TOKYO J. MATH.
VOL. 16, NO. 1, 1993

On Vector Bundles on P^n Which Have σ-Transition Matrices

Hiroshi TANGO

Kyoto University of Education
(Communicated by K. Ogiue)

0. Introduction.

When $n \geq 4$ and $n-1 \geq r \geq 2$, only few indecomposable vector bundles of rank r on n-dimensional projective space P^n are known. Some of them are very special vector bundles. That is, they have special system of transition matrices. In this article, we study such vector bundles which have special system of transition matrices, namely, system of σ-transition matrices.

In section 2, we give a general theory of a system of σ-transition matrices and its initial datum. Initial datum of a system of σ-transition matrices furnishes us with complete information about the system of σ-transition matrices.

In section 3, we give a theory of standard σ-matrix. Using a standard σ-matrix we can construct a system of σ-transition matrices and its initial datum.

In section 4, we give an example of standard σ-matrix which determines essentially the cotangent bundle Ω_{P^n} of P^n. And we give an example of initial datum of a system of σ-transition matrices, which determines the tangent bundle T_{P^n} of P^n.

In section 5, using the theory of system of σ-transition matrices, we reconstruct the Horrocks bundle of rank 3 on P^5 (cf. [3]).

In section 6, we reconstruct the vector bundle of rank 2 on P^5 which is given in [10] in characteristic two. Using the theory of system of σ-transition matrices, we give an example of non-constant morphism from P^5 to the Grassmann variety $Gr(5, 1)$ which parametrizes lines contained in P^5. And we give an example of indecomposable vector bundle of rank 4 on P^5 which is essentially different from null-correlation bundle and the bundle given in [9].

In section 7, we give several remarks on vector bundles and matrices which appeared in sections 4, 5 and 6.

Recently, H. Kaji showed that the Horrocks-Mumford bundle, which is an indecomposable vector bundle of rank 2 on P^4, has a system of σ-transition matrices

Received March 30, 1989
Revised December 2, 1992

(cf. [5]).

The author wishes to thank Professor N. Sasakura for his valuable suggestion and encouragement.

1. Notations.

Let k be a field. We denote by $\boldsymbol{P}^n$ the n-dimensional projective space defined over k. We denote by $k[x_0, x_1, \cdots, x_n]$ the ring of polynomials in $n+1$-variables with coefficients in k, and we denote its quotients field by K. The field of rational functions on $\boldsymbol{P}^n$ can be naturally identified with the subfield of K. We denote this subfield of K by K_0. We denote by S_{n+1} the symmetric group of permutations of $n+1$ objects $0, 1, 2, \cdots, n$. Let σ be a cyclic permutation of length $n+1$ and G be the cyclic subgroup $\langle\sigma\rangle$ of S_{n+1} generated by σ.

Throughout this article, we use the following notation.

(1) All components of matrices and vectors are elements of K.

(2) We use the letters h, i, j as indices running the ranges $0 \leq h, i, j \leq n$.

(3) We denote $\{V_i\}_{0 \leq i \leq n}$ simply by $\{V_i\}$ and similarly denote $\{V_{ij}\}_{0 \leq i, j \leq n}$ by $\{V_{ij}\}$.

(4) We denote the affine open set $\{(a_0, a_1, \cdots, a_n) \mid a_i \neq 0\}$ of $\boldsymbol{P}^n$ by U_i. $\{U_i\}$ is an affine open covering of $\boldsymbol{P}^n$.

(5) We say that matrices and vectors are regular on U_i if all their components are regular functions on U_i. Similarly we say that they are regular on $U_i \cap U_j$ if all their components are regular functions on $U_i \cap U_j$.

(6) We use the letters τ, η, ξ as arbitrary elements of G.

2. System of σ-transition matrices.

Let E be a vector bundle of rank r on $\boldsymbol{P}^n$ such that $E|_{U_i}$ are trivial on U_i for all i, and let $\{e_{i1}, e_{i2}, \cdots, e_{ir}\}$ be a basis of $E|_{U_i}$. Then there exist relations

$$(e_{j1}, e_{j2}, \cdots, e_{jr}) = (e_{i1}, e_{i2}, \cdots, e_{ir})G_{ij} \tag{1}$$

on $U_i \cap U_j$, where $\{G_{ij}\}$ is a system of non-singular $r \times r$ matrices such that both G_{ij} and G_{ij}^{-1} are regular on $U_i \cap U_j$. Then $\{G_{ij}\}$ is called a system of transition matrices of E with respect to the given bases $\{\{e_{i1}, e_{i2}, \cdots, e_{ir}\}\}$. Suppose that G_{ij} is regular on U_i for any pair i and j. Then, we can write

$$e_{js} = f_1 e_{i1} + f_2 e_{i2} + \cdots + f_r e_{ir}$$

where $f_1, f_2, \cdots, f_r$ are regular functions on U_i. This shows that any section e_{js} extends to a section on U_i. Thus, in this case, section e_{js} extends to a global section of E. This shows that E is generated by its global sections.

Now let $\{\hat{e}_{i1}, \hat{e}_{i2}, \cdots, \hat{e}_{ir}\}$ be another basis of $E|_{U_i}$. Let $\{\hat{G}_{ij}\}$ be the system of transition matrices of E with respect to these bases, i.e., the relations

$$(\hat{e}_{j1}, \hat{e}_{j2}, \cdots, \hat{e}_{jr}) = (\hat{e}_{i1}, \hat{e}_{i2}, \cdots, \hat{e}_{ir})\hat{G}_{ij} \tag{2}$$

hold. Since $\{e_{i1}, e_{i2}, \cdots, e_{ir}\}$ and $\{\hat{e}_{i1}, \hat{e}_{i2}, \cdots, \hat{e}_{ir}\}$ are both bases of $E|_{U_i}$, there exists a set of matrices $\{C_i\}$ such that C_i and C_i^{-1} are regular on U_i and the relations

$$(e_{i1}, e_{i2}, \cdots, e_{ir}) = (\hat{e}_{i1}, \hat{e}_{i2}, \cdots, \hat{e}_{ir})C_i \tag{3}$$

hold. By virtue of the formulae (1), (2), (3), we have

$$\hat{G}_{ij} = C_i G_{ij} C_j^{-1} . \tag{4}$$

Furthermore, suppose that E'' be the subbundle of E and $\{\hat{e}_{ir'+1}, \hat{e}_{ir'+2}, \cdots, \hat{e}_{ir}\}$ be a basis of $E''|_{U_i}$, then there exist three families of matrices $\{G'_{ij}\}$, $\{D_{ij}\}$, $\{G''_{ij}\}$ of type $r' \times r'$, $(r-r') \times r'$ and $(r-r') \times (r-r')$ respectively, which satisfy

$$\hat{G}_{ij} = \begin{pmatrix} G'_{ij} & O \\ D_{ij} & G''_{ij} \end{pmatrix} . \tag{5}$$

In this case, $\{G''_{ij}\}$ is a system of transition matrices of E'' and $\{G'_{ij}\}$ is a system of transition matrices of the quotient bundle E/E''.

DEFINITION. We say that a set of matrices $\{g_{ij}\}$ of rank r be *a system of transition matrices* of rank r if it satisfies the following conditions:
(1) g_{ij} is a regular matrix on $U_i \cap U_j$ for any i and j.
(2) $g_{ii} = I_r$ for all i, where I_r is the unit matrix of rank r.
(3) $g_{hi} g_{ij} = g_{hj}$ for all h, i and j.

The following are well-known results.

LEMMA 1. *Let $\{g_{ij}\}$ be a system of transition matrices of rank r. Then*

(1) *$\{g_{ij}\}$ is a system of transition matrices of a vector bundle of rank r on $\boldsymbol{P}^n$ with respect to some set of bases. If g_{ij} are regular on U_i, for all i and j, then this vector bundle is generated by its global sections.*

(2) *Suppose that $\{h_i\}$ be a set of $r \times r$ matrices such that h_i and h_i^{-1} are regular on U_i. Then $\{h_i g_{ij} h_j^{-1}\}$ is a system of transition matrices.*

(3) *Suppose that $\{g'_{ij}\}$, $\{d_{ij}\}$ and $\{g''_{ij}\}$ be sets of matrices of type $r' \times r'$, $r'' \times r'$ and $r'' \times r''$ respectively such that $g_{ij} = \begin{pmatrix} g'_{ij} & O \\ d_{ij} & g''_{ij} \end{pmatrix}$ for all i and j. Then $\{g'_{ij}\}$ and $\{g''_{ij}\}$ are systems of transition matrices of rank r' and r'' respectively. Let E, E' and E'' be vector bundles on $\boldsymbol{P}^n$ which have systems of transition matrices $\{g_{ij}\}$, $\{g'_{ij}\}$ and $\{g''_{ij}\}$ respectively. Then there exists an exact sequence of vector bundles*

$$0 \longrightarrow E'' \longrightarrow E \longrightarrow E' \longrightarrow 0 .$$

Let $\varphi = \varphi(x_0, x_1, \cdots, x_n)$ be an element of K, then for any element τ of G we denote

$\varphi(x_{\tau(0)}, x_{\tau(1)}, \cdots, x_{\tau(n)})$ by ${}_\tau\varphi$. Similarly when $\begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1r} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{s1} & \varphi_{s2} & \cdots & \varphi_{sr} \end{pmatrix}$ is a $s \times r$ matrix, we denote by ${}_\tau A$ the $s \times r$ matrix $\begin{pmatrix} {}_\tau\varphi_{11} & {}_\tau\varphi_{12} & \cdots & {}_\tau\varphi_{1r} \\ {}_\tau\varphi_{21} & {}_\tau\varphi_{22} & \cdots & {}_\tau\varphi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ {}_\tau\varphi_{s1} & {}_\tau\varphi_{s2} & \cdots & {}_\tau\varphi_{sr} \end{pmatrix}$. It is easy to verify the following facts.

LEMMA 2. (1) *τ induces an automorphism of K.*

(2) ${}_{(\eta\tau)}\varphi = {}_\eta({}_\tau\varphi)$.

(3) ${}_{(\eta\tau)}A = {}_\eta({}_\tau A)$ *for any matrix A.*

(4) ${}_\tau(BA) = {}_\tau B {}_\tau A$ *for any two matrices A and B.*

(5) ${}_\tau(B^{-1}) = ({}_\tau B)^{-1}$ *for any non-singular matrix B.*

(6) ${}_\tau\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}_\tau A & {}_\tau B \\ {}_\tau C & {}_\tau D \end{pmatrix}$.

(7) *If φ be an element of K_0 which is regular on U_i, then ${}_\tau\varphi$ is regular on $U_{\tau(i)}$.*

(8) *If A be a matrix which is regular on U_i, then ${}_\tau A$ is regular on $U_{\tau(i)}$.*

(9) *If A be a non-singular matrix such that A^{-1} is regular on U_i, then $({}_\tau A)^{-1}$ is regular on $U_{\tau(i)}$.*

DEFINITION. Suppose that a system of transition matrices $\{B_{ij}\}$ satisfies the condition

$${}_\tau B_{ij} = B_{\tau(i)\tau(j)} \qquad \text{for any } \tau \in G \quad \text{and} \quad \text{for any } i \text{ and } j\,.$$

Then $\{B_{ij}\}$ is called *a system of σ-transition matrices* or *a system of G-transition matrices.* $B_{\sigma(0)0}$ is called the *initial datum* of this system of σ-transition matrices $\{B_{ij}\}$.

Let $\{B_{ij}\}$ be a system of σ-transition matrices of rank r and B be its initial datum. Then for any i and j, there exist two natural numbers p and q such that

$$j = \sigma^p(0) \quad \text{and} \quad i = \sigma^q(j) = \sigma^{q+p}(0)\,.$$

Therefore we have

$$\begin{aligned} B_{\sigma^q(0)0} &= B_{\sigma^q(0)\sigma^{q-1}(0)} B_{\sigma^{q-1}(0)\sigma^{q-2}(0)} \cdots B_{\sigma(0)0} \\ &= {}_{\sigma^{q-1}}(B_{\sigma(0)0})_{\sigma^{q-2}}(B_{\sigma(0)0}) \cdots {}_\sigma(B_{\sigma(0)0})(B_{\sigma(0)0}) = {}_{\sigma^{q-1}}B_{\sigma^{q-2}}B \cdots {}_\sigma BB\,. \end{aligned}$$

Thus

$$B_{ij} = B_{\sigma^{p+q}(0)\sigma^p(0)} = {}_{\sigma^p}B_{\sigma^q(0)0} = {}_{\sigma^{p+q-1}}B_{\sigma^{p+q-2}}B \cdots {}_{\sigma^{p+1}}B_{\sigma^p}B\,. \tag{6}$$

This shows that a system of σ-transition matrices $\{B_{ij}\}$ is determined by its initial datum B.

Suppose that B $(= B_{\sigma(0)0})$ is regular on $U_{\sigma(0)}$. Then $B_{\sigma^2(0)\sigma(0)} = {}_\sigma B_{\sigma(0)0}$ is regular on $U_{\sigma^2(0)}$. Therefore $B_{\sigma^2(0)0} = B_{\sigma^2(0)\sigma(0)} B_{\sigma(0)0}$ is regular on $U_{\sigma^2(0)} \cap U_{\sigma(0)}$. Since, by definition,

$B_{\sigma^2(0)0}$ is regular on $U_{\sigma^2(0)}\cap U_0$, this is regular on $U_{\sigma^2(0)}\cap(U_0\cup U_{\sigma(0)})$. This shows that $B_{\sigma^2(0)0}$ is regular on $U_{\sigma^2(0)}$. Since $B_{\sigma^3(0)0}={}_{\sigma}B_{\sigma^2(0)0}B_{\sigma(0)0}$ is regular on $U_{\sigma^3(0)}\cap(U_0\cup U_{\sigma(0)})$, this is regular on $U_{\sigma^3(0)}$. In the same way as above, we can show that B_{i0} is regular on U_i. For any j, there exists an element $\tau\in G$ such that $j=\tau(0)$. Since $B_{\tau^{-1}(i)0}$ is regular on $U_{\tau^{-1}(i)}$, B_{ij} $(={}_{\tau}B_{\tau^{-1}(i)0})$ is regular on U_i. In this case, the vector bundle determined by this system of σ-transition matrices is generated by its global sections (cf. Lemma 1 (1)).

Next, suppose that B is regular on U_0. In a similar way as above, we can show that B_{ij} is regular on U_j for any pair i and j. Let E be the vector bundle determined by $\{B_{ij}\}$. Then it is easy to verify that $\{{}^tB_{ij}^{-1}\}$ is a system of σ-transition matrices with initial datum ${}^tB_{\sigma(0)0}^{-1}$ which determines the dual vector bundle $\check{E}$. Since ${}^tB_{\sigma(0)0}^{-1}={}^tB_{0\sigma(0)}$, this matrix is regular on $U_{\sigma(0)}$. Hence $\check{E}$ is generated by its global sections. Thus, in this case, the vector bundle determined by this system of σ-transition matrices $\{B_{ij}\}$ is a subbundle of a trivial bundle.

Let C be a non-singular matrix such that C and C^{-1} are regular on U_0. For arbitrary element η of G let us denote ${}_{\eta}C$ by $C_{\eta(0)}$. Then it is easy to see that the relations ${}_{\tau}C_i=C_{\tau(i)}$ hold for all i and τ. Since C_0 and C_0^{-1} are regular on U_0, C_i and C_i^{-1} are regular on U_i (cf. Lemma 2 (8), (9)). Therefore, we see that

$$\begin{aligned}
&C_iB_{ij}C_j^{-1} \quad \text{is regular on} \quad U_i\cap U_j\,,\\
&C_iB_{ii}C_i^{-1}=I_r\,,\\
&(C_hB_{hi}C_i^{-1})(C_iB_{ij}C_j^{-1})=C_hB_{hj}C_j^{-1} \quad \text{and}\\
&{}_{\tau}(C_iB_{ij}C_j^{-1})={}_{\tau}C_i\,{}_{\tau}B_{ij}\,{}_{\tau}C_j^{-1}=C_{\tau(i)}B_{\tau(i)\tau(j)}C_{\tau(j)}^{-1}\,.
\end{aligned}$$

These imply that $\{C_iB_{ij}C_j^{-1}\}$ is a system of σ-transition matrices and $C_{\sigma(0)}B_{\sigma(0)0}C_0^{-1}$ is its initial datum.

Let m be an arbitrary integer. Then it is easy to check that

$$\begin{aligned}
&(x_i/x_j)^mB_{ij} \quad \text{is regular on} \quad U_i\cap U_j\,,\\
&(x_i/x_i)^mB_{ii}=I_r\,,\\
&((x_h/x_i)^mB_{hi})((x_i/x_j)^mB_{ij})=(x_h/x_j)^mB_{hj} \quad \text{and}\\
&{}_{\tau}((x_i/x_j)^mB_{ij})=(x_{\tau(i)}/x_{\tau(j)})^mB_{\tau(i)\tau(j)}\,.
\end{aligned}$$

These imply that $(x_i/x_j)^mB_{ij}$ is a system of σ-transition matrices and $(x_{\sigma(0)}/x_0)^mB_{\sigma(0)0}$ is its initial datum. Suppose that $B=\begin{pmatrix} B' & O\\ D & B''\end{pmatrix}$, where B', D and B'' are matrices of type $r'\times r'$, $r''\times r'$ and $r''\times r''$ respectively. Then formula (6) shows that there exist three families of matrices $\{B'_{ij}\}$, $\{D_{ij}\}$ and $\{B''_{ij}\}$ such that $B_{ij}=\begin{pmatrix} B'_{ij} & O\\ D_{ij} & B''_{ij}\end{pmatrix}$. This and Lemma 1 (3) and Lemma 2 (6) show that $\{B'_{ij}\}$ is a system of σ-transition matrices with initial datum B' and $\{B''_{ij}\}$ is a system of σ-transition matrices with initial datum B''.

We summarise these results as follows:

LEMMA 3. *Let $\{B_{ij}\}$ be a system of σ-transition matrices of rank r with initial datum B. Then*

(1) *Initial datum B furnishes us with complete information about the system of σ-transition matrices $\{B_{ij}\}$.*

(2) *When B is regular on $U_{\sigma(0)}$, the vector bundle determined by $\{B_{ij}\}$ is generated by its global sections.*

(3) *When B is regular on U_0, the vector bundle determined by $\{B_{ij}\}$ is a subbundle of a trivial vector bundle.*

(4) *When C is a non-singular $r\times r$ matrix such that C and C^{-1} are regular on U_0, there exists a system of σ-transition matrices with initial datum ${}_{\sigma}CBC^{-1}$.*

(5) *For any integer m, there exists a system of σ-transition matrices with initial datum $(x_{\sigma(0)}/x_0)^m B_{\sigma(0)0}$.*

(6) *Suppose that $B=\begin{pmatrix} B' & O \\ D & B'' \end{pmatrix}$, where B', D and B'' are matrices of type $r'\times r'$, $r''\times r'$ and $r''\times r''$ respectively. Then there exist two systems of σ-transition matrices with intial datum B' and B''.*

3. Standard σ-matrix.

Let s, which may depend on n, be a positive integer. We denote the K-vector space $\{(\psi_0\psi_1\cdots\psi_s)\,|\,K\ni\psi_i\}$ by K^{s+1}. Let us consider an action of G to K^{s+1} which satisfies the following conditions

$$ {}^{\tau}(c_1\vec{\psi}_1+c_2\vec{\psi}_2)=c_1{}^{\tau}\vec{\psi}_1+c_2{}^{\tau}\vec{\psi}_2 \tag{7}$$

$$ {}^{\tau}(\varphi\vec{\psi})={}_{\tau}\varphi{}^{\tau}\vec{\psi} \tag{8}$$

$$ {}^{1}\vec{\psi}=\vec{\psi} \tag{9}$$

$$ {}^{(\eta\tau)}\vec{\psi}={}^{\eta}({}^{\tau}\vec{\psi}) \tag{10}$$

where $k\ni c_1, c_2$, $K\ni\varphi$, $K^{s+1}\ni\vec{\psi}, \vec{\psi}_1, \vec{\psi}_2$ and $G\ni\eta, \tau$, $1=\sigma^0$.

When $A=\begin{pmatrix}\vec{\psi}_1\\ \vec{\psi}_2\\ \vdots\\ \vec{\psi}_r\end{pmatrix}$ is an $r\times(s+1)$ matrix, we denote $\begin{pmatrix}{}^{\tau}\vec{\psi}_1\\ {}^{\tau}\vec{\psi}_2\\ \vdots\\ {}^{\tau}\vec{\psi}_r\end{pmatrix}$ by ${}^{\tau}A$. It is easy to verify the following lemma.

LEMMA 4. *Let A be an $r\times(s+1)$ matrix and B be an $r\times r$ matrix. Then for any elements η and τ of G we have*

(1) $\operatorname{rank}A=\operatorname{rank}{}^{\tau}A$,

(2) ${}^{\eta\tau}A={}^{\eta}({}^{\tau}A)$,

(3) ${}^{\tau}(BA)={}_{\tau}B\,{}^{\tau}A$.

DEFINITION. Let A be an $r\times(s+1)$ matrix of rank r. If there exists an $r\times r$ matrix B which satisfies

$$^{\sigma}A=BA\ , \tag{11}$$

we say that A is *a σ-stable*, or *G-stable*, *matrix* of rank r and we say that B is *the initial datum* of σ-stable matrix A. For the sake of brevity, we sometimes say: "(A; B) is a σ-stable, or G-stable, matrix of rank r" instead of saying: "A is a σ-stable matrix of rank r with initial datum B".

LEMMA 5. *Let (A; B) be a σ-stable matrix of rank r. For every element ξ of G, we denote $^{\xi}A$ by $A_{\xi(0)}$. Then we have*

(1) $^{\tau}A_i=A_{\tau(i)}$.

(2) *There exists uniquely a system of $r\times r$-matrices $\{B_{ij}\}$ which satisfies*

$$A_i=B_{ij}A_j\ .$$

PROOF. (1) Let η be the element of G which satisfies $\eta(0)=i$, then we have

$$^{\tau}A_i={}^{\tau}A_{\eta(0)}={}^{\tau\eta}A=A_{\tau\eta(0)}=A_{\tau(i)}\quad\text{(cf. Lemma 4)}\ .$$

(2) Let p and q be two positive integers such that $j=\sigma^p(0)$ and $i=\sigma^q(j)=\sigma^{p+q}(0)$. Then we have

$$\begin{aligned}{}^{\sigma^{p+q}}A&={}^{\sigma^{p+q-1}}({}^{\sigma}A)={}^{\sigma^{p+q-1}}(BA)={}_{\sigma^{p+q-1}}B\,{}^{\sigma^{p+q-1}}A\quad\text{(cf. Lemma 4)}\\&={}_{\sigma^{p+q-1}}B\,{}_{\sigma^{p+q-2}}B\cdots{}_{\sigma^p}B\,{}^{\sigma^p}A\ .\end{aligned}\tag{12}$$

Put

$$B_{ij}={}_{\sigma^{p+q-1}}B\,{}_{\sigma^{p+q-2}}B\cdots{}_{\sigma^p}B\,{}^{\sigma^p}A\ . \tag{13}$$

Since $A_i=A_{\sigma^{p+q}(0)}={}^{\sigma^{p+q}}A$ and $A_j={}^{\sigma^p}A$, formula (13) implies $A_i=B_{ij}A_j$. Since $\operatorname{rank}A_i=\operatorname{rank}A=r=\operatorname{rank}A_j$, it is easy to show the uniqueness of $\{B_{ij}\}$.

DEFINITION. Under the same notation as in Lemma 5, we say that $\{A_i, B_{ij}\}$ is the *σ-data* of σ-stable matrix A.

LEMMA 6. *Let (A; B) be a σ-stable matrix of rank r and $\{A_i, B_{ij}\}$ be the σ-data of A. Then*

(1) $B_{ii}=I_r$.

(2) $B_{hi}B_{ij}=B_{hj}$.

(3) ${}_{\tau}B_{ij}=B_{\tau(i)\tau(j)}$.

(4) $B=B_{\sigma(0)0}$.

(5) *$\{B_{ij}\}$ is determined only by the initial datum B of A.*

PROOF. By virtue of the uniqueness of $\{B_{ij}\}$ and the following equations

$$A_i=I_rA_i\ ,\quad A_h=B_{hi}A_i=B_{hi}B_{ij}A_j\ ,\quad A_{\sigma(0)}={}^{\sigma}A=BA=BA_0\ ,\quad\text{and}$$
$$A_{\tau(i)}={}^{\tau}A_i={}^{\tau}(B_{ij}A_j)={}_{\tau}B_{ij}\,{}^{\tau}A_j={}_{\tau}B_{ij}A_{\tau(j)}\ ,$$

we have (1)–(4). Formula (13) implies (5).

Formulae (6) and (13) show that:

PROPOSITION 1. *Let $(A;B)$ be a σ-stable matrix and $\{A_i, B_{ij}\}$ be the σ-data of A. If $\{B'_{ij})$ is a system of σ-transition matrices with initial datum B. Then*

$$\{B_{ij}\}=\{B'_{ij}\}.$$

Let $(A;B)$ be a σ-stable matrix and $\{A_i, B_{ij}\}$ be the σ-data of A. Lemma 6 shows that $\{B_{ij}\}$ is a system of σ-transition matrices if and only if B_{ij} is regular on $U_i\cap U_j$, for every i and j. Suppose that for arbitrary h, B_{h0} is regular on $U_h\cap U_0$. Let η be an element of G such that $j=\eta(0)$. Then $B_{ij}={}_\eta B_{\eta^{-1}(i)0}$ is regular on $U_{\eta\eta^{-1}(i)}\cap U_{\eta(0)}$, i.e., B_{ij} is regular on $U_i\cap U_j$. Thus we have

DEFINITION. Let A be a σ-stable matrix with initial datum B and let $\{A_i, B_{ij}\}$ be the set of σ-data of A. We say that A is *a σ-matrix* or $(A;B)$ is a σ-matrix if $\{B_{ij}\}$ is a system of σ-transition matrices.

THEOREM 1. *Let A be a σ-stable matrix and $\{A_i, B_{ij}\}$ be the σ-data of A. Then A is a σ-matrix if and only if B_{i0} is regular on $U_i\cap U_0$ for every i.*

It is a simple matter to verify that the following lemmas hold.

LEMMA 7. *Let A be a σ-stable matrix of rank r and let $\{A_i, B_{ij}\}$ be the σ-data of A. Let C be a non-singular matrix of rank r. Then $(CA;{}_\sigma CBC^{-1})$ is a σ-stable matrix of rank r and $\{C_iA_i, C_iB_{ij}C_j^{-1}\}$ is the σ-data of CA. Suppose that A be a σ-matrix and C and C^{-1} be regular on U_0, then CA is a σ-matrix and hence $\{C_iB_{ij}C_j^{-1}\}$ is a system of σ-transition matrices.*

LEMMA 8. *Let $(A;B)$ be a σ-matrix of rank r and $(A';B')$ be a σ-matrix of rank r'. Let C be an $r\times r$-matrix such that C and C^{-1} are regular on U_0. Assume that there exists an $r''\times(s+1)$-matrix A'' such that $CA=\begin{pmatrix}A'\\A''\end{pmatrix}$. Then there exist two matrices D and B'' of type $r''\times r'$ and $r''\times r''$ respectively such that ${}_\sigma CBC^{-1}=\begin{pmatrix}B'_{ij} & O\\ D_{ij} & B''_{ij}\end{pmatrix}$. Then there exists a system of σ-transition matrices which has B'' as its initial datum.*

LEMMA 9. *Let A' and A'' be two σ-matrices with the same initial datum B. Then, under the new action of G to $(A'\ A'')$ defined by ${}^\sigma(A'\ A'')=({}^\sigma A'\ {}^\sigma A'')$, $(A'\ A'')$ becomes a σ-matrix with initial datum B.*

In the following, let $s=n$ and let the action of G to K^{n+1} be the standard one, i.e., which is defined by

$${}^\tau(\psi_0\ \psi_1\ \cdots\ \psi_n)=({}_\tau\psi_{\tau^{-1}(0)}\ {}_\tau\psi_{\tau^{-1}(1)}\ \cdots\ {}_\tau\psi_{\tau^{-1}(n)})\,. \tag{14}$$

It is a simple matter to verify that this action satisfies the conditions (7)–(10).

DEFINITION. We shall say that a σ-stable matrix A is *a standard σ-stable matrix* if the action of G to A is the standard one. Similarly we shall say that a σ-matrix A is *a standard σ-matrix* if the action of G to A is the standard one.

THEOREM 2. *Let* $\begin{pmatrix} \varphi_{10} & \varphi_{11} & \cdots & \varphi_{1n} \\ \varphi_{20} & \varphi_{21} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{r0} & \varphi_{r1} & \cdots & \varphi_{rn} \end{pmatrix}$ *be a standard σ-stable matrix of rank r. Assume that all components of A are homogeneous polynomials of degree m and there exist r integers $s_1, s_2, \cdots, s_r$ such that*

$$\begin{pmatrix} \varphi_{1s_1} & \varphi_{1s_2} & \cdots & \varphi_{1s_r} \\ \varphi_{2s_1} & \varphi_{2s_2} & \cdots & \varphi_{2s_r} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{rs_1} & \varphi_{rs_2} & \cdots & \varphi_{rs_r} \end{pmatrix} = x_0^m I_r \,. \tag{15}$$

Then A is a standard σ-matrix with initial datum

$$\begin{pmatrix} \psi_{1t_1} & \psi_{1t_2} & \cdots & \psi_{1t_r} \\ \psi_{2t_1} & \psi_{2t_2} & \cdots & \psi_{2t_r} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{rt_1} & \psi_{rt_2} & \cdots & \psi_{rt_r} \end{pmatrix} \quad \textit{where } \psi_{it_j} = {}_\sigma\varphi_{i\sigma^{-1}(s_j)} \,. \tag{16}$$

PROOF. Let τ be an arbitrary element of G. Put $\varphi'_{ij} = {}_\tau\varphi_{i\tau^{-1}(s_j)}$. Since ${}_\tau A = B_{\tau(0)0}A$, we have

$$\begin{pmatrix} \varphi'_{10} & \varphi'_{11} & \cdots & \varphi'_{1n} \\ \varphi'_{20} & \varphi'_{21} & \cdots & \varphi'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi'_{r0} & \varphi'_{r1} & \cdots & \varphi'_{rn} \end{pmatrix} = B_{\tau(0)0} \begin{pmatrix} \varphi_{10} & \varphi_{11} & \cdots & \varphi_{1n} \\ \varphi_{20} & \varphi_{21} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{r0} & \varphi_{r1} & \cdots & \varphi_{rn} \end{pmatrix},$$

in particular we have

$$\begin{pmatrix} \varphi'_{1s_1} & \varphi'_{1s_2} & \cdots & \varphi'_{1s_r} \\ \varphi'_{2s_1} & \varphi'_{2s_2} & \cdots & \varphi'_{2s_r} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi'_{rs_1} & \varphi'_{rs_2} & \cdots & \varphi'_{rs_r} \end{pmatrix} = B_{\tau(0)0} \begin{pmatrix} \varphi_{1s_1} & \varphi_{1s_2} & \cdots & \varphi_{1s_r} \\ \varphi_{2s_1} & \varphi_{2s_2} & \cdots & \varphi_{2s_r} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{rs_1} & \varphi_{rs_2} & \cdots & \varphi_{rs_r} \end{pmatrix} = x_0^m B_{\tau(0)0} \,.$$

This shows that

$$B_{\tau(0)0} = x_0^{-m} \begin{pmatrix} \varphi'_{1s_1} & \varphi'_{1s_2} & \cdots & \varphi'_{1s_r} \\ \varphi'_{2s_1} & \varphi'_{2s_2} & \cdots & \varphi'_{2s_r} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi'_{rs_1} & \varphi'_{rs_2} & \cdots & \varphi'_{rs_r} \end{pmatrix}.$$

Since this matrix is regular on U_0, Theorem 1 asserts that A is a σ-matrix. Since the initial datum of this matrix is $B_{\sigma(0)0}$, the latter half of the theorem is obvious.

The following is proven similary as above.

COROLLARY 1. *Let $A', A'', \cdots$ be standard σ-stable matrices of rank r with the same initial datum B. Assume that all components of $A', A'', \cdots$ are homogeneous polynomials of degree m and some $r \times r$ submatrix of $(A' \; A'' \cdots)$ has cx_0^{mr} as its determinant, where c is a non-zero constant. Then $A', A'', \cdots$ are σ-matrices.*

We can construct a standard σ-matrix using a given system of σ-transition matrices. In the following we give a method of construction.

PROPOSITION 2. *Let $\{B_{ij}\}$ be a system of σ-transition marices of rank r with initial datum B. Let us fix an integer s such that $1 \leq s \leq n$ and fix an integer m. For each i, let us denote the s-th column of B_{0i} by $\vec{b}_i^{(s)}$. Put $\vec{a}_i^{(s)} = x_i^m \vec{b}_i^{(s)}$. Then*

$$A^{(s)} = (\vec{a}_0^{(s)} \; \vec{a}_1^{(s)} \; \vec{a}_2^{(s)} \cdots \vec{a}_n^{(s)})$$

is a standard σ-matrix with initial datum B, if $\operatorname{rank} A^{(s)} = r$.

PROOF. Since

$${}_\sigma B_{0\sigma^{-1}(i)} = B_{\sigma(0)i} = B_{\sigma(0)0} B_{0i} = BB_{0i} ,$$

we have

$${}_\sigma \vec{b}_{\sigma^{-1}(i)}^{(s)} = B\vec{b}_i^{(s)} .$$

This shows that

$${}_\sigma \vec{a}_{\sigma^{-1}(i)}^{(s)} = {}_\sigma(x_{\sigma^{-1}(i)}^m \vec{b}_{\sigma^{-1}(i)}^{(s)}) = x_i^m {}_\sigma \vec{b}_{\sigma^{-1}(i)}^{(s)} = x_i^m B\vec{b}_i^{(s)} = B\vec{a}_i^{(s)} .$$

Therefore, under the standard action, we have

$$\begin{aligned}
{}^\sigma A^{(s)} &= {}^\sigma(\vec{a}_0^{(s)} \; \vec{a}_1^{(s)} \; \vec{a}_2^{(s)} \cdots \vec{a}_n^{(s)}) \\
&= ({}_\sigma \vec{a}_{\sigma^{-1}(0)}^{(s)} \; {}_\sigma \vec{a}_{\sigma^{-1}(1)}^{(s)} \; {}_\sigma \vec{a}_{\sigma^{-1}(2)}^{(s)} \cdots {}_\sigma \vec{a}_{\sigma^{-1}(n)}^{(s)}) \\
&= (B\vec{a}_0^{(s)} \; B\vec{a}_1^{(s)} \; B\vec{a}_2^{(s)} \cdots B\vec{a}_n^{(s)}) \\
&= B(\vec{a}_0^{(s)} \; \vec{a}_1^{(s)} \; \vec{a}_2^{(s)} \cdots \vec{a}_n^{(s)}) \\
&= BA^{(s)} .
\end{aligned}$$

When $\operatorname{rank} A^{(s)} = r$, this shows that $A^{(s)}$ is a standard σ-stable matrix with initial datum B. Then by virtue of Proposition 1 we have $\{B'_{ij}\} = \{B_{ij}\}$. This means that $A^{(s)}$ is a standard σ-matrix with initial datum B. The proof of the last half of this proposition is similar to the proof of Theorem 2.

4. Example 1.

Let σ be the cyclic permutation $(0\ 1\ 2 \cdots n)$ and let us consider the standard action of G. Let

$$A' = (x_0\ x_1\ x_2 \cdots x_n)\,.$$

Then easily we have

$${}^{\sigma}A' = A'\,.$$

This and Theorem 2 show that A' is a standard σ-matrix with initial datum 1. Let us consider the following matrix A which has A' as submatrix.

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_n \\ 0 & x_0 & 0 & \cdots & 0 \\ 0 & 0 & x_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_0 \end{pmatrix}.$$

The relation ${}^{\sigma}A = ({}^{\sigma}AA^{-1})A$ and Theorem 2 show that A is a standard σ-matrix with initial datum ${}^{\sigma}AA^{-1}$. Now let us compute the matrix ${}^{\sigma}AA^{-1}$.

$${}^{\sigma}AA^{-1} = x_0^{-2} \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_n \\ 0 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_1 \\ x_1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0 & -x_1 & -x_2 & \cdots & -x_n \\ 0 & x_0 & 0 & \cdots & 0 \\ 0 & 0 & x_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_0 \end{pmatrix}$$

$$= x_0^{-2} \begin{pmatrix} x_0^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_0x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x_0x_1 \\ x_0x_1 & -x_1^2 & -x_1x_2 & \cdots & -x_1x_{n-1} & -x_1x_n \end{pmatrix}. \tag{17}$$

Lemma 3 (4) shows that there exists a system of σ-transition matrices which has

$$x_0^{-2} \begin{pmatrix} 0 & x_0x_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_0x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x_0x_1 \\ -x_1^2 & -x_1x_2 & -x_1x_3 & \cdots & -x_1x_{n-1} & -x_1x_n \end{pmatrix} \tag{18}$$

as its initial datum. Hence, there exists a vector bundle of rank n on $\boldsymbol{P}^n$ which is determined by this matrix (18) (cf. sect. 7 *4-1).

Let us consider the following matrix

$$A''=\begin{pmatrix} -x_1 & x_0 & 0 & 0 & \cdots & 0 & 0 \\ -x_2 & 0 & x_0 & 0 & \cdots & 0 & 0 \\ -x_3 & 0 & 0 & x_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1} & 0 & 0 & 0 & \cdots & x_0 & 0 \\ -x_n & 0 & 0 & 0 & \cdots & 0 & x_0 \end{pmatrix}.$$

Then we have

$${}^{\sigma}A''=\begin{pmatrix} 0 & -x_2 & x_1 & 0 & 0 & \cdots & 0 \\ 0 & -x_3 & 0 & x_1 & 0 & \cdots & 0 \\ 0 & -x_4 & 0 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -x_n & 0 & 0 & 0 & \cdots & x_1 \\ x_1 & -x_0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$B'=\begin{pmatrix} -x_2 & x_1 & 0 & 0 & \cdots & 0 \\ -x_3 & 0 & x_1 & 0 & \cdots & 0 \\ -x_4 & 0 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_n & 0 & 0 & 0 & \cdots & x_1 \\ -x_0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then it is easy to verify the following relation holds.

$${}^{\sigma}A''=B'A''.$$

Theorem 2 shows that A'' is a standard σ-matrix of rank n with initial datum B'. Let $\{A_i''; B_{ij}''\}$ be the σ-data of A''. Then $\{B_{ij}''\}$ is a system of σ-transition matrices which has B' as its initial datum. $\{B_{ij}''\}$ determines a vector bundle of rank n on P^n (cf. sect. 7 *4-2).

5. Example 2.

In this section, using the terms of system of σ-transition matrices, we will reconstruct the Horrocks bundle of rank 3 on P^5 (cf. Horrocks [3]).

Let $n=5$, and σ be the cyclic permutation (0 4 2 3 1 5), and the action of G be the standard action. Let

$$A=\begin{pmatrix} -x_1 & x_0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & x_0 & 0 & 0 \\ -x_4 & 0 & 0 & 0 & x_0 & 0 \\ -x_5 & 0 & 0 & 0 & 0 & x_0 \end{pmatrix}.$$

Then

$$^{\sigma}A=\begin{pmatrix} 0 & 0 & 0 & 0 & -x_5 & x_4 \\ 0 & 0 & 0 & x_4 & -x_3 & 0 \\ 0 & x_4 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & x_4 & 0 & -x_2 & 0 \\ x_4 & 0 & 0 & 0 & -x_0 & 0 \end{pmatrix}.$$

Let

$$B=x_0^{-1}\begin{pmatrix} 0 & 0 & 0 & -x_5 & x_4 \\ 0 & 0 & x_4 & -x_3 & 0 \\ x_4 & 0 & 0 & -x_1 & 0 \\ 0 & x_4 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & -x_0 & 0 \end{pmatrix}.$$

Then it is easy to verify the formula

$$^{\sigma}A=BA$$

holds. Hence, Theorem 2 shows that B is the initial datum of the standard σ-marix A. Let

$$C=x_0^{-1}\begin{pmatrix} -x_4 & -x_5 & x_0 & x_1 & x_2 \\ x_0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}.$$

Then we have

$$CA=\begin{pmatrix} -x_3 & -x_4 & -x_5 & x_0 & x_1 & x_2 \\ -x_1 & x_0 & 0 & 0 & 0 & 0 \\ -x_2 & 0 & x_0 & 0 & 0 & 0 \\ -x_4 & 0 & 0 & 0 & x_0 & 0 \\ -x_5 & 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$$

and

$$C^{-1}=x_0^{-1}\begin{pmatrix} 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ x_0 & x_4 & x_5 & -x_1 & -x_2 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}.$$

Note the fact that C and C^{-1} are regular on U_0 and

$${}^{\sigma}(-x_3\ -x_4\ -x_5\ x_0\ x_1\ x_2)=-(-x_3\ -x_4\ -x_5\ x_0\ x_1\ x_2)\,.$$

Let us compute ${}_{\sigma}CBC^{-1}$.

$${}_{\sigma}CBC^{-1}=$$

$$x_0^{-2}x_4^{-1}\begin{pmatrix} -x_2 & -x_0 & x_4 & x_5 & x_3 \\ x_4 & 0 & 0 & 0 & 0 \\ 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & x_4 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -x_5 & x_4 \\ 0 & 0 & x_4 & -x_3 & 0 \\ x_4 & 0 & 0 & -x_1 & 0 \\ 0 & x_4 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & -x_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ x_0 & x_4 & x_5 & -x_1 & -x_2 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$$

$$=x_0^{-2}\begin{pmatrix} x_4 & x_5 & -x_0 & -x_1 & -x_2 \\ 0 & 0 & 0 & -x_5 & x_4 \\ 0 & 0 & x_4 & -x_3 & 0 \\ 0 & x_4 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & -x_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 \\ x_0 & x_4 & x_5 & -x_1 & -x_2 \\ 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & x_0 \end{pmatrix}$$

$$=x_0^{-2}\begin{pmatrix} -x_0^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_0x_5 & x_0x_4 \\ x_0x_4 & x_4^2 & x_4x_5 & -x_1x_4-x_0x_3 & -x_2x_4 \\ 0 & 0 & x_0x_4 & -x_0x_2 & 0 \\ 0 & 0 & 0 & -x_0^2 & 0 \end{pmatrix}.$$

These and Lemma 3 (4) show that there exists a system of σ-transition matrices which has

$$x_0^{-2}\begin{pmatrix} 0 & 0 & -x_0x_5 & x_0x_4 \\ x_4^2 & x_4x_5 & -x_1x_4-x_0x_3 & -x_2x_4 \\ 0 & x_0x_4 & -x_0x_2 & 0 \\ 0 & 0 & -x_0^2 & 0 \end{pmatrix} \tag{19}$$

as its initial datum (cf. sect. 7 *5-1). Using the method of Proposition 2, we have the following two standard σ-matrices which have the matrix (19) as their initial datum:

$$\begin{pmatrix} x_0^2 & 0 & -x_1x_2 & -x_1^2 & x_0x_2 & 0 \\ 0 & x_0x_1 & -x_2^2 & -x_1x_2 & 0 & x_0^2 \\ 0 & x_0x_5 & -x_2x_4 & x_0x_3-x_1x_4 & 0 & 0 \\ 0 & 0 & x_0x_3-x_2x_5 & -x_1x_5 & x_0x_4 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & x_1x_5 & -x_0x_3-x_1x_4 & x_0x_5 \\ 0 & 0 & 0 & x_0x_3+x_2x_5 & -x_2x_4 & -x_0x_4 \\ 0 & -x_0^2 & x_0x_2 & x_4x_5 & -x_4^2 & 0 \\ x_0^2 & 0 & -x_0x_1 & x_5^2 & -x_4x_5 & 0 \end{pmatrix}.$$

Let us consider two times exterior product of the matrix (19):

$$x_0^{-3}\begin{pmatrix} 0 & x_4^2x_5 & -x_4^3 & x_4x_5^2 & -x_4^2x_5 & x_4\tilde{x} \\ 0 & 0 & 0 & x_0x_4x_5 & -x_0x_4^2 & x_0x_2x_4 \\ 0 & 0 & 0 & 0 & 0 & x_0^2x_4 \\ x_4^3 & -x_2x_4^2 & 0 & x_4(x_0x_3+x_1x_4-x_2x_5) & x_2x_4^2 & -x_2^2x_4 \\ 0 & -x_0x_4^2 & 0 & -x_0x_4x_5 & 0 & -x_0x_2x_4 \\ 0 & 0 & 0 & -x_0^2x_4 & 0 & 0 \end{pmatrix} \tag{20}$$

where $\tilde{x}=x_0x_3+x_1x_4+x_2x_5$. Let us denote this matrix by $B^{(1)}$. Since the matrix (19) is an initial datum of a system of σ-transition matrices, so is this matrix $B^{(1)}$.

For a moment, we will assume that k is a field of characteristic not equal to two. Let us consider the matrix

$$C^{(1)}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and its inverse

$$C^{(1)^{-1}}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Both matrices $C^{(1)}$ and $C^{(1)^{-1}}$ are regular on U_0. Computing ${}_\sigma C^{(1)}B^{(1)}C^{(1)^{-1}}$, we have the following matrix:

$$x_0^{-3}\begin{pmatrix} 0 & x_4^2x_5 & -x_4^3 & x_4x_5^2 & x_4\tilde{x} & 0 \\ 0 & x_0x_4^2 & 0 & 2x_0x_4x_5 & 2x_0x_2x_4 & 0 \\ 0 & 0 & 0 & 0 & x_0^2x_4 & 0 \\ x_4^3 & -x_2x_4^2 & 0 & x_4(x_0x_3+x_1x_4-x_2x_5) & -x_2^2x_4 & 0 \\ 0 & 0 & 0 & -x_0^2x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_0^2x_4 \end{pmatrix}.$$

Lemma 3 (4) shows that the matrix

$$x_0^{-3}\begin{pmatrix} 0 & x_4^2x_5 & -x_4^3 & x_4x_5^2 & x_4\tilde{x} \\ 0 & x_0x_4^2 & 0 & 2x_0x_4x_5 & 2x_0x_2x_4 \\ 0 & 0 & 0 & 0 & x_0^2x_4 \\ x_4^3 & -x_2x_4^2 & 0 & x_4(x_0x_3+x_1x_4-x_2x_5) & -x_2^2x_4 \\ 0 & 0 & 0 & -x_0^2x_4 & 0 \end{pmatrix} \tag{21}$$

is an initial datum of a system of σ-transition matrices. Let us denote this matrix by $B^{(2)}$. We proved that this matrix $B^{(2)}$ is an initial datum of a system of σ-transition matrices, under the assumption that k is a field of characteristic not equal to two. But this fact shows that, in the case when k is a field of characteristic two, $B^{(2)}$ is an initial datum of a system of σ-transition matrices. In the following, we assume that k is a field of arbitrary characteristic. Furthermore, let

$$C^{(2)}=x_0^{-2}\begin{pmatrix} x_0^2 & -x_4x_5 & x_4^2 & -x_5^2 & -\tilde{x} \\ 0 & x_0^2 & 0 & 0 & 0 \\ 0 & 0 & x_0^2 & 0 & 0 \\ 0 & 0 & 0 & x_0^2 & 0 \\ 0 & 0 & 0 & 0 & x_0^2 \end{pmatrix}.$$

Then $C^{(2)}$ and $C^{(2)^{-1}}$ are regular on U_0. Computing ${}_\sigma C^{(2)}B^{(2)}C^{(2)^{-1}}$, we have

$$x_0^{-5}\begin{pmatrix} -x_0^4x_4 & 0 & 0 & 0 & 0 \\ 0 & x_0^3x_4^2 & 0 & 2x_0^3x_4x_5 & 2x_0^3x_2x_4 \\ 0 & 0 & 0 & 0 & x_0^4x_4 \\ x_0^2x_4^3 & -x_0^2x_2x_4^2+x_4^4x_5 & -x_4^5 & x_0^2x_4\tilde{x}-2x_0^2x_1x_4x_5+x_4^3x_5^2 & -x_0^2x_2^2x_4+x_4^3\tilde{x} \\ 0 & 0 & 0 & -x_0^4x_4 & 0 \end{pmatrix}.$$

Therefore, Lemma 3 (4) shows that there exists a system of σ-transition matrix which has the matrix

$$X_0^{-5}\begin{pmatrix} x_0^3x_4^2 & 0 & 2x_0^3x_4x_5 & 2x_0^3x_2x_4 \\ 0 & 0 & 0 & x_0^4x_4 \\ -x_0^2x_2x_4^2+x_4^4x_5 & -x_4^5 & x_0^2x_4\tilde{x}-2x_0^2x_1x_4x_5+x_4^3x_5^2 & -x_0^2x_2^2x_4+x_4^3\tilde{x} \\ 0 & 0 & -x_0^4x_4 & 0 \end{pmatrix} \tag{22}$$

as its ititial datum (cf. sect. 7 *5-2). We denote this matrix by $B^{(3)}$. Finally, let us consider the following matrix

$$C^{(3)}=x_0^{-2}\begin{pmatrix} x_0^2 & 0 & 0 & -2x_4x_5 \\ 0 & x_0^2 & 0 & -x_5^2 \\ 0 & 0 & x_0^2 & x_4^2 \\ 0 & 0 & 0 & x_0^2 \end{pmatrix}.$$

Then $C^{(3)}$ and $C^{(3)^{-1}}$ are regular on U_0, and ${}_\sigma C^{(3)}B^{(3)}C^{(3)^{-1}}$ is equal to

$$x_0^{-5}x_4^{-1}\begin{pmatrix} x_0^3x_4^3 & 0 & 2(x_0^3x_4^2x_5+x_0^5x_2) & 0 \\ 0 & 0 & x_0^6 & 0 \\ -x_0^2x_2x_4^3+x_4^5x_5 & -x_4^6 & x_0^2x_4^2\tilde{x}-2x_0^2x_2x_4^2x_5+x_4^4x_5^2-x_0^4x_2^2 & 0 \\ 0 & 0 & -x_4^6 & x_0^2x_4^4 \end{pmatrix}.$$

By virtue of Lemma 3 (3) and (4), there exists a system of σ-transition matrices which has

$$x_0^{-6}\begin{pmatrix} x_0^3x_4^3 & 0 & 2(x_0^3x_4^2x_5+x_0^5x_2) \\ 0 & 0 & x_0^6 \\ -x_0^2x_2x_4^3+x_4^5x_5 & -x_4^6 & x_0^2x_4^2\tilde{x}-2x_0^2x_2x_4^2x_5+x_4^4x_5^2-x_0^4x_2^2 \end{pmatrix} \tag{23}$$

as its initial datum. Hence there exists a vector bundle of rank 3 on $\boldsymbol{P}^5$. This vector bundle is essentially the Horrocks bundle (cf. sect. 7 *5-3).

Assume that k is a field of characteristic two. Then the matrix (23) shows that there exists a system of σ-transition matrices which has

$$x_0^{-6}\begin{pmatrix} 0 & x_0^6 \\ x_4^6 & x_0^2x_4^2\tilde{x}+x_4^4x_5^2+x_0^4x_2^2 \end{pmatrix} \tag{24}$$

as its initial datum. Therefore, in this case, there exists a rank 2 bundle on $\boldsymbol{P}^5$ determined by this system of transition matrices (cf. sect. 7 *5-4).

6. Example 3.

In this section, let $n=5$ and σ be the cyclic permutation (0 4 2 3 1 5) and the action of G be the standard one. Throughout this section, we assume that k is a field of characteristic two. In [10], we gave an example of indecomposable vector bundle E of rank 2 on $\boldsymbol{P}^5$, using a non-constant morphism from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 2)$, where $\boldsymbol{Gr}(5, 2)$ is the Grassmann variety which parametrizes planes contained in $\boldsymbol{P}^5$. In this section we reconstruct this vector bundle E in terms of system of σ-transition matrices and using this we give an example of non-constant morphism from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 1)$, where $\boldsymbol{Gr}(5, 1)$ is the Grassmann variety which parametrizes lines contained in $\boldsymbol{P}^5$. Using this we give an example of rank 4 on $\boldsymbol{P}^5$ of new type.

Let us recall the method of construction of the vector bundle E. Let Q_6 be the quadratic hypersurface of $\boldsymbol{P}^7$ defined by

$$X_0X_1+X_2X_3+X_4X_5+X_6X_7=0 .$$

Let ψ be the morphism from Q_6 to $\boldsymbol{Gr}(5, 2)$ defined by

$$\psi(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)= \begin{pmatrix} x_0^2 & 0 & 0 & x_2^2 & x_2x_4+x_0x_7 & x_2x_6-x_0x_5 \\ 0 & x_0^2 & 0 & x_2x_4-x_0x_7 & x_4^2 & x_4x_6+x_0x_3 \\ 0 & 0 & x_0^2 & x_2x_6+x_0x_5 & x_4x_6-x_0x_3 & x_6^2 \end{pmatrix}^{\sim} ,$$

where

$$\begin{pmatrix} x_0^2 & 0 & 0 & x_2^2 & x_2x_4+x_0x_7 & x_2x_6-x_0x_5 \\ 0 & x_0^2 & 0 & x_2x_4-x_0x_7 & x_4^2 & x_4x_6+x_0x_3 \\ 0 & 0 & x_0^2 & x_2x_6+x_0x_5 & x_4x_6-x_0x_3 & x_6^2 \end{pmatrix}^{\sim}$$

denotes the point of $\boldsymbol{Gr}(5, 2)$ which corresponds to the plane spanned by the three points given by the three rows of this matrix. This morphism is defined over the field of arbitrary characteristic. Under the assumption that the field k is a field of characteristic two, let us consider the morphism φ from $\boldsymbol{P}^5$ to Q_6 defined by

$$\varphi(x_0, x_1, x_2, x_3, x_4, x_5)=(x_0^2, x_3^2, x_5^2, x_2^2, x_4^2, x_1^2, \tilde{x}, \tilde{x})$$

where $\tilde{x}=x_0x_3+x_1x_4+x_2x_5$. Let us denote $x_5^2x_4^2+x_0^2\tilde{x}$ by g and denote ${}_{\tau}g$ by $g_{\tau(0)}$. Hence,

$$g_0=x_5^2x_4^2+x_0^2\tilde{x} , \quad g_4=x_0^2x_2^2+x_4^2\tilde{x} , \quad g_2=x_4^2x_3^2+x_2^2\tilde{x} , \quad g_3=x_2^2x_1^2+x_3^2\tilde{x} ,$$
$$g_1=x_3^2x_5^2+x_1^2\tilde{x} \quad \text{and} \quad g_5=x_1^2x_0^2+x_5^2\tilde{x} .$$

Then $\psi\circ\varphi$ is a non-constant morphism from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 2)$ which is given by

$$\psi\circ\varphi(x_0, x_1, x_2, x_3, x_4, x_5)= \begin{pmatrix} x_0^4 & 0 & 0 & x_5^4 & g_0 & g_5 \\ 0 & x_0^4 & 0 & g_0 & x_4^4 & g_4 \\ 0 & 0 & x_0^4 & g_5 & g_4 & \tilde{x}^2 \end{pmatrix}^{\sim} . \tag{25}$$

It is a simple matter to verify that among g_i the following relations hold:

LEMMA 10.
(1) $g_4^2+x_4^4\tilde{x}^2+x_0^4x_2^4=0$,
(2) $g_4g_0+x_4^4g_5+x_0^4g_2=0$,
(3) $g_0^2+x_0^4\tilde{x}^2+x_5^4x_4^4=0$,
(4) $x_4^2g_5+x_5^2g_4+x_0^2\tilde{x}^2=x_0^4x_3^2$,
(5) $x_4^2x_5^4+x_5^2g_0+x_0^2g_5=x_0^4x_1^2$,
(6) $x_4^2g_0+x_5^2x_4^4+x_0^2g_4=x_0^4x_2^2$.

Arranging the columns of matrix (25), let us consider the following matrix

$$\hat{A}=\begin{pmatrix} g_5 & x_0^4 & 0 & 0 & x_5^4 & g_0 \\ g_4 & 0 & x_0^4 & 0 & g_0 & x_4^4 \\ \tilde{x}^2 & 0 & 0 & x_0^4 & g_5 & g_4 \end{pmatrix}.$$

Then, we have

$$^{\sigma}\hat{A}=\begin{pmatrix} g_4 & 0 & x_0^4 & 0 & g_0 & x_4^4 \\ x_2^4 & 0 & g_4 & x_4^4 & g_2 & 0 \\ g_2 & x_4^4 & g_0 & 0 & \tilde{x}^2 & 0 \end{pmatrix}.$$

Now put

$$\hat{B}=x_0^{-4}\begin{pmatrix} 0 & x_0^4 & 0 \\ 0 & g_4 & x_4^4 \\ x_4^4 & g_0 & 0 \end{pmatrix}. \tag{26}$$

By virtue of Lemma 10 (1), (2) and (3), we can easily verify that the formula

$$^{\sigma}\hat{A}=\hat{B}\hat{A}$$

holds. Theorem 2 shows that $\hat{A}$ is a standard σ-matrix with initial datum $\hat{B}$. Therefore, $\hat{B}$ is an initial datum of a system of σ-transition matrices of rank 3, which determines a vector bundle of rank 3 on $\boldsymbol{P}^5$ (cf. sect. 7 *6-1).

Let us consider the following matrix

$$\hat{C}=x_0^{-2}\begin{pmatrix} x_4^2 & x_5^2 & x_0^2 \\ x_0^2 & 0 & 0 \\ 0 & x_0^2 & 0 \end{pmatrix}.$$

Then by virtue of Lemma 10 (4), (5) and (6), we have

$$\hat{C}\hat{A}=\begin{pmatrix} x_0^2x_3^2 & x_0^2x_4^2 & x_0^2x_5^2 & x_0^4 & x_0^2x_1^2 & x_0^2x_2^2 \\ g_5 & x_0^4 & 0 & 0 & x_5^4 & g_0 \\ g_4 & 0 & x_0^4 & 0 & g_0 & x_4^4 \end{pmatrix}.$$

Note that

$$^{\sigma}(x_0^2x_3^2\ x_0^2x_4^2\ x_0^2x_5^2\ x_0^4\ x_0^2x_1^2\ x_0^2x_2^2)=(x_4^2x_3^2\ x_4^4\ x_4^2x_5^2\ x_4^2x_0^2\ x_4^2x_1^2\ x_4^2x_2^2)$$

(cf. Lemma 8). Since

$$\hat{C}^{-1}=x_0^{-2}\begin{pmatrix} 0 & x_0^2 & 0 \\ 0 & 0 & x_0^2 \\ x_0^2 & x_4^2 & x_5^2 \end{pmatrix},$$

$\hat{C}$ and $\hat{C}^{-1}$ are regular on U_0. Computing $_{\sigma}\hat{C}\hat{B}\hat{C}^{-1}$, we have the following matrix

$$x_0^{-6}\begin{pmatrix} x_0^4x_4^2 & 0 & 0 \\ 0 & 0 & x_0^6 \\ x_0^2x_4^4 & x_4^6 & x_4^4x_5^2+x_0^2g_4 \end{pmatrix}. \tag{27}$$

Lemma 8 shows that there exists a system of σ-transition matrices which has the following matrix as its initial datum:

$$x_0^{-6}\begin{pmatrix} 0 & x_0^6 \\ x_4^6 & x_4^4x_5^2+x_0^2g_4 \end{pmatrix}. \tag{28}$$

We denote this system of σ-transition matrices by $\{B_{ij}\}$ and its initial datum by B. The vector bundle which is determined by this system of transition matrices is essentially equal to E (cf. sect. 7 *6-2).

In order to compute the standard σ-matrix which has B as its initial datum, let us consider the following polynomials and relations. Let

$$h=x_0^4x_1^2+x_5^2g_0=x_0^4x_1^2+x_5^4x_4^2+x_5^2x_0^2\tilde{x} \quad \text{and}$$
$$f=x_0^2x_1^2x_2^2+x_3^2x_4^2x_5^2+(x_0^2x_3^2+x_2^2x_5^2)\tilde{x}\,.$$

For all element τ of G we denote ${}_\tau h$ by $h_{\tau(0)}$ and denote ${}_\tau f$ by $f_{\tau(0)}$. Then accordingly ${}_\tau h_i=h_{\tau(i)}$ and ${}_\tau f_i=f_{\tau(i)}$ hold. By direct calculation we have the following two relations

$$h_4h_2+x_0^6x_2^6+x_4^6f_0=0 \quad \text{and} \quad x_3^6h_4+x_2^6h_5+h_3f_0=0\,.$$

Acting all elements of G to these polynomials and relations, we have the following relations:

Relations.

$$h_0=x_0^4x_1^2+x_5^4x_4^2+x_5^2x_0^2\tilde{x}\,,\quad f_0=x_0^2x_1^2x_2^2+x_3^2x_4^2x_5^2+(x_0^2x_3^2+x_2^2x_5^2)\tilde{x}=f_3\,,$$
$$h_4=x_4^4x_5^2+x_0^4x_2^2+x_0^2x_4^2\tilde{x}\,,\quad f_4=x_0^2x_1^2x_2^2+x_3^2x_4^2x_5^2+(x_4^2x_1^2+x_3^2x_0^2)\tilde{x}=f_1\,,$$
$$h_2=x_2^4x_0^2+x_4^4x_3^2+x_4^2x_2^2\tilde{x}\,,\quad f_2=x_0^2x_1^2x_2^2+x_3^2x_4^2x_5^2+(x_2^2x_5^2+x_1^2x_4^2)\tilde{x}=f_5\,,$$
$$h_3=x_3^4x_4^2+x_2^4x_1^2+x_2^2x_3^2\tilde{x}\,,\quad h_1=x_1^4x_2^2+x_3^4x_5^2+x_3^2x_1^2\tilde{x}\,,$$
$$h_5=x_5^4x_3^2+x_1^4x_0^2+x_1^2x_5^2\tilde{x}\,,$$
$$h_4h_2+x_0^6x_2^6+x_4^6f_0=0\,,\quad x_3^6h_4+x_2^6h_5+h_3f_0=0\,,$$
$$h_2h_3+x_4^6x_3^6+x_2^6f_4=0\,,\quad x_1^6h_2+x_3^6h_0+h_1f_4=0\,,$$
$$h_3h_1+x_2^6x_1^6+x_3^6f_2=0\,,\quad x_5^6h_3+x_1^6h_4+h_5f_2=0\,,$$
$$h_1h_5+x_3^6x_5^6+x_1^6f_3=0\,,\quad x_0^6h_1+x_5^6h_2+h_0f_3=0\,,$$
$$h_5h_0+x_1^6x_0^6+x_5^6f_1=0\,,\quad x_4^6h_5+x_0^6h_3+h_4f_1=0\,,$$
$$h_0h_4+x_5^6x_4^6+x_0^6f_5=0\,,\quad x_2^6h_0+x_4^6h_1+h_2f_5=0\,.$$

Using these relations, matrix (28) and relation (1), we can calculate $\{B_{ij}\}$. Hence by virtue of the method of Proposition 2, we have the following standard σ-matrix A

which has B as its initial datum:

$$A=\begin{pmatrix} x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \\ 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \end{pmatrix}. \tag{29}$$

Actually since

$${}^\sigma A=\begin{pmatrix} 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \\ x_4^6 & f_5 & x_2^6 & h_3 & 0 & h_4 \end{pmatrix},$$

we can easily verify that the formula

$${}^\sigma A=BA$$

holds.

Let $\{A_i, B_{ij}\}$ be the σ-data of A, and let us compute $\{A_i\}$.

$$A_0= A =\begin{pmatrix} x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \\ 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \end{pmatrix},$$

$$A_4={}^\sigma A_0=\begin{pmatrix} 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \\ x_4^6 & f_5 & x_2^6 & h_3 & 0 & h_4 \end{pmatrix},$$

$$A_2={}^\sigma A_4=\begin{pmatrix} x_4^6 & f_5 & x_2^6 & h_3 & 0 & h_4 \\ h_2 & h_1 & 0 & x_3^6 & x_2^6 & f_0 \end{pmatrix},$$

$$A_3={}^\sigma A_2=\begin{pmatrix} h_2 & h_1 & 0 & x_3^6 & x_2^6 & f_0 \\ f_4 & x_1^6 & x_3^6 & 0 & h_3 & h_5 \end{pmatrix},$$

$$A_1={}^\sigma A_3=\begin{pmatrix} f_4 & x_1^6 & x_3^6 & 0 & h_3 & h_5 \\ h_0 & 0 & h_1 & x_1^6 & f_2 & x_5^6 \end{pmatrix},$$

$$A_5={}^\sigma A_1=\begin{pmatrix} h_0 & 0 & h_1 & x_1^6 & f_2 & x_5^6 \\ x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \end{pmatrix}.$$

By the relation

$$A_i=B_{ij}A_j$$

we can easily compute $\{B_{ij}\}$. For example,

$$\begin{pmatrix} h_0 & 0 & h_1 & x_1^6 & f_2 & x_5^6 \\ x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \end{pmatrix}=B_{53}\begin{pmatrix} h_2 & h_1 & 0 & x_3^6 & x_2^6 & f_0 \\ f_4 & x_1^6 & x_3^6 & 0 & h_3 & h_5 \end{pmatrix}$$

shows that

$$B_{53}=x_3^{-6}\begin{pmatrix} x_1^6 & h_1 \\ h_5 & f_3 \end{pmatrix}.$$

Collecting all different rows of $\{A_i\}$, we have the following matrix of rank 2:

$$\acute{A}=\begin{pmatrix} x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \\ 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \\ x_4^6 & f_5 & x_2^6 & h_3 & 0 & h_4 \\ h_2 & h_1 & 0 & x_3^6 & x_2^6 & f_0 \\ f_4 & x_1^6 & x_3^6 & 0 & h_3 & h_5 \\ h_0 & 0 & h_1 & x_1^6 & f_2 & x_5^6 \end{pmatrix}. \tag{30}$$

In general, when D is an $s\times(n+1)$ matrix such that $\operatorname{rank} D=r$ and all components of D are homogeneous polynomials (D has no null rows), for general point $(x_0\,x_1\cdots x_n)$, we denote by $D^{\sim}$ the points of $\boldsymbol{Gr}(n, r-1)$ which corresponds to the linear $(n-1)$-space of $\boldsymbol{P}^n$ spanned by s-points given by s-rows of D.

Under this notation, we have

$$\acute{A}^{\sim}=A^{\sim}=A_0^{\sim}=A_4^{\sim}=A_2^{\sim}=A_3^{\sim}=A_1^{\sim}=A_5^{\sim}\,.$$

This shows that the rational map Ψ from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 1)$ which is defined by

$$\Psi(x_0\,x_1\cdots x_n)=A^{\sim}$$

is a morphism. Thus we have

THEOREM 3. *There exists a non-constant morphism Ψ from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 1)$ which is defined by*

$$\Psi(x_0\,x_1\cdots x_n)=A^{\sim}\,,$$

where A is the matrix given in (29) *and the defining field is a field of characteristic two* (cf. sect. 7 *6-3).

Theorem 3 suggests that there exists a system of σ-transition matrices of rank 4. In order to seek such matrices, let us consider the following matrix A' of rank 6 which has A as its submatrix, and its inverse A'^{-1}. Let

$$A'=\begin{pmatrix} x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \\ 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \\ 0 & x_0^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0^6 & 0 & 0 \\ 0 & 0 & x_0^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0^6 & 0 \end{pmatrix}. \tag{31}$$

Since

$$x_0^{-12}\begin{pmatrix} x_0^6 & x_5^6 & f_3 & h_5 & h_4 & 0 \\ 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \\ 0 & x_0^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0^6 & 0 & 0 \\ 0 & 0 & x_0^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0^6 & 0 \end{pmatrix}\begin{pmatrix} x_0^6 & 0 & x_5^6 & h_5 & f_3 & h_4 \\ 0 & 0 & x_0^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0^6 & 0 \\ 0 & 0 & 0 & x_0^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0^6 \\ 0 & x_0^6 & h_0 & f_1 & h_2 & x_4^6 \end{pmatrix}=I_6,$$

we have

$$A'^{-1}=x_0^{-2}\begin{pmatrix} x_0^6 & 0 & x_5^6 & h_5 & f_3 & h_4 \\ 0 & 0 & x_0^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0^6 & 0 \\ 0 & 0 & 0 & x_0^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0^6 \\ 0 & x_0^6 & h_0 & f_1 & h_2 & x_4^6 \end{pmatrix}.$$

The relation

$${}^\sigma A'=({}^\sigma A'A'^{-1})A'$$

shows that A' is a standard σ-stable matrix with initial datum ${}^\sigma A'A'^{-1}$. Corollary 1 of Theorem 2 shows that A' is a σ-matrix. Let us compute its initial datum ${}^\sigma A'A'^{-1}$:

$$x_0^{-12}\begin{pmatrix} 0 & h_0 & h_2 & f_1 & x_4^6 & x_0^6 \\ x_4^6 & f_5 & x_2^6 & h_3 & 0 & h_4 \\ 0 & 0 & 0 & 0 & 0 & x_4^6 \\ 0 & x_4^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4^6 & 0 & 0 \\ 0 & 0 & x_4^6 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_0^6 & 0 & x_5^6 & h_5 & f_3 & h_4 \\ 0 & 0 & x_0^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0^6 & 0 \\ 0 & 0 & 0 & x_0^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0^6 \\ 0 & x_0^6 & h_0 & f_1 & h_2 & x_4^6 \end{pmatrix}$$

$$=x_0^{-12}\begin{pmatrix} 0 & x_0^{12} & 0 & 0 & 0 & 0 \\ x_0^6x_4^6 & x_0^6h_4 & 0 & 0 & 0 & 0 \\ 0 & x_4^6x_0^6 & x_4^6h_0 & x_4^6f_1 & x_4^6h_2 & x_4^{12} \\ 0 & 0 & x_4^6x_0^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4^6x_0^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_4^6x_0^6 & 0 \end{pmatrix}. \tag{32}$$

Lemma 3 (3) and (4) show that there exists a system of σ-transition matrices which has the following matrix B'' as its initial datum:

$$B''=x_0^{-6}\begin{pmatrix} h_0 & f_1 & h_2 & x_4^6 \\ x_0^6 & 0 & 0 & 0 \\ 0 & x_0^6 & 0 & 0 \\ 0 & 0 & x_0^6 & 0 \end{pmatrix}.$$

Therefore, there exists a vector bundle of rank 4 on P^5 (cf. sect. 7 *6-4). Using the method of Proposition 2, we have the following three standard σ-matrices which have B'' as their initial datum:

$$A''^{(1)}=\begin{pmatrix} x_0^6 & x_3^6 & 0 & 0 & 0 & h_5 \\ 0 & h_1 & 0 & x_2^6 & 0 & x_5^6 \\ 0 & x_1^6 & x_4^6 & h_3 & 0 & 0 \\ 0 & 0 & h_2 & x_3^6 & x_0^6 & 0 \end{pmatrix},$$

$$A''^{(2)}=\begin{pmatrix} 0 & h_3 & 0 & 0 & x_4^6 & f_3 \\ x_0^6 & f_2 & 0 & h_2 & 0 & 0 \\ 0 & 0 & h_4 & f_4 & 0 & x_5^6 \\ 0 & x_1^6 & f_0 & 0 & h_0 & 0 \end{pmatrix},$$

$$A''^{(3)}=\begin{pmatrix} 0 & f_1 & x_2^6 & 0 & 0 & h_4 \\ 0 & h_0 & 0 & f_3 & x_4^6 & 0 \\ x_0^6 & 0 & f_2 & h_5 & 0 & 0 \\ 0 & 0 & h_1 & 0 & f_4 & x_5^6 \end{pmatrix}. \tag{33}$$

Finally let us consider the following matrix $\bar{A}$ which has $\hat{A}$ as a submatrix:

$$\bar{A}=\begin{pmatrix} g_5 & x_0^4 & 0 & 0 & x_5^4 & g_0 \\ g_4 & 0 & x_0^4 & 0 & g_0 & x_4^4 \\ \tilde{x}^2 & 0 & 0 & x_0^4 & g_5 & g_4 \\ x_0^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0^4 \end{pmatrix}.$$

Computing ${}^{\sigma}\bar{A}\cdot\bar{A}^{-1}$, we have

$$x_0^{-8}\begin{pmatrix} g_4 & 0 & x_0^4 & 0 & g_0 & x_4^4 \\ x_2^4 & 0 & g_4 & x_4^4 & g_2 & 0 \\ g_2 & x_4^4 & g_0 & 0 & \tilde{x}^2 & 0 \\ 0 & 0 & 0 & 0 & x_4^4 & 0 \\ 0 & 0 & x_4^4 & 0 & 0 & 0 \\ x_4^4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & x_0^4 & 0 & 0 \\ x_0^4 & 0 & 0 & g_5 & x_5^4 & g_0 \\ 0 & x_0^4 & 0 & g_4 & g_0 & x_4^4 \\ 0 & 0 & x_0^4 & \tilde{x}^2 & g_5 & g_4 \\ 0 & 0 & 0 & 0 & x_0^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0^4 \end{pmatrix}$$

$$=x_0^{-8}\begin{pmatrix} 0 & 0 & x_0^8 & 0 & 0 & 0 \\ 0 & x_0^4 g_4 & x_0^4 x_4^4 & 0 & 0 & 0 \\ x_0^4 x_4^4 & x_0^4 g_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_4^4 x_0^4 & 0 \\ 0 & x_4^4 x_0^4 & 0 & x_4^4 g_4 & x_4^4 g_0 & x_4^8 \\ 0 & 0 & 0 & x_4^4 x_0^4 & 0 & 0 \end{pmatrix}. \tag{34}$$

Lemma 3 (3) and (4) show that there exists a system of σ-transition matrices which has the following matrix $\bar{B}$ as its initial datum (cf. sect. 7 *6-5):

$$\bar{B}=x_0^{-4}\begin{pmatrix} 0 & x_0^4 & 0 \\ g_4 & g_0 & x_4^4 \\ x_0^4 & 0 & 0 \end{pmatrix}.$$

It is remarkable that

$${}^t\bar{B}^{-1}=x_4^{-4}\begin{pmatrix} g_0 & x_4^4 & g_4 \\ 0 & 0 & x_0^4 \\ x_0^4 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\bar{B}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}=x_0^{-4}\begin{pmatrix} g_0 & x_4^4 & g_4 \\ 0 & 0 & x_0^4 \\ x_0^4 & 0 & 0 \end{pmatrix}$$

(cf. sect. 7 *6-6).

7. Remarks.

In this section we give some remarks without proofs. We use the following notation.

When E is a vector bundle of rank r on P^n, and c_i is the degree of the i-th Chern class of E, we denote the polynomial

$$1+c_1t+c_2t^2+\cdots+c_rt^r$$

by $c(E)$ and call it the total Chern class of E.

Let E be a vector bundle of rank 4 on P^n, where $3\leq n$, and let

$$c(E)=1+c_1t+c_2t^2+c_3t^3+c_4t^4 .$$

When $8c_2-3c_1^2\neq 0$ we denote

$$\frac{(8c_3-4c_1c_2+c_1^3)^{2/3}}{8c_2-3c_1^2}$$

by $V_3^4(E)$. $V_3^4(E)$ is an invariant of E which has the following properties.

(1) $V_3^4(\check{E})=V_3^4(E)$ where $\check{E}$ is the dual vector bundle of E.

(2) $V_3^4(E(m))=V_3^4(E)$ for an arbitrary integer m.

(3) $V_3^4(\psi^*E)=V_3^4(E)$ for an arbitrary non-constant morphism ψ from P^n to P^n.

These properties show that if E and E' be two vector bundles of rank 4 on P^n and $V_3^4(E)\neq V_3^4(E')$, then E differs essentially from E'.

We denote by L_4 the line in P^5 defined by

$$x_1=x_2=x_3=x_5=0\,,$$

by L_2 the line in $\boldsymbol{P}^5$ defined by

$$x_1=x_3=x_4=x_5=0\,,$$

and by L_3 the line in $\boldsymbol{P}^5$ defined by

$$x_1=x_2=x_4=x_5=0\,.$$

If L is a line in $\boldsymbol{P}^5$, we denote by $E|_L$ the restriction of E to L.

(*4-1) This bundle is Ω_{P^n}. This initial datum shows that $\Omega_{P^n}(2)$ is generated by its global sections. Matrix (17) shows that there exists the following exact sequence of vector bundles

$$0\longrightarrow\Omega_{P^n}\longrightarrow\bigoplus^{6}O(-1)\longrightarrow O\longrightarrow 0\,.$$

(*4-2) This bundle is the tangent bundle $T_{P^n}(-2)$.

(*5-1) This bundle is $S(-1)$, where S is the null-correlation bundle on $\boldsymbol{P}^5$. This bundle has the following properties:

$$c(S)=1+t^2+t^4=(1-t+t^2)(1+t+t^2)\,,$$
$$V_3^4(S)=0\,,$$
$$S|_{L_4}=O(1)\oplus O\oplus O\oplus O(-1)\,,$$
$$S|_{L_3}=O\oplus O\oplus O\oplus O\,.$$

(Hence S is indecomposable.)

(*5-2) Let us denote by E_1 the vector bundle of rank 4 on $\boldsymbol{P}^5$ which is defined by the initial datum (22). Then we have

$$c(E_1)=1-9t+33t^2-59t^3+42t^4=(1-3t)(1-2t)(1-4t+7t^2)\,,$$
$$V_3^4(E_1)=\sqrt[3]{9}/21\,,$$
$$(E_1)|_{L_4}=O(-1)\oplus O(-1)\oplus O(-2)\oplus O(-5)\,.$$

(*5-3) Let us denote this bundle by E_2. Then

$$c(E_2)=1-9t+30t^2-36t^3=(1-3t)(1-6t+12t^2)\,.$$

There exists the following exact sequence of vector bundles

$$0\longrightarrow O(-3)\longrightarrow E_1\longrightarrow E_2(1)\longrightarrow 0\,.$$

(*5-4) (characteristic of k is two). Let us denote this bundle by E_3. Then

$$c(E_3)=1-6t+12t^2\,,$$
$$(E_3)|_{L_4}=O\oplus O(-6)\,,$$
$$(E_3)|_{L_2}=O(-2)\oplus O(-4)\,.$$

There exists the following exact sequence of vector bundles

$$0\longrightarrow E_3\longrightarrow E_2\longrightarrow O(-3)\longrightarrow 0\,.$$

(*6-1) (characteristic of k is two). Let us denote this bundle by E_4. Then

$$c(E_4)=1-8t+24t^2-24t^3\,,$$
$$c(E_4(4))=1+4t+8t^2+8t^3\,.$$

$E_4(4)$ is generated by its global sections, and using these global sections we can construct a non-constant morphism from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 2)$.

$$(E_4)|_{L_4}=O\oplus O(-4)\oplus O(-4)\,,$$
$$(E_2)|_{L_2}=O(-2)\oplus O(-2)\oplus O(-4)\,,$$
$$(E_2)|_{L_3}=O(-2)\oplus O(-3)\oplus O(-3)\,.$$

Hence E_4 is indecomposable.

(*6-2) (characteristic of k is two). This bundle is E_3. $E_3(6)$ is generated by its global sections. Using these global sections we obtained the morphism ψ from $\boldsymbol{P}^5$ to $\boldsymbol{Gr}(5, 1)$, which appeared in Theorem 3. Matrix (27) shows that there exists the following exact sequence of vector bundles

$$0\longrightarrow E_3\longrightarrow E_4\longrightarrow O(-2)\longrightarrow 0\,.$$

(*6-3) (characteristic of k is two). By virtue of this theorem, the following holds: When $n>2d>0$ and $10>d$, there exists non-constant morphism form $\boldsymbol{P}^n$ to $\boldsymbol{Gr}(n, d)$ if and only if

$$(n, d)=(3, 1) \text{ or } (5, 1) \text{ or } (5, 2)$$

(cf. [8], [10] and [11]).

(*6-4) (characteristic of k is two). Let us denote this bundle by E_5. Matrix (32) shows that there exists the following exact sequence of vector bundles

$$0\longrightarrow E_5(-6)\longrightarrow \overset{6}{\bigoplus} O(-6)\longrightarrow E_3\longrightarrow 0\,.$$

Matrix (33) shows that there exists the following exact sequence of vector bundles

$$\overset{18}{\bigoplus} O(-6)\longrightarrow E_5\longrightarrow 0\,.$$

E_5 has the properties

$$c(E_5)=1-6t+24t^2-72t^3+144t^4=(1-6t+12t^2)(1+12t^2)\,,$$
$$V_3^4(E_5)=3/7\,.$$

The dual vector bundle $\check{E}_5$ of E_5 is generated by its global sections and

$$c(\check{E}_5)=(1+6t+12t^2)(1+12t^2)\,.$$

Since there exists no vector bundle which is generated by its global sections and which has Chern polynomial $1+12t^2$ (cf. [8]), E_5 is indecomposable.

(*6-5) (characteristic of k is two). Let us denote by E_4' the vector bundle determined by this matrix $\bar{B}$. Matrix (34) shows that there exists the following exact sequence of vector bundles

$$0 \longrightarrow E_4'(-4) \longrightarrow \bigoplus^6 O(-4) \longrightarrow E_4 \longrightarrow 0\,.$$

(*6-6) (characteristic of k is two). That equation shows that $\check{E}_4=E_4'(4)$. Thus if we put $E=E_4(4)$, we have the following type of exact sequence of vector bundles

$$0 \longrightarrow \check{E} \longrightarrow \bigoplus^6 O \longrightarrow E \longrightarrow 0\,.$$

(* -) In [9], we gave an example of indecomposable vector bundle of rank 4 on P^5. Let us denote this bundle by E_6. Then

$$c(E_6)=1+4t+9t^2+14t^3+14t^4\,,$$
$$V_3^4(E_6)=\sqrt[3]{2}/3\,.$$

Thus, four vector bundles S, E_1, E_5 and E_6 are essentially different from each other.

References

[1] A. GROTHENDIECK, Sur la classification de fibres holomorphismes sur la sphere de Riemann, Amer. J. Math., **79** (1957), 12–138.

[2] R. HARTSHORNE, Algebraic vector bundles on projective spaces; a problem list, Topology, **18** (1979), 117–128.

[3] G. HORROCKS, Example of rank three vector bundles on five dimensional projective space, J. London Math. Soc. (2), **18** (1978), 15–27.

[4] G. HORROCKS and D. MUMFORD, A rank 2 vector bundle on P^4 with 15,000 symmetries, Topology, **12** (1973), 63–81.

[5] H. KAJI, Example of σ-transition matrices defining the Horrocks-Mumford bundle, Tokyo J. Math., **12** (1989), 21–32.

[6] M. MARUYAMA, On a family of algebraic vector bundles, *Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki*, Kinokuniya (1973), 95–146.

[7] J. SIMONIS, A class of indecomposable algebraic vector bundles, Math. Ann., **192** (1971), 262–278.

[8] H. TANGO, On $(n-1)$-dimensional projective spaces contained in the Grassmann variety $Gr(n, d)$, J.

Math. Kyoto Univ., **14** (1974), 415–460.

[9] H. Tango, An example of indecomposable vector bundle of rank $n-1$ on $\boldsymbol{P}^n$, J. Math. Kyoto Univ., **16** (1976), 137–141.

[10] H. Tango, On morphisms from projective space $\boldsymbol{P}^n$ to the Grassmann variety $\boldsymbol{Gr}(n, d)$, J. Math. Kyoto Univ., **16** (1976), 201–207.

[11] H. Tango, On morphisms from projective space $\boldsymbol{P}^n$ to the Grassmann variety $\boldsymbol{Gr}(n, d)$ II, Bull. Kyoto Univ. of Educ. Ser. B, **64** (1984), 1–20.

[12] H. Tango, Remark on low rank indecomposable vector bundles on projective space I, Bull. Kyoto Univ. of Educ. Ser. B, **66** (1985), 1–24.

[13] H. Tango, Remark on low rank indecomposable vector bundles on projective space II (On σ-transition matrices), Bull. Kyoto Univ. of Educ. Ser. B, **68** (1986), 1–18.

Present Address:
Department of Mathematics, Faculty of Education, Kyoto University of Education
Fushimi-ku, Kyoto 612, Japan