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q-Analogues of Determinants and Symmetric Chain Decompositions

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Abstract. We introduce multivariable q-analogues of the determinant and show a sufficient condition for an interval of the symmetric group to have a symmetric chain decomposition in the Bruhat order by using the expansion formulas.

§0. Introduction.

The purpose of this article is to show a sufficient condition for an interval of the symmetric group to have a symmetric chain decomposition (see Def. 3.1.1) in the Bruhat order by using expansion formulas of multivariable **q**-determinants.

In §1 we introduce multivariable q-analogues of the determinant which contains several parameters q_{ij} and we investigate their expansion formulas. This specializes to the quantum determinant (see [NYM]). In §2 we then introduce four kinds of orders on the symmetric group and study the relationship with the Bruhat order. In §3 we give the proof of the following result (Theorem 3.1.7) which contains a sufficient condition for an interval of the symmetric group to have a symmetric chain decomposition in the Bruhat order. First, we define some notation. We denote the symmetric group of degree n by $\mathfrak{S}_n$. Let $x, x' \in \mathfrak{S}_n$. We put $[x', x]_B := \{y \in \mathfrak{S}_n ; x' \leq_B y \leq_B x\}$, where $\leq_B$ denotes the Bruhat order (see §2). For $i \in [n] := \{1, 2, \cdots, n\}$, we put $l_i^{(1)}(x) := \#\{j ; i < j, x(i) > x(j)\}$ and we define $x \leq_1 y \Leftrightarrow l_i^{(1)}(x) \leq l_i^{(1)}(y)$ for all $i \in [n]$. We define two conditions as follows: (t_1) $l_i^{(1)}(x) \leq l_{i+1}^{(1)}(x) + 1$ for $\forall i \in [n-1]$, (t_1') $l_{i+1}^{(1)}(x') \leq l_i^{(1)}(x')$ for $\forall i \in [n-1]$. For $x', x \in \mathfrak{S}_n$, if x' satisfies (t_1'), x satisfies (t_1) and $x' \leq_1 x$, then the interval $([x', x]_B, \leq_B)$ has a symmetric chain decomposition. In particular, the rank generating function is given by $\sum_{i=1}^n (1 + q + q^2 + \cdots + q^{l_i^{(1)}(x) - l_i^{(1)}(x')})$. For the other three orders defined in Def. 2.1.1, we can find similar sufficient conditions (see Prop. 3.1.4).

Let us give a brief review of results known on this problem. Kung-Wei Yang defined a q-analogue of the determinant over a field of characteristic 0 and got the

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expansion formula for 1 row (column) and last row (column), but there was no comment about other rows and columns ([Y]). We found the expansion formulas for all rows and columns. It is known that Weyl groups of type A, B and D have symmetric chain decompositions in the Bruhat order (cf. [S2]). Let x' and x be elements of a Coxeter group satisfying $x' \leq_B x$. For $l(x)-l(x')\leq 3$, it can be easily inferred from [B] that the interval $([x', x]_B, \leq_B)$ has a symmetric chain decomposition. In general, the interval does not always have symmetric chain decomposition and no general theorems seems to be known. Our results give the sufficient conditions.

§1. q-analogues of the determinant.

In this section, we first make a refinement of the inversion number and introduce multivariable q-analogues of the determinant. We state their relations and expansion formulas.

1.1. The Q-determinant.

DEFINITION 1.1.1. For $w\in\mathfrak{S}_n$, $i\in[n]$ and $p\in[4]$, we define an inversion number of the p-th kind at i (denoted by $l_i^{(p)}(w)$) as follows: $l_i^{(1)}(w):=\#\{j\,;\,i<j,\,w(i)>w(j)\}$, $l_i^{(2)}(w):=\#\{k\,;\,k<i,\,w(k)>w(i)\}$, $l_i^{(3)}(w):=l_{w^{-1}(i)}^{(1)}(w)$, $l_i^{(4)}(w):=l_{w^{-1}(i)}^{(2)}(w)$. Then, for $p\in[4]$, the inversion number of w (denoted by $l(w)$ here) is expressed by $\sum_{i=1}^{n} l_i^{(p)}(w)$. (Note that the inversion number of w equals the length of w for all $w\in\mathfrak{S}_n$.) Of course, $l_n^{(1)}(w)=l_1^{(2)}(w)=l_1^{(3)}(w)=l_n^{(4)}(w)=0$ for any $w\in\mathfrak{S}_n$.

There are many relations in these refined inversion numbers. One can easily show the following.

LEMMA 1.1.2. *For $w\in\mathfrak{S}_n$ and $i\in[n]$, we have*

$$l_i^{(1)}(w)=l_{n+1-i}^{(2)}(w_0ww_0)=l_{n+1-i}^{(3)}(w_0w^{-1}w_0)=l_i^{(4)}(w^{-1}),$$

where w_0 is the longest element of $\mathfrak{S}_n$ (i.e. $w_0:=\begin{pmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{pmatrix}$).

We introduce multivariable q-analogues of the determinant.

DEFINITION 1.1.3. Let K be a commutative ring, q_i $(1\leq i\leq n)$ and $q_{i,j}$ $(1\leq i<j\leq n)$ be commutative variables. We put $q_{j,i}:=q_{i,j}^{-1}$, $q_{i,i}:=1$ $(1\leq i<j\leq n)$ and $\mathbf{Q}:=(q_{i,j})\in M(n, K[q_{i,j}, 1\leq i,j\leq n])$. For $w\in\mathfrak{S}_n$, we put $\mathbf{Q}_w^-:=\prod_{i<j,\,w(i)>w(j)}(-q_{i,j})$ and $\mathbf{Q}_w:=\prod_{i<j,\,w(i)>w(j)} q_{i,j}$. Let $A=(a_{i,j})\in M(n, K)$. When there does not occur confusion, we denote $q_{i,j}$ and $a_{i,j}$ by q_{ij} and a_{ij}, respectively. We define the $\mathbf{Q}$-determinant of A (denoted by $\det_{\mathbf{Q}}A$) by

$$\sum_{w\in\mathfrak{S}_n}\mathbf{Q}_w^- a_{1w(1)}a_{2w(2)}\cdots a_{nw(n)}=\sum_{w\in\mathfrak{S}_n}\mathbf{Q}_{w^{-1}}^- a_{w(1)1}a_{w(2)2}\cdots a_{w(n)n}\,.$$

Also, we define the **Q**-permanent of A (denoted by $\mathrm{per}_{\mathbf{Q}} A$) by

$$\sum_{w\in\mathfrak{S}_n} \mathbf{Q}_w a_{1w(1)}a_{2w(2)}\cdots a_{nw(n)} = \sum_{w\in\mathfrak{S}_n} \mathbf{Q}_{w^{-1}} a_{w(1)1}a_{w(2)2}\cdots a_{w(n)n}\,.$$

Moreover we put $\mathbf{q}:=(q_1, q_2, \cdots, q_n)$, $\det_{\mathbf{q}} A := \det_{\mathbf{Q}} A|_{q_{ij}=q_i(1\le i,j\le n)}$ and $\mathrm{per}_{\mathbf{q}} A := \det_{-\mathbf{q}} A$. We call $\det_{\mathbf{q}} A$ (resp. $\mathrm{per}_{\mathbf{q}} A$) the **q**-determinant of A (resp. the **q**-permanent of A).

EXAMPLE 1.1.4.

$$\det_{\mathbf{Q}}\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = aei - q_{23}afh - q_{12}bdi + q_{13}q_{23}bfg + q_{12}q_{13}cdh - q_{12}q_{13}q_{23}ceg\,.$$

REMARKS. (i) For $A=(a_{ij})\in M(n, K)$ and $p\in[4]$, we have

$$\det_{\mathbf{q}} A = \sum_{w\in\mathfrak{S}_n} (-q_1)^{l_1^{(1)}(w)}(-q_2)^{l_2^{(1)}(w)}\cdots(-q_n)^{l_n^{(1)}(w)} a_{1w(1)}a_{2w(2)}\cdots a_{nw(n)}\,,$$

$$\mathrm{per}_{\mathbf{q}} A = \sum_{w\in\mathfrak{S}_n} q_1^{l_1^{(1)}(w)} q_2^{l_2^{(1)}(w)}\cdots q_n^{l_n^{(1)}(w)} a_{1w(1)}a_{2w(2)}\cdots a_{nw(n)}\,.$$

(ii) For $A=(a_{ij})\in M(n, K)$, $\det_{(1,1,\cdots,1)} A$ equals the determinant of A and $\mathrm{per}_{(1,1,\cdots,1)} A$ equals the permanent of A which is defined by $\sum_{w\in\mathfrak{S}_n} a_{1w(1)}a_{2w(2)}\cdots a_{nw(n)}$ ([M]). (iii) Both the **Q**, **q**-determinants and the **Q**, **q**-permanents are multilinear with respect to columns and rows. (iv) The determinant can be defined in terms of the exterior algebra. In our case, one can define q-analogues of the exterior algebra to obtain **Q**-determinant (see the proof of Prop. 1.2.2).

1.2. Expansion formulas. We will prove expansion formulas into minors for the **Q**-determinant and the **Q**-permanent. By virtue of Def. 1.1.3, there is no loss of generality by restricting ourselves to the **Q**-permanent.

First, we introduce q-analogues of the complementary submatrix.

DEFINITION 1.2.1. Let $n\ge 2$, $A=(a_{ij})\in M(n, K)$ and $1\le i, j\le n$. We define $A_{ij}(\mathbf{Q})$, $A_{ij}(\mathbf{q})\in M(n-1, K[q_i, q_{ij}\,;\, 1\le i,j\le n])$ as follows:

$$A_{ij}(\mathbf{Q}) := \left(\begin{array}{c|c} a_{\mu\nu}\begin{pmatrix}\mu<i\\ \nu<j\end{pmatrix} & q_{\mu i}a_{\mu\nu}\begin{pmatrix}\mu<i\\ \nu>j\end{pmatrix}\\ \hline q_{i\mu}a_{\mu\nu}\begin{pmatrix}\mu>i\\ \nu<j\end{pmatrix} & a_{\mu\nu}\begin{pmatrix}\mu>i\\ \nu>j\end{pmatrix}\end{array}\right),\quad A_{ij}(\mathbf{q}) := A_{ij}(\mathbf{Q})|_{q_{ij}=q_i\,(1\le i,j\le n)}\,.$$

Then, we have the following expansion formulas.

PROPOSITION 1.2.2. *For $A=(a_{ij})\in M(n, K)$ and $j\in[n]$, we have*

$$\mathrm{per}_{\mathbf{Q}}A=\sum_{k=1}^{n}a_{kj}\,\mathrm{per}_{\mathbf{Q}_{kk}}A_{kj}(\mathbf{Q})=\sum_{k=1}^{n}a_{jk}\,\mathrm{per}_{\mathbf{Q}_{jj}}A_{jk}(\mathbf{Q})\,,$$

where $\mathbf{Q}_{ij}$ *is the ordinary* (i,j)*-complementary submatrix (i.e.* $\mathbf{Q}_{ij}$ *is the* $n-1$ *square matrix given by removing* i *row and* j *column from* $\mathbf{Q}$*). In particular, we have*

$$\mathrm{per}_{\mathbf{q}}A=\sum_{k=1}^{n}a_{kj}\,\mathrm{per}_{(q_1,q_2,\cdots,\widehat{q_k},\cdots,q_n)}A_{kj}(\mathbf{q})=\sum_{k=1}^{n}a_{jk}\,\mathrm{per}_{(q_1,q_2,\cdots,\widehat{q_j},\cdots,q_n)}A_{jk}(\mathbf{q})\,,$$

where $(q_1,q_2,\cdots,\widehat{q_k},\cdots,q_n):=(q_1,q_2,\cdots,q_{k-1},q_{k+1},\cdots,q_n)$.

PROOF. Let M be a K-algebra with generators $x_1,x_2,\cdots,x_n$ whose basic relations are given by $x_ix_i=0$ and $x_jx_i=q_{ij}x_ix_j$ $(1\le i,j\le n)$. We put $q(r_1,r_2,\cdots,r_s):=\prod_{r_i<r_j,\,i>j}q_{r_ir_j}$ for $r_1,r_2,\cdots,r_s\in N^s$. For $A=(a_{ij})\in M(n,K)$ and $j\in[n]$, let $s_j=\sum_{i=1}^{n}a_{ij}x_i\in M$. Then, we can easily see that $s_1s_2\cdots s_n=(\mathrm{per}_{\mathbf{Q}}A)x_1x_2\cdots x_n$. Also, we can get

$$\begin{aligned}s_1s_2\cdots s_n&=\sum_{i_j}a_{i_jj}\sum_{i_1,i_2,\cdots,\widehat{i_j},\cdots,i_n}q_{i_ji_1}q_{i_ji_2}\cdots q_{i_ji_{j-1}}q_{1i_j}q_{2i_j}\cdots q_{i_j-1\,i_j}\\&\quad\cdot q(i_1,i_2,\cdots,\widehat{i_j},\cdots,i_n)a_{i_11}a_{i_22}\cdots\widehat{a_{i_jj}}\cdots a_{i_nn}x_1x_2\cdots x_n\\&=\sum_{i_j}a_{i_jj}\,\mathrm{per}_{\mathbf{Q}_{i_ji_j}}A_{i_jj}(\mathbf{Q})x_1x_2\cdots x_n\,.\end{aligned}$$
□

REMARK. Prop. 1.2.2 is an extension of the results of K. W. Yang ([Y]).

We shall use expansion formulas in the following form from now on.

COROLLARY 1.2.3. *For* $A=(a_{ij})\in M(n,K)$*, we have*

$$\mathrm{per}_{\mathbf{q}}A=\sum_{j=1}^{n}q_1^{j-1}a_{1j}\,\mathrm{per}_{(q_2,q_3,\cdots,q_n)}A_{1j}(1,1,\cdots,1)\,.$$

REMARK. There exists an analogue of the Laplace expansion formula for the $\mathbf{q}$-determinant (the detailed statement is noted in [T]).

§2. The Bruhat order and other orders.

Let us recall the definition of the Bruhat order for the symmetric group $\mathfrak{S}_n$. Let $T=\{(i,j)\,;\,1\le i<j\le n\}$, where (i,j) is the permutation switching the number i and j and leaving the other numbers fixed. For $x',x\in\mathfrak{S}_n$, we call x' is a reduction of x (denoted $x'<'x$ here) if and only if there exists $t=(i,j)\in T$ such that $x(i)>x(j)$ and $x'=xt$. Then, the Bruhat order denoted by $\le_B$ is defined as follows: $x'<_Bx\Leftrightarrow\exists z_1,z_2,\cdots,z_k\in\mathfrak{S}_n$, s.t. $x'=z_k<'z_{k-1}<'\cdots<'z_1<'x$. For the Bruhat decomposition $GL(n,C)=\coprod_{w\in\mathfrak{S}_n}BwB$, where B denotes the set of upper triangular matrix, it is known that $BwB/B\subset\overline{Bw'B/B}\Leftrightarrow w\le_Bw'$ (cf. [H]).

In this section, we will introduce other four kinds of orders $\leq_1$, $\leq_2$, $\leq_3$ and $\leq_4$ on $\mathfrak{S}_n$, and investigate the relations with the Bruhat order.

2.1. Definition of $(\mathfrak{S}_n, \leq_p)$ and its relations with $(\mathfrak{S}_n, \leq_B)$.

DEFINITION-PROPOSITION 2.1.1. *For $p \in [4]$, we define an order $\leq_p$ on $\mathfrak{S}_n$ by*

$$x \leq_p y \Leftrightarrow l_i^{(p)}(x) \leq l_i^{(p)}(y) \text{ for all } i \in [n] .$$

PROOF. We can easily see that $\leq_1$ is well defined from the following equation.

$$\sum_{w \in \mathfrak{S}_n} q_1^{l_1^{(1)}(w)} q_2^{l_2^{(1)}(w)} \cdots q_n^{l_n^{(1)}(w)} = \mathrm{per}_{\mathbf{q}} H = \prod_{i=1}^{n} (1 + q_i + q_i^2 + \cdots + q_i^{n-i}) ,$$

where H is an n-square matrix whose entries are all 1. For other cases, we can get the similar equations from Lem. 1.1.2. □

REMARKS. (i) These five kinds of orders on $\mathfrak{S}_n$ are mutually distinct. (ii) The identity element e is minimal and the element w_0 is maximal in the poset $(\mathfrak{S}_n, \leq_r)$ for each $r \in \{1, 2, 3, 4, B\}$.

DEFINITION 2.1.2. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be posets. We denote $x \lessdot_P y$ if y covers x in $(P, \leq_P)$ (i.e. $x \leq_P z \leq_P y \Rightarrow z = x$ or y). Q is called a cover suborder of P if and only if P equals Q as a set and the order $\leq_Q$ of Q is generated by a subset of the covering relation of P (i.e. for $x, y \in Q$, $x \lessdot_Q y \Rightarrow x \lessdot_P y$). We define the poset $P \times Q$ called the direct product of P and Q as follows: $P \times Q = \{(x, y)\,;\, x \in P, y \in Q\}$ as a set and $(x, y) \leq_{P \times Q} (x', y')$ if and only if $x \leq_P x'$ and $y \leq_Q y'$. For $r \in \{1, 2, 3, 4, B\}$ and $x, y \in \mathfrak{S}_n$ with $x \leq_r y$, we put $[x, y]_r := \{z \in \mathfrak{S}_n\,;\, x \leq_r z \leq_r y\}$. Let a and b be integers with $a \leq b$. $\underline{[a, b]}$ is the totally ordered set on $[a, b] := \{a, a+1, \cdots, b\}$ with ordinary order of $\boldsymbol{Z}$.

The following proposition immediately follows from Def.-Prop. 2.1.1.

PROPOSITION 2.1.3. *For $p \in [4]$, we have*

$$(\mathfrak{S}_n, \leq_p) \simeq \underline{[0, n-1]} \times \underline{[0, n-2]} \times \cdots \times \underline{[0, 0]} .$$

We have the following relations between $(\mathfrak{S}_n, \leq_B)$ and $(\mathfrak{S}_n, \leq_p)$ ($p \in [4]$). Note that the case $p = 3$ is described in [S2].

PROPOSITION 2.1.4. *$(\mathfrak{S}_n, \leq_p)$ is a cover suborder of $(\mathfrak{S}_n, \leq_B)$ for $p \in [4]$.*

PROOF. Let $x, y \in \mathfrak{S}_n$. We can see the following easily. $x \lessdot_1 y \Leftrightarrow \exists i, j \in [n]$ such that $i < j$, $y(i) > y(j)$, $y(k) \geq y(i)$ or $y(j) \geq y(k)$ for all $k \in [i, n]$ and $x = y(i, j)$. Hence, we can see this proposition in case $p = 1$. Similarly, we can show this one in other cases. □

2.2. Definition and properties of $S_n(\mathbf{a}'; \mathbf{a})$. In this place, we define a subset of $\mathfrak{S}_n$ and we show that it is equal to a certain interval with respect to the Bruhat order as a set. This result plays a crucial role in the proof of the main theorem.

DEFINITION 2.2.1. Let $a_1, a_2, \cdots, a_n$ and $a'_1, a'_2, \cdots, a'_n$ be natural numbers satisfying $1 \leq a'_n \leq a'_{n-1} \leq \cdots \leq a'_1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n$ and $a_i - a'_i \geq i-1$ for $\forall i \in [n]$. We put $\mathbf{a} := (a_1, a_2, \cdots, a_n)$ and $\mathbf{a}' := (a'_1, a'_2, \cdots, a'_n)$. Then, we define subset $S_n(\mathbf{a}'; \mathbf{a})$ of $\mathfrak{S}_n$ by $\{w \in \mathfrak{S}_n ; a'_i \leq w(i) \leq a_i$ for $\forall i \in [n]\}$. For $1 \leq i, j \leq n$, we put $m_{ij} := 1$ if $a'_i \leq j \leq a_i$, otherwise $m_{ij} := 0$, and $M_n\begin{pmatrix} a'_1 & a'_2 & \cdots & a'_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} := (m_{ij}) \in M(n, \boldsymbol{R})$.

Then, we have the following lemma.

LEMMA 2.2.2.

$$(S_n(\mathbf{a}'; \mathbf{a}), \leq_1) \simeq \underline{[a'_1 - 1, a_1 - 1]} \times \underline{[a'_2 - 1, a_2 - 2]} \times \cdots \times \underline{[a'_n - 1, a_n - n]}\,.$$

PROOF. From Cor. 1.2.3, we have

$$\sum_{w \in S_n(\mathbf{a}'; \mathbf{a})} q_1^{l_1^{(1)}(w)} q_2^{l_2^{(1)}(w)} \cdots q_n^{l_n^{(1)}(w)} = \mathrm{per}_{\mathbf{q}} M_n\begin{pmatrix} a'_1 & a'_2 & \cdots & a'_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} = \prod_{i=1}^{n} (q_i^{a'_i - 1} + q_i^{a'_i} + \cdots + q_i^{a_i - i})\,.$$

So, the function μ from $(S_n(\mathbf{a}'; \mathbf{a}), \leq_1)$ to $\underline{[a'_1 - 1, a_1 - 1]} \times \underline{[a'_2 - 1, a_2 - 2]} \times \cdots \times \underline{[a'_n - 1, a_n - n]}$ defined by $\mu(w) := (l_1^{(1)}(w), l_2^{(1)}(w), \cdots, l_n^{(1)}(w))$ is isomorphism. □

REMARK. We put $w = (1, 2)(1, 3) \cdots (1, l_1 + 1)(2, 3)(2, 4) \cdots (2, l_2 + 2) \cdots (n-1, l_{n-1} + n - 1)$. Then, we have $\mu(w) = (l_1, l_2, \cdots, l_{n-1}, l_n)$, where μ is defined in Lem. 2.2.2 and $l_n := 0$.

We can easily see the following corollary from Lem. 2.2.2 and its proof.

COROLLARY 2.2.3. *We use the same notation as in the proof of Lem.* 2.2.2. *We put*

$$x'_1 := \mu^{-1}(a'_1 - 1, a'_2 - 1, \cdots, a'_n - 1)\,, \qquad x_1 := \mu^{-1}(a_1 - 1, a_2 - 2, \cdots, a_n - n)\,.$$

Then, we have $S_n(\mathbf{a}'; \mathbf{a}) = [x'_1, x_1]_1$.

From now on, x'_1 and x_1 denote the elements defined in Cor. 2.2.3.

Next, we have the following proposition, which follows from Lem. 2.2.5.

PROPOSITION 2.2.4. *We have* $[x'_1, x_1]_1 = [x'_1, x_1]_B$.

LEMMA 2.2.5. *For* $x \in [x'_1, w_0]_1$ *and* $y \in [e, x_1]_1$, *we can see the following.*

(i) *When* x *is one of the reductions of* $z \in \mathfrak{S}_n$ (*i.e.* $x <' z$), *we have* $z \in [x'_1, w_0]_1$.

(ii) *When* $z \in \mathfrak{S}_n$ *is a reduction of* y (*i.e.* $z <' y$), *we have* $z \in [e, x_1]_1$.

(iii) *When* $z \in [x'_1, x_1]_B$, *we have* $z \in [x'_1, x_1]_1$.

PROOF. Note that $[e, x_1]_1 = \{w \in \mathfrak{S}_n ; w(i) \leq a_i$ for $\forall i \in [n]\}$ and $[x'_1, w_0]_1 = \{w \in \mathfrak{S}_n ; a'_i \leq w(i)$ for $\forall i \in [n]\}$. (i) From $x <' z$, there exist $i, j \in [n]$ such that $1 \leq i < j \leq n$, $z(i) > z(j)$ and $x = z(i, j)$. So, from $x \in [x'_1, w_0]_1$, we have $a'_j \leq a'_i \leq x(i) < x(j)$. Hence, for any $k \in [n]$, we have $a'_k \leq z(k)$. So, we have (i). We can obtain (ii) similarly to (i), and we can easily see (iii) from (i), (ii). □

§3. A sufficient condition for symmetric chain decomposition.

In this section, we show a sufficient condition for an interval of $\mathfrak{S}_n$ to have a symmetric chain decomposition.

3.1. Main theorem. First, we state the definition of a symmetric chain decomposition.

DEFINITION 3.1.1. Let P be a poset and C be a subposet of P. We call that C is a chain if C is a totally ordered set. Let C be a chain of P. C is called saturated if there is no $z \in P \setminus C$ such that $x <_P z <_P y$ for some $x, y \in C$ and that $C \cup \{z\}$ is a chain. $\sharp C - 1$ is called the length of C and we define rank of $P := \max\{\sharp C - 1\,;\, C$ is a chain of $P\}$. If there is no chain $C'(\neq C)$ of P such that $C \subset C'$, then C is called a maximal chain of P. If every maximal chain of P has the same length, P is called a graded poset, and then we can define the function ρ from P to $\{0, 1, 2, \cdots\}$ satisfying $\rho(z) = 0$ if z is a minimal element of P and $\rho(x) = \rho(y) + 1$ if x uniquely covers y in P. We say that a graded poset P with rank function ρ has a symmetric chain decomposition if and only if there exist saturated chains $C_1, C_2, \cdots, C_r$ of P satisfying $P = \coprod_{i=1}^{r} C_i$ (disjoint union) and $\rho(x_i) + \rho(y_i) =$ rank of P for all $i \in [r]$, where x_i is the minimal element of C_i and y_i is the maximal element of C_i for each $i \in [r]$.

Since the direct product of posets with symmetric chain decompositions has a symmetric chain decomposition, Lem. 2.2.2 and Prop. 2.1.3 imply the following.

PROPOSITION 3.1.2. *$(S_n(\mathbf{a}'; \mathbf{a}), \leq_1)$ has a symmetric chain decomposition. In particular, so does $(\mathfrak{S}_n, \leq_p)$ for each $p \in [4]$.*

Next, we will show the key point of this article.

PROPOSITION 3.1.3. *$(S_n(\mathbf{a}'; \mathbf{a}), \leq_B)$ has a symmetric chain decomposition.*

PROOF. Since an interval of graded poset is a graded poset, $(S_n(\mathbf{a}'; \mathbf{a}), \leq_B) = ([x'_1, x_1]_B, \leq_B)$ is a graded poset. Also, we can easily see that $(S_n(\mathbf{a}'; \mathbf{a}), \leq_1)$ is a graded poset. So, from Prop. 2.1.4, $(S_n(\mathbf{a}'; \mathbf{a}), \leq_1)$ is a cover suborder of $(S_n(\mathbf{a}'; \mathbf{a}), \leq_B)$ and their rank functions are identical. Hence, we can easily see that $(S_n(\mathbf{a}'; \mathbf{a}), \leq_B)$ has a symmetric chain decomposition because $(S_n(\mathbf{a}'; \mathbf{a}), \leq_1)$ has a symmetric chain decomposition.

□

Then, we have the following proposition.

PROPOSITION 3.1.4. *Let $x', x \in \mathfrak{S}_n$. For $p \in [4]$, we put the conditions (t_p) and (t'_p) as follows: When $p = 1, 4$, (t_p) $l_i^{(p)}(x) \leq l_{i+1}^{(p)}(x) + 1$ for $\forall i \in [n-1]$, (t'_p) $l_{i+1}^{(p)}(x') \leq l_i^{(p)}(x')$ for $\forall i \in [n-1]$. When $p = 2, 3$, (t_p) $l_{i+1}^{(p)}(x) \leq l_i^{(p)}(x) + 1$ for $\forall i \in [n-1]$, (t'_p) $l_i^{(p)}(x') \leq l_{i+1}^{(p)}(x')$ for $\forall i \in [n-1]$. If there exists some $p \in [4]$ such that x satisfies (t_p), x' satisfies (t'_p) and $x' \leq_p x$, then $([x', x]_B, \leq_B)$ has a symmetric chain decomposition, and its rank generating function is given by*

$$\prod_{i=1}^{n}(1+q+q^2+\cdots+q^{l_i^{(p)}(x)-l_i^{(p)}(x')}).$$

PROOF. We will prove in case $p=1$. For each $i\in[n]$, we put $a_i:=l_i^{(1)}(x)+i$, $a_i':=l_i^{(1)}(x')+1$, $\mathbf{a}:=(a_1,a_2,\cdots,a_n)$ and $\mathbf{a}':=(a_1',a_2',\cdots,a_n')$. Then, from the assumptions, we have $1\le a_n'\le a_{n-1}'\le\cdots\le a_1'\le a_1\le a_2\le\cdots\le a_n\le n$ and $a_i-a_i'\ge i-1$ for $\forall i\in[n]$. So, $\mathbf{a}'$ and $\mathbf{a}$ satisfy the condition of Def. 2.2.1. Hence, we can consider $S_n(\mathbf{a}';\mathbf{a})$, and we have $S_n(\mathbf{a}';\mathbf{a})=[x',x]_1=[x',x]_B$. It follows that $([x',x]_B,\le_B)$ has a symmetric chain decomposition from Prop. 3.1.3. Since rank function of $([x',x]_B,\le_B)$ equals the rank function of $(S_n(\mathbf{a}';\mathbf{a}),\le_1)$, we can immediately obtain the rank generating function of $([x',x]_B,\le_B)$ from Lem. 2.2.2. From Lem. 1.1.2, we can easily see the following. x satisfies $(t_1)\Leftrightarrow w_0xw_0$ satisfies $(t_2)\Leftrightarrow w_0x^{-1}w_0$ satisfies $(t_3)\Leftrightarrow x^{-1}$ satisfies (t_4), x' satisfies $(t_1')\Leftrightarrow w_0x'w_0$ satisfies $(t_2')\Leftrightarrow w_0x'^{-1}w_0$ satisfies $(t_3')\Leftrightarrow x'^{-1}$ satisfies (t_4'), $x'\le_1 x\Leftrightarrow w_0x'w_0\le_2 w_0xw_0\Leftrightarrow w_0x'^{-1}w_0\le_3 w_0x^{-1}w_0\Leftrightarrow x'^{-1}\le_4 x^{-1}$. Hence, we get the other cases from the following facts: We define bijections g_1 and g_2 on $(\mathfrak{S}_n,\le_B)$ by $g_1(w):=w_0ww_0$ and $g_2(w):=w^{-1}$. Then, g_1 and g_2 are isomorphisms. □

We will state the main theorem after showing two lemmas.

LEMMA 3.1.5. *For $x\in\mathfrak{S}_n$, we have the following.*
(i) *x satisfies (t_1) (resp. (t_2)) $\Leftrightarrow$ x satisfies (t_3) (resp. (t_4)).*
(ii) *x satisfies (t_1') (resp. (t_2')) $\Leftrightarrow$ x satisfies (t_4') (resp. (t_3')).*

PROOF. (i) From Lem. 1.1.2, we may show only the following case. x satisfies $(t_3)\Rightarrow x$ satisfies (t_1). We suppose that there exists $i\in[n-1]$ such that $l_i^{(1)}(x)>l_{i+1}^{(1)}(x)+1$. Then, we can find $k>i+1$ satisfying $x(i+1)<x(k)<x(i)$ and $x^{-1}(x(k)+1)(=:r)\le i$. In this case, we have $x=\begin{pmatrix}\cdots & r & \cdots & i & i+1 & \cdots & k & \cdots\\ \cdots & x(k)+1 & \cdots & x(i) & x(i+1) & \cdots & x(k) & \cdots\end{pmatrix}$. Hence, we have $l_{x(k)+1}^{(3)}(x)\ge l_{x(k)}^{(3)}(x)+2$. This is a contradiction. We can show (ii) similarly to (i). □

LEMMA 3.1.6. *Let $x',x\in\mathfrak{S}_n$ with $x'\le_B x$. For each $p\in[4]$, we have the following. If x satisfies (t_p) or x' satisfies (t_p'), then $x'\le_p x$.*

PROOF. When x satisfies (t_p), we have $[e,x]_p=[e,x]_B$ from the fact that e satisfies (t_p'), $e\le_p x$ and the proof of Prop. 3.1.4. Hence, $x'\in[e,x]_B=[e,x]_p$. Similarly, we can show the other case. □

Finally, we can obtain the main theorem.

THEOREM 3.1.7. *For $x',x\in\mathfrak{S}_n$, if x' satisfies (t_1') or (t_2'), x satisfies (t_1) or (t_2) and $x'\le_r x$ for some $r\in\{1,2,3,4,B\}$, then $([x',x]_B,\le_B)$ has a symmetric chain decomposition. Also, we put $k:=1$ if x' satisfies (t_1') and x satisfies (t_1), $k:=4$ if x' satisfies (t_1') and x satisfies (t_2), $k:=3$ if x' satisfies (t_2') and x satisfies (t_1), $k:=2$ if x' satisfies (t_2') and x satisfies (t_2), then its rank generating function is given by*

$$\sum_{i=1}^{n}(1+q+q^2+\cdots+q^{l_i^{(k)}(x)-l_i^{(k)}(x')}).$$

REMARKS. (i) Prop. 3.1.4 and Theorem 3.1.7 are the same statement. (ii) Let $w, z \in \mathfrak{S}_n$ with $w \le_B z$. We can easily see the following. If there exists $k \in [n-1]$ satisfying at least one of the following four conditions, then $\exists w', z' \in \mathfrak{S}_{n-k}$ such that $([z, w]_B, \le_B) \simeq ([z', w']_B, \le_B)$. (1) $x(i)=y(i)$ for $\forall i \in [k]$, (2) $x(n+1-i)=y(n+1-i)$ for $\forall i \in [k]$, (3) $x^{-1}(i)=y^{-1}(i)$ for $\forall i \in [k]$, (4) $x^{-1}(n+1-i)=y^{-1}(n+1-i)$ for $\forall i \in [k]$. (iii) There are 205 posets which have symmetric chain decompositions in the 213 intervals of $(\mathfrak{S}_4, \le_B)$, and we know that 127 intervals in it have symmetric chain decompositions from Theorem 3.1.7. Also, we can know further 50 intervals have symmetric chain decompositions from (ii).

3.2. Some corollaries of the main theorem. We can easily obtain the following from Theorem 3.1.7.

COROLLARY 3.2.1. *For $x \in \mathfrak{S}_n$, we have the following.*
(i) *If x satisfies (t_1) or (t_2), then $([e, x]_B, \le_B)$ has a symmetric chain decomposition.*
(ii) *If x satisfies (t_1') or (t_2'), then $([x, w_0]_B, \le_B)$ has a symmetric chain decomposition.*

PROPOSITION 3.2.2. *If $x \in \mathfrak{S}_n$ satisfies the following condition (u) or (u'), then $([e, x]_B, \le_B)$ and $([x, w_0]_B, \le_B)$ have symmetric chain decompositions.*

$$(u) \qquad x(1)<\cdots<x(k)>\cdots>x(n) \qquad \text{for some} \quad k \in [n],$$

$$(u') \qquad x(1)>\cdots>x(k)<\cdots<x(n) \qquad \text{for some} \quad k \in [n].$$

PROOF. We can see that x satisfies (t_2') and (t_1) (resp. (t_1') and (t_2)) if x satisfies (u) (resp. (u')). So, we have this proposition from Cor. 3.2.1. □

COROLLARY 3.2.3. *For $x', x \in \mathfrak{S}_n$ satisfying $x' \le_r x$ for some $r \in \{1, 2, 3, 4, B\}$, if x' satisfies (u) or (u') and x satisfies (u) or (u'), then $([x', x]_B, \le_B)$ has a symmetric chain decomposition.*

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