{D}. \end{tikzcd}$$

For any given commutative diagram in $\mathscr{C}(A)$:

$$\begin{tikzcd} \operatorname{Spec} R_1 \arrow[r] \arrow[dr, "f_{R_1}"'] & \operatorname{Spec} R_2 \arrow[d, "f_{R_2}"] \\ & X , \end{tikzcd}$$

we put $f(R) = f_R(m(R))$ for $R \in Z$. Then $f : \boldsymbol{D} \to X$ is continuous, dominant and $f_R = f \circ j_R$.

(ii) For any object X of $\mathscr{C}(K|A)$, we shall define a morphism $\Phi_X : \boldsymbol{D} \to X$. First we define $f_K : \operatorname{Spec} K \to X$ by $f_K(0) = \xi$; where ξ is the generic point of X. Since X is valuative-proper, there exists a morphism $f_R : \operatorname{Spec} R \to X$ such that

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "f_K"'] & \operatorname{Spec} R \arrow[d] \arrow[dl, "f_R"] \\ X \arrow[r] & \operatorname{Spec} A , \end{tikzcd}$$

for any $R \in Z$. By (i), there exists a morphims $\Phi_X : \boldsymbol{D} \to X$ such that $f_R = \Phi_X \circ j_R$ for any $R \in Z$. Then we obtain that $\Phi_X(R) = x$ if and only if R dominates $\mathcal{O}_{X,x}$. Therefore $\Phi_X : \boldsymbol{D} \to X$ is a morphism of $\mathscr{C}(K|A)$. Q.E.D.

Finally, we deal with the topology $\boldsymbol{T}(0)$, and consider the relationship to schemes. We introduce the following two conditions for a local ringed space $(X, \mathcal{O}_X)$:

(35) The topology on X is generated by $\{\operatorname{dom}(\alpha) \mid \alpha \in \operatorname{Rat} X\}$.

(36) For any $x \in X$, $\mathcal{O}_{X,x}$ is a valuation ring.

The following theorem gives a characterization of Zariski ringed space, induced from (15), (27′), Lemmas 10, 15 and 16.

THEOREM 1. *Let A be an integral domain.*

(i) *Let K be a field containing A. If we put $Z = \operatorname{Zar}(K|A)$, then $(Z, \boldsymbol{T}(0), \mathcal{O}_Z)$ satisfies the conditions* (29), (30), (32), (33), (35), (36) *and* $K \simeq \operatorname{Rat}(Z, \boldsymbol{T}(0), \mathcal{O}_Z)$.

(ii) *Conversely, suppose that $(X, \mathcal{O}_X)$ satisfies the conditions* (29), (30), (32), (33), (35) *and* (36). *If we put $K = \operatorname{Rat}(X, \mathcal{O}_X)$, then K contains A and $(X, \mathcal{O}_X) \simeq (Z, \boldsymbol{T}(0), \mathcal{O}_Z)$, where $Z = \operatorname{Zar}(K|A)$.*

Thus the functors Zar *and* Rat *give a contravariant equivalence between the category of fields containing A and the category of local ringed spaces satisfying the above conditions with dominant morphisms.*

DEFINITION. Let A be a subring of a field K satisfying the following condition:

(37) A is noetherian and K is a finitely generated extension over QA.

Then we denote by $\mathscr{C}_1(K|A)$ (resp. $\mathscr{C}_2(K|A)$) the category of proper (resp. projective) integral schemes $(X, \mathcal{O}_X)$ over A satisfying the condition (31) and morphisms (f, θ) satisfying (34).

Then we obtain

(38) $\mathscr{C}_2(K|A)$ is a full subcategory of $\mathscr{C}_1(K|A)$,

(38′) $\mathscr{C}_1(K|A)$ is a full subcategory of $\mathscr{C}(K|A)$.

If we put

(39) $\boldsymbol{P}=\{B \mid A\subset B\subset K,\ B$ is of finite type over $A,\ QB=K\}$,

then $\Sigma_0=\{\mathrm{Zar}(K|B) \mid B\in\boldsymbol{P}\}$ is an open base of $\boldsymbol{T}(0)$.

We consider the following condition for a full subcategory $\mathscr{C}$ of $\mathscr{C}_1(K|A)$:

(40) For any $B\in\boldsymbol{P}$, there exists an object X of $\mathscr{C}$ and an affine open subset V of X such that $B=\mathcal{O}_X(V)$.

It is well known that $\mathscr{C}_1(K|A)$ and $\mathscr{C}_2(K|A)$ satisfy the condition (40). Therefore, the following theorem implies Main Theorem as a corollary.

THEOREM 2. *Let K be a field, A a subring of K satisfying the condition* (37), *and* $Z=\mathrm{Zar}(K|A)$.

(i) *Let $\mathscr{C}$ be a full subcategory of $\mathscr{C}_1(K|A)$ satisfying the condition* (40). *Then we obtain*

$$(Z, \boldsymbol{T}(0), \mathcal{O}_Z) = \mathrm{proj.lim}\,\mathscr{C} \qquad in \quad \mathscr{C}(K|A).$$

(ii) *The category $\mathscr{C}_1(K|A)$ has an initial object if and only if $(Z, \boldsymbol{T}(0), \mathcal{O}_Z)$ is an object of $\mathscr{C}_1(K|A)$. Then we obtain that* $\dim Z\le 1$.

(iii) *Suppose that K and A satisfy one of the following three conditions*:

(a) *A is of finite type over a subfield of K.*

(b) *A is integrally closed in K.*

(c) *A is an integrally closed integral domain and K is separable over QA.*

Then $\mathscr{C}_1(K|A)$ has an initial object if and only if $\dim Z\le 1$. *Moreover, $\Phi_X : (Z, \boldsymbol{T}(0), \mathcal{O}_Z)\to X$ are the normalizations of X for any objects X of $\mathscr{C}_1(K|A)$.*

PROOF. (i) We shall prove in the following three steps:

Step 1. The topology $\boldsymbol{T}(0)$ of Z is the weakest topology among the topologies such that $\Phi_X : Z\to X$ is continuous for any object X of $\mathscr{C}$.

Let X be an object of $\mathscr{C}$ and V an affine open subset of X. Then we have $\Phi_X^{-1}(V)=\mathrm{Zar}(K|\mathcal{O}_X(V))$ and $\mathcal{O}_X(V)\in\boldsymbol{P}$. Therefore Φ_X is continuous with respect to $\boldsymbol{T}(0)$. By (40), we see that $\boldsymbol{T}(0)$ is weakest.

Thus we obtain a morphism $\Phi_X : (Z, \boldsymbol{T}(0), \mathcal{O}_Z)\to(X, \mathcal{O}_X)$ of $\mathscr{C}(K|A)$, for any object X of $\mathscr{C}_1(K|A)$.

Step 2. Let $R_1, R_2\in Z$. If $\Phi_X(R_1)=\Phi_X(R_2)$ holds for any object X of $\mathscr{C}$, then $R_1=R_2$.

If $R_1\ne R_2$, then we may assume $m(R_1)\cap R_2^\times\ne\varnothing$ without loss of generality. Hence there exist $\alpha \in m(R_1) \cap R_2^\times$ and $B \in \boldsymbol{P}$ such that $\alpha \in B \subset R_1 \cap R_2$. Then we have $R_1, R_2 \in \mathrm{Zar}(K|B)$ and $\Phi_{K|B}(R_1) \neq \Phi_{K|B}(R_2)$. By (40), there exists an object X of $\mathscr{C}$ such that $\Phi_X(R_1) \neq \Phi_X(R_2)$.

Step 3. Let Y be an object of $\mathscr{C}(K|A)$. If morphisms $Y \to X$ of $\mathscr{C}(K|A)$ are given for any objects X of $\mathscr{C}$, then there exists a topology $\boldsymbol{T}$ on Z such that $\boldsymbol{T}(0) \subset \boldsymbol{T} \subset \boldsymbol{T}(1)$ and $(Y, \mathcal{O}_Y) \simeq (Z, \boldsymbol{T}, \mathcal{O}_Z)$.

By Lemma 16, the morphism $\Phi_Y : Z \to Y$ is defined. Then Φ_Y is bijective by Step 2. Let $\boldsymbol{T}$ be the topology of Z such that Φ_Y is a homeomorphism. By Step 1, we have that $\boldsymbol{T}(0) \subset \boldsymbol{T} \subset \boldsymbol{T}(1)$, and by (40), we have that $\Phi_Y(R) = y$ implies $\mathcal{O}_{Y,y} = R$ for any $R \in Z$. Therefore we have $(Y, \mathcal{O}_Y) \simeq (Z, \boldsymbol{T}, \mathcal{O}_Z)$.

Thus we obtain a morphism $(Y, \mathcal{O}_Y) \to (Z, \boldsymbol{T}(0), \mathcal{O}_Z)$ of $\mathscr{C}(K|A)$.

(ii) Since the "if" part is clear, we shall prove the "only if" part. Let $(Y, \mathcal{O}_Y)$ be the initial object of $\mathscr{C}_1(K|A)$. By Step 3 in (i), we have that Φ_Y is an isomorphism. Therefore $(Z, \boldsymbol{T}(0), \mathcal{O}_Z)$ is an object of $\mathscr{C}_1(K|A)$. By Theorem 20 in [3], we obtain that $\dim Z \leq 1$.

(iii) It is sufficient to prove the "if" part only. We remark that $\dim A + \text{tr-deg}_{QA} K \leq 1$.

(a) First we assume $\dim A = 0$. Then we have $\text{tr-deg}_{QA} K \leq 1$. If $\text{tr-deg}_{QA} K = 0$, then $\mathrm{Zar}(K|A) = \mathrm{Zar}(K|K) \simeq \mathrm{Spec}\, K$ is an initial object of $\mathscr{C}_1(K|A)$. If $\text{tr-deg}_{QA} K = 1$, then $\mathrm{Zar}(K|A)$ is an algebraic curve over A and is an initial object of $\mathscr{C}_1(K|A)$. Next we assume $\dim A = 1$. Let A' be the integral closure of A in K. Then A' is a Dedekind domain with quotient field K. Therefore $\mathrm{Zar}(K|A) \simeq \mathrm{Spec}\, A'$ is an initial object of $\mathscr{C}_1(K|A)$.

(b) We may assume $\dim A = 1$. Since K is an extension of finite degree over QA, we have $K = QA$. Therefore $\mathrm{Zar}(K|A) \simeq \mathrm{Spec}\, A$ is an initial object of $\mathscr{C}_1(K|A)$.

(c) We may assume $\dim A = 1$. Let A' be the integral closure of A in K. Then A' is a Dedekind domain with quotient field K. Therefore $\mathrm{Zar}(K|A) \simeq \mathrm{Spec}\, A'$ is an initial object of $\mathscr{C}_1(K|A)$. Q.E.D.

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