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On Certain Infinite Series

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§0. Introduction.

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

where $n^s=\exp(s\,\mathrm{Log}\,n)$ and $\mathrm{Log}\,z$ denotes the principal branch of $\log z$. The series is locally uniformly convergent for $\mathrm{Re}(s)>1$, so that $\zeta(s)$ represents a regular function of s there. It is known that $\zeta(s)$ possesses an analytic continuation into the whole s-plane which is regular except for a simple pole at $s=1$ with residue 1 and that $\zeta(s)$ has the Laurent expansion at $s=1$ of the form

(1) $$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma(n)(s-1)^n,$$

where

$$\gamma(n)=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{\mathrm{Log}^n k}{k}-\frac{\mathrm{Log}^{n+1}N}{n+1}\right\}$$

for all values of n. In particular,

(2) $$\gamma=\gamma(0)=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{1}{k}-\mathrm{Log}\,N\right\}$$

is called the Euler constant. The above expansion has been discovered independently by Briggs and Chowla [1] and a lot of mathematicians. It is also known that

(3) $$\zeta(0)=-\frac{1}{2},\quad \zeta(-2m)=0 \quad\text{and}\quad \zeta(1-2m)=-\frac{B_{2m}}{2m}$$

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for any positive integer m, where B_k denotes the k-th Bernoulli number defined by

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k .$$

In the present paper, we shall show that similar results to the above hold for a class of infinite series.

Let $h(z)$ be a complex valued function which is regular and non-vanishing in the half plane $\mathrm{Re}(z)>\alpha$, where α is a real number. We consider the infinite series of the form

$$\zeta_h(s)=\sum_{n=M}^{\infty}\frac{h'(n)}{h(n)^s},$$

where $M=[\alpha+1/2]+1$, $[x]$ denotes the integral part of x and $h'(z)$ stands for the derivative of $h(z)$. Here and in what follows, $h(z)^s=\exp(s\log h(z))$ with a fixed branch of $\log h(z)$. Moreover, we assume the following conditions:

(A.1) $\zeta_h(s)$ converges for all sufficiently large real values of s.

(A.2) $\log|h(z)|$, $\arg h(z)\ll\log|z|$ $(|z|\gg 0)$, where $\arg h(z)$ stands for the argument of $h(z)$.

(A.3) $|h(z)|\to\infty$ as $|z|\to\infty$.

Then we obtain

THEOREM 1. *Under the above hypothesis, $\zeta_h(s)$ is extended to a meromorphic function of s in the whole complex s-plane which is regular except for a simple pole at $s=1$ with residue* 1.

The usefulness of this theorem is shown by the fact that we can deduce from it a lot of old and new results on analytic continuations for certain infinite series.

EXAMPLE 1. Let $h(z)$ be a non-constant polynomial of z with complex coefficients. Take an integer M such that $h(z)$ has no zeros in $\mathrm{Re}(z)>M-1$ and put $\alpha=M-1$. Then the infinite series $\zeta_h(s)$ is absolutely convergent for $s>1$ and for any fixed branch of $\log h(z)$. Hence, in view of Theorem 1, $\zeta_h(s)$ is extended to a meromorphic function of s in the whole s-plane which is holomorphic except for a simple pole at $s=1$ with residue 1. In particular, if we take $h(z)=z$, $M=1$ and the principal branch of $\log z$, then $\zeta_h(s)$ coincides with the Riemann zeta function $\zeta(s)$. Further, if we take $h(z)=z+a$ $(0<a\leqq 1)$, $M=1$ and the principal branch of $\log(z+a)$, then $\zeta_h(s)$ coincides with $\zeta(s,a)-a^{-s}$, where $\zeta(s,a)$ denotes the Hurwitz zeta function defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\qquad(\mathrm{Re}(s)>1)$$

and reduces to $\zeta(s)$ in the case of $a=1$.

More generally, we have

EXAMPLE 2. Let $a(z)$ and $b(z)$ be polynomials of z with complex coefficients satisfying $\deg a(z) > \deg b(z)$. Take an integer M such that both $a(z)$ and $b(z)$ have no zeros in $\mathrm{Re}(z) > M-1$. Putting $h(z) = a(z)/b(z)$ and $\alpha = M-1$, the infinite series

$$\zeta_h(s) = \sum_{n=M}^{\infty} \frac{\{a'(n)b(n) - a(n)b'(n)\}/b^2(n)}{\{a(n)/b(n)\}^s}$$

is absolutely convergent for $s > 1$ and for any fixed branch of $\log\{a(z)/b(z)\}$. Thus, by virtue of Theorem 1, $\zeta_h(s)$ can be continued analytically to a meromorphic function of s in the whole s-plane which is regular except for a simple pole at $s=1$ with residue 1.

EXAMPLE 3. Let $g(x, y)$ be a non-constant polynomial in x and y with complex coefficients. Take a positive integer M such that $g(z, \log z)$ has no zeros in $\mathrm{Re}(z) > M-1$, where any fixed branch is taken for the logarithm. If we take $h(z) = g(z, \log z)$ and $\alpha = M-1$, then the infinite series

$$\zeta_h(s) = \sum_{n=M}^{\infty} \frac{g_x(n, \log n) + (1/n) g_y(n, \log n)}{\{g(n, \log n)\}^s}$$

is absolutely convergent for $s > 1$ and for any fixed branch of $\log h(z)$. Therefore, by Theorem 1, $\zeta_h(s)$ is extended to a meromorphic function of s in the whole s-plane which is holomorphic except for a simple pole at $s=1$ with residue 1.

EXAMPLE 4. Let $h(z)$, $\log h(z)$ and α be as in Theorem 1. Take an integer M such that both $h(z)$ and $\log h(z)$ have no zeros in $\mathrm{Re}(z) > M-1 \geqq \alpha$. We set

$$\zeta_{\log h}(s) = \sum_{n=M}^{\infty} \frac{h'(n)}{h(n)\{\log h(n)\}^s};$$

where $\{\log h(z)\}^s = \exp\{s \log(\log h(z))\}$ with a fixed branch of $\log(\log h(z))$. Suppose that $\zeta_{\log h}(s)$ is convergent for all sufficiently large real values of s. Then it follows easily from Theorem 1 that $\zeta_{\log h}(s)$ has an analytic continuation into the whole s-plane which is regular except for a simple pole at $s=1$ with residue 1. In particular, when $h(z) = z$, $\mathrm{Log}\, z$, $\mathrm{Log}\,\mathrm{Log}\, z, \cdots$ and the principal branch is taken for $\log h(z)$, $\zeta_{\log h}(s)$ was also studied by Hurwitz [3].

The organization of the paper is as follows. In Section 1, we will establish the proof of Theorem 1. We shall discuss in Section 2 the Laurent coefficients for $\zeta_h(s)$ at $s=1$. Section 3 is devoted to prove Kronecker limit formula for $\zeta_h(s)$ when $h(z)$ is a rational function of z. In Section 4, we will study the values of $\zeta_h(s)$ at non-positive integers when $h(z)$ is a polynomial of z.

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§1. Proof of Theorem 1.

Let C be the rectangle in the z-plane consisting of the line segments C_1, C_2, C_3 and C_4, joining $\xi-Ni$, $(N+1/2)-Ni$, $(N+1/2)+Ni$, $\xi+Ni$ and $\xi-Ni$, where $\xi=M-1/2$ and N is a sufficiently large integer. We note here that $\xi>\alpha$. Consider the integral

$$I(s)=\int_C f(s, z)dz\,,$$

where

$$f(s, z)=\frac{e(z)}{\{e(z)-1\}^2}h(z)^{1-s}\,,$$

$e(z)$ denotes an abbreviation of $\exp(2\pi iz)$ and s is a sufficiently large real number. By the residue theorem, we have

$$I(s)=2\pi i\sum_{n=M}^{N}\operatorname*{Res}_{z=n} f(s, z)\,.$$

Noticing that

$$\frac{e(z)}{\{e(z)-1\}^2}=\frac{1}{(2\pi i)^2}\cdot\frac{1}{(z-n)^2}-\frac{1}{12}+\cdots$$

and

$$h(z)^{1-s}=h(n)^{1-s}-(s-1)\frac{h'(n)}{h(n)^s}(z-n)+\cdots\,,$$

we obtain

$$I(s)=-\frac{s-1}{2\pi i}\sum_{n=M}^{N}\frac{h'(n)}{h(n)^s}\,. \tag{4}$$

On the other hand, we see that

$$\begin{aligned} I(s)&=\left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right)f(s, z)dz \\ &=I_1+I_2+I_3+I_4\,, \qquad \text{say.} \end{aligned} \tag{5}$$

Since $s>1$ and $|h(z)^{1-s}|=|h(z)|^{1-s}$, given any $\varepsilon>0$, in view of (A.3) we can find a number N_0 depending on ε such that

$$|h(x-Ni)^{1-s}|<\varepsilon \qquad (N>N_0)$$

for any real number $x\geqq\xi$, so that

$$|I_1|<\frac{\varepsilon e^{2\pi N}}{(e^{2\pi N}-1)^2}\int_{\xi}^{N+1/2}dx<\varepsilon .$$

This implies that $I_1\to 0$ as $N\to\infty$. Similarly, $I_2, I_3\to 0$ as $N\to\infty$. Letting $N\to\infty$, we infer from (4) and (5) that

$$\frac{s-1}{2\pi i}\zeta_h(s)=\int_{\xi-i\infty}^{\xi+i\infty}f(s, z)dz ,$$

which yields

$$(s-1)\zeta_h(s)=2\pi\int_0^{\infty}v(s, x)dx , \tag{6}$$

where

$$v(s, x)=\frac{e^{2\pi x}}{(e^{2\pi x}+1)^2}\{h(\xi+ix)^{1-s}+h(\xi-ix)^{1-s}\} .$$

Hence the formula (6) holds for all sufficiently large real values of s. Now we study the behavior of the above integral in the whole plane of the complex variable $s=\sigma+it$. Since

$$|h(z)^{1-s}|=\exp\{(1-\sigma)\log|h(z)|+t\arg h(z)\} ,$$

if $|s|\leqq R$ and R is large enough, then by (A.2), there exists a positive integer K depending on R satisfying

$$|h(\xi+ix)^{1-s}|\ll x^K \qquad (|x|\gg 0) .$$

Thus, when $q>p$ and p is large enough,

$$\left|\int_p^q v(s, x)dx\right|\leqq 2\int_p^q e^{-2\pi x}x^K dx<2\int_p^q e^{-x}e^{-x}x^K dx$$

$$<2K!\int_p^q e^{-x}dx<2K!e^{-p}\to 0 \qquad \text{as} \quad p\to\infty .$$

This shows that the integral of (6) converges uniformly in any finite region of the complex s-plane and so defines an integral function of s. Therefore the formula (6) provides the analytic continuation of $\zeta_h(s)$ over the whole s-plane. It is easily verified that the only possible singularity is a simple pole at $s=1$ with residue 1. Thus we have completed the proof of Theorem 1.

§2. The Laurent coefficients for $\zeta_h(s)$ at $s=1$.

Let $h(z)$ and $\log h(z)$ be as in Theorem 1. Then the Laurent expansion of $\zeta_h(s)$ at

$s=1$ can be written in the form

(7) $$\zeta_h(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_h(n)(s-1)^n .$$

The purpose of this section is to investigate these coefficients $\gamma_h(n)$. First we have

PROPOSITION 1. *For any non-negative integer n,*

$$\gamma_h(n)=-\frac{2\pi}{n+1}\int_0^{\infty}\frac{e^{2\pi x}}{(e^{2\pi x}+1)^2}\{\log^{n+1}h(\xi+ix)+\log^{n+1}h(\xi-ix)\}dx .$$

PROOF. We note that

$$h(\xi\pm ix)^{1-s}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\log^n h(\xi\pm ix)\cdot(s-1)^n .$$

Then substituting this into (6) and integrating term by term, the assertion is immediately derived from (7) provided the process is justifiable. But, from (A.2) it is easily justified.

The next corollary follows at once from (6) and the above proposition.

COROLLARY 1. *Let $h(z)$ and $\zeta_{\log h}(s)$ be as in Example 4. Then $\gamma_h(n)=\zeta_{\log h}(-n)$ for any non-negative integer n.*

If we put $h(z)=z$ in Proposition 1, then from (1) and (2), we obtain a new representation of the Euler constant γ.

COROLLARY 2.

$$\gamma=\mathrm{Log}2-8\int_0^{\infty}\frac{1}{e^{2\pi x}+1}\cdot\frac{x}{4x^2+1}dx .$$

Hereafter, we add the following condition:

(A.4) There exist positive constants ε and η such that

(8) $$\frac{h''(x)}{h(x)^s},\ \frac{\{h'(x)\}^2}{h(x)^{s+1}}\ll\frac{1}{x^{1+\varepsilon}}\qquad(x\gg 0)$$

for all s on the line segment from $(1-\eta)-i$ to $(1-\eta)+i$.

We remark that every function $h(z)$ of Examples 1, 2 and 3 in the previous section satisfies the above condition.

By virtue of (A.3), the next lemma is obvious.

LEMMA 1. *Let ε and η be as above. The estimate* (8) *also holds for all s in the region H_η defined by $1-\eta<\mathrm{Re}(s)$ and $-1<\mathrm{Im}(s)<1$.*

Now applying Euler summation formula to the function $h'(x)/h(x)^s$, we get

$$\sum_{n=M}^{N} \frac{h'(n)}{h(n)^s} = \int_M^N \frac{h'(x)}{h(x)^s} dx + \frac{1}{2}\left(\frac{h'(M)}{h(M)^s} + \frac{h'(N)}{h(N)^s}\right)$$

$$+ \int_M^N \left(x - [x] - \frac{1}{2}\right) w(s, x) dx ,$$

where N is an integer greater than M and

$$w(s, x) = \frac{h''(x)}{h(x)^s} - s\frac{\{h'(x)\}^2}{h(x)^{s+1}} .$$

Let s be any sufficiently large real number. When $N \to \infty$, in view of (A.1), $h'(N)/h(N)^s$ tends to 0, so that we have

$$\zeta_h(s) = \frac{1}{s-1} h(M)^{1-s} + \frac{1}{2} \frac{h'(M)}{h(M)^s} + E(s) ,$$

where

$$E(s) = \int_M^\infty \left(x - [x] - \frac{1}{2}\right) w(s, x) dx .$$

Since

$$\frac{h'(M)}{h(M)^s} = \frac{h'(M)}{h(M)} h(M)^{1-s}$$

and

$$h(M)^{1-s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \log^n h(M) \cdot (s-1)^n ,$$

we see that

$$\zeta_h(s) - \frac{1}{s-1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \log^{n+1} h(M) \cdot (s-1)^n \tag{9}$$

$$+ \frac{1}{2} \frac{h'(M)}{h(M)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \log^n h(M) \cdot (s-1)^n + E(s)$$

for all sufficiently large real values of s. Lemma 1 shows that $E(s)$ is a regular function of s in H_η. By Theorem 1, the left-hand side of (9) represents an integral function of s, so the equality (9) also holds in H_η. To obtain the Laurent expansion for $\zeta_h(s)$ at $s=1$, we have only to investigate the Taylor expansion for $E(s)$ at $s=1$.

LEMMA 2. *The Taylor expansion for $E(s)$ at $s=1$ is given by*

$$E(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}C(n)(s-1)^n\,,$$

where

$$C(n)=\frac{1}{n+1}\log^{n+1}h(M)-\frac{1}{2}\frac{h'(M)}{h(M)}\log^n h(M)$$
$$+\lim_{N\to\infty}\left\{\sum_{k=M}^{N}\frac{h'(k)}{h(k)}\log^n h(k)-\frac{1}{n+1}\log^{n+1}h(N)\right\}$$

for all values of n.

PROOF. Since

$$\left[\frac{\partial^n}{\partial s^n}w(s,x)\right]_{s=1}=(-1)^n u'(x)$$

where

$$u(x)=\frac{h'(x)}{h(x)}\log^n h(x)\,,$$

we get

$$E^{(n)}(1)=(-1)^n\int_M^{\infty}\left(x-[x]-\frac{1}{2}\right)u'(x)dx$$
$$=(-1)^n\lim_{N\to\infty}\int_M^N\left(x-[x]-\frac{1}{2}\right)u'(x)dx\,.$$

Therefore, by Euler summation formula, we find that

$$E^{(n)}(1)=(-1)^n\lim_{n\to\infty}\left\{\sum_{k=M}^{N}u(k)-\int_M^N u(x)dx-\frac{u(M)+u(N)}{2}\right\}. \tag{10}$$

Applying Lemma 1 to the case of $s=1$, we have

$$\frac{h'(N)}{h(N)}\ll\frac{1}{\sqrt{N}}\,.$$

Hence, by virtue of (A.2), we see that $u(N)\to 0$ as $N\to\infty$. Then the assertion follows easily from (10).

With the help of (7), (9) and Lemma 2, we obtain another representation of $\gamma_h(n)$.

PROPOSITION 2. *Under the above hypothesis,*

$$\gamma_h(n)=\lim_{N\to\infty}\left\{\sum_{k=M}^{N}\frac{h'(k)}{h(k)}\log^n h(k)-\frac{1}{n+1}\log^{n+1}h(N)\right\}$$

for any non-negative integer n.

§3. Kronecker limit formula for $\zeta_h(s)$.

Let $h(z)=a(z)/b(z)$, M and $\zeta_h(s)$ be as in Example 2. Then the Laurent expansion of $\zeta_h(s)$ at $s=1$ is given by

$$\zeta_h(s)=\frac{1}{s-1}+\gamma_h(0)+\cdots .$$

In the present section, we study the constant term $\gamma_h(0)$. By Proposition 2, $\gamma_h(0)$ may be written as

(11)
$$\gamma_h(0)=\lim_{N\to\infty}\left\{\sum_{k=M}^{N}\frac{h'(k)}{h(k)}-\log h(N)\right\}.$$

For brevity, we assume that M is a positive integer. Let the decompositions into linear factors be

$$a(z)=a_0\prod_{j=1}^{D}(z-\alpha_j)$$

and

$$b(z)=b_0\prod_{p=1}^{d}(z-\beta_p),$$

where D and d denote the degrees of $a(z)$ and $b(z)$, respectively, satisfying $D>d$. We take a branch of $\log h(z)$ such that

(12)
$$\lim_{N\to\infty}\{\log h(N)-(D-d)\operatorname{Log}N\}=\operatorname{Log}(a_0/b_0).$$

First of all, we notice that

$$\sum_{k=M}^{N}\frac{h'(k)}{h(k)}=\sum_{j=1}^{D}\sum_{k=M}^{N}\frac{1}{k-\alpha_j}-\sum_{p=1}^{d}\sum_{k=M}^{N}\frac{1}{k-\beta_p},$$

where N is a sufficiently large integer. For any real number $t<M$, we set

$$A(t,N)=\begin{cases}\displaystyle\sum_{k=M}^{N}\left(\frac{1}{k-t}-\frac{1}{k}\right) & \text{if}\quad t\in \mathbf{N}_0,\\ \displaystyle\sum_{k=1}^{N}\left(\frac{1}{k-t}-\frac{1}{k}\right) & \text{if}\quad t\notin \mathbf{N}_0,\end{cases}$$

and

$$B(t, M)=\begin{cases} 0 & \text{if } M=1\,, \\ \sum_{k=1}^{M-1} \frac{1}{k} & \text{if } M\geqq 2,\ t\in \boldsymbol{N}_0\,, \\ \sum_{k=1}^{M-1} \frac{1}{k-t} & \text{if } M\geqq 2,\ t\notin \boldsymbol{N}_0, \end{cases}$$

where $\boldsymbol{N}_0$ stands for the set of all non-negative integers. Then we have

$$\begin{aligned} (13) \qquad & \sum_{k=M}^{N} \frac{h'(k)}{h(k)} - \log h(N) \\ &= \sum_{j=1}^{D} \{A(\alpha_j, N) - B(\alpha_j, M)\} - \sum_{p=1}^{d} \{A(\beta_p, N) - B(\beta_p, M)\} \\ &\quad + (D-d)\left\{\sum_{k=1}^{N} \frac{1}{k} - \operatorname{Log} N\right\} - \left\{\log h(N) - (D-d)\operatorname{Log} N\right\}. \end{aligned}$$

We introduce the Gauss ψ-function defined by

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right).$$

If we put

$$A(t) = \lim_{N\to\infty} A(t, N)\,,$$

then

$$A(t)=\begin{cases} 0 & \text{if } t=0\,, \\ \sum_{k=M}^{M+t-1} \frac{1}{k-t} & \text{if } t\in \boldsymbol{N}\,, \\ -\psi(-t) + \frac{1}{t} - \gamma & \text{if } t\notin \boldsymbol{N}_0\,, \end{cases}$$

where, as usual, $\boldsymbol{N}$ denotes the set of all positive integers. Hence, by letting $N\to\infty$, from (2), (11), (12) and (13) we obtain

THEOREM 2. *Under the above hypothesis,*

$$\gamma_h(0) = \sum_{j=1}^{D} \{A(\alpha_j) - B(\alpha_j, M)\} - \sum_{p=1}^{d} \{A(\beta_p) - B(\beta_p, M)\} + (D-d)\gamma \dot{-} \operatorname{Log}(a_0/b_0)\,.$$

In particular, when $b(z)=1$, we have

COROLLARY 3. *If* $h(z)=a(z)$, *then*

$$\gamma_h(0)=\sum_{j=1}^{D}\{A(\alpha_j)-B(\alpha_j, M)\}+D\gamma-\mathrm{Log}\, a_0 .$$

To calculate some examples, we need the following facts (cf. [2], p. 522).

(i) $\psi(1)=-\gamma$.

(ii) $\psi(z+1)=\psi(z)+1/z$.

(iii) $\psi(1-z)=\psi(z)+\pi\cot(\pi z)$.

(iv) For positive integers r and t with $t<r$,

$$\psi\left(\frac{t}{r}\right)=-\gamma-\mathrm{Log}\, r-\frac{1}{2}\pi\cot\left(\frac{t\pi}{r}\right)+\sum_{j=1}^{r-1}\cos\left(\frac{2\pi jt}{r}\right)\mathrm{Log}\left(2\sin\frac{j\pi}{r}\right).$$

EXAMPLE 5. Take $h(z)=z^a(z+1)^b$, $M=1$ and the principal branch of $\log h(z)$, where a is a positive integer and b is a non-negative integer. Then $\gamma_h(0)=(a+b)\gamma-b$. In particular, when $h(z)=z$, $\zeta_h(s)=\zeta(s)$ and $\gamma_h(0)=\gamma$.

EXAMPLE 6. Take $h(z)=z^a(2z+1)^b$, $M=1$ and the principal branch of $\log h(z)$, where a is a non-negative integer and b is a positive integer. Then $\gamma_h(0)=b\,\mathrm{Log}\,2+(a+b)\gamma-2b$.

EXAMPLE 7. Take $h(z)=z^a(3z-1)^b$, $M=2$ and the principal branch of $\log h(z)$, where a is a non-negative integer and b is a positive integer. Then

$$\gamma_h(0)=\frac{b}{2}\mathrm{Log}\,3+(a+b)\gamma-\frac{\sqrt{3}}{6}b\pi-a-\frac{3}{2}b .$$

EXAMPLE 8. Take $h(z)=z^a(z+\delta+1)/(z+\delta)$, $M=1$ and the principal branch of $\log h(z)$, where a is a positive integer and δ is a positive number. Then $\gamma_h(0)=a\gamma-1/(1+\delta)$.

EXAMPLE 9. Take $h(z)=z^a(z+1-\delta)/(z+\delta)$, $M=1$ and the principal branch of $\log h(z)$, where a is a positive integer and δ is a positive number satisfying $0<\delta<1$. Then

$$\gamma_h(0)=a\gamma-\pi\cot(\pi\delta)+\frac{1}{\delta}-\frac{1}{1-\delta} .$$

§4. On the values of $\zeta_h(s)$ at non-positive integers.

Let $h(z)$, M and $\zeta_h(s)$ be as in Example 1. Let F be a subfield of the field of complex numbers. Suppose that all coefficients of $h(z)$ are contained in F. For any non-negative integer m, we put

$$h(M-1+z)^{m+1}=\sum_{k=0}^{D(m+1)}a_k(m)z^k , \tag{14}$$

where D is the degree of $h(z)$. We note here that all coefficients $a_k(m)$ belong to the field F. Then we obtain

THEOREM 3. *Under the above hypothesis,*

$$-(m+1)\zeta_h(-m)=a_0(m)+\frac{a_1(m)}{2}+\sum_{1\leqq k\leqq D(m+1)/2} a_{2k}(m)B_{2k}$$

for any non-negative integer m, where B_{2k} denotes the $2k$-th Bernoulli number.

PROOF. Putting $h(z)=z$ in (6), we get

$$(s-1)\zeta(s)=2\pi\int_0^\infty \frac{e^{2\pi x}}{(e^{2\pi x}+1)^2}\left\{\left(\frac{1}{2}+ix\right)^{1-s}+\left(\frac{1}{2}-ix\right)^{1-s}\right\}dx\,,$$

so that

(15) $$-(m+1)\zeta(-m)=2\pi\int_0^\infty \frac{e^{2\pi x}}{(e^{2\pi x}+1)^2}\left\{\left(\frac{1}{2}+ix\right)^{m+1}+\left(\frac{1}{2}-ix\right)^{m+1}\right\}dx$$

for any non-negative integer m. It also follows from (6) that

$$-(m+1)\zeta_h(-m)=2\pi\int_0^\infty \frac{e^{2\pi x}}{(e^{2\pi x}+1)^2}\left\{h\left(M-\frac{1}{2}+ix\right)^{m+1}+h\left(M-\frac{1}{2}-ix\right)^{m+1}\right\}dx\,.$$

Therefore, from (14) and (15), we have

$$\begin{aligned}-(m+1)\zeta_h(-m)&=\sum_{k=0}^{D(m+1)} a_k(m)\cdot 2\pi\int_0^\infty \frac{e^{2\pi x}}{(e^{2\pi x}+1)^2}\left\{\left(\frac{1}{2}+ix\right)^{k}+\left(\frac{1}{2}-ix\right)^{k}\right\}dx\\ &=a_0(m)+\sum_{k=1}^{D(m+1)} a_k(m)(-k)\zeta(1-k)\,.\end{aligned}$$

Then the assertion follows immediately from (3).

As an immediate consequence of Theorem 3, we get

COROLLARY 4. *Under the same assumptions as in Theorem* 3, *the value $\zeta_h(-m)$ is an element of F for any non-negative integer m.*

From this, we have at once

COROLLARY 5. *For any non-negative integer m, the value $\zeta(-m, a)$ belongs to F if a is an element of F, where $\zeta(s, a)$ stands for the Hurwitz zeta function defined in Example* 1.

By the functional equation for $\zeta(s, a)$ (cf. [4], p. 37), we can prove

COROLLARY 6. *For any positive integer r, both values*

$$\frac{1}{\pi^{2r}}\sum_{m=1}^{\infty}\frac{\cos(2m\pi a)}{m^{2r}}$$

and

$$\frac{1}{\pi^{2r+1}}\sum_{m=1}^{\infty}\frac{\sin(2m\pi a)}{m^{2r+1}}$$

are elements of ***F*** *if* a *is a real number belonging to* ***F***.

EXAMPLE 10. Take $h(z)=z^2+a$ and $M=1$, where a is a non-negative number. Then

$$\zeta_h(0)=-\frac{6a+1}{6}, \qquad \zeta_h(-1)=-\frac{30a^2+10a-1}{60}\,.$$

and

$$\zeta_h(-2)=-\frac{210a^3+105a^2-21a+5}{630}\,.$$

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