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Some Remarks on the Characterization of the Poisson Kernels for the Hyperbolic Spaces

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Introduction.

Let G be a classical connected simple Lie group of real rank 1: i.e. G is one of the groups $SO_0(1,n)$, $SU(1,n)$ and $Sp(1,n)$ corresponding to the fields $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ respectively. Let $G=KAN$ be an Iwasawa decompositon and M be the centralizer of A in K. Denoting by $\boldsymbol{F}$ the field corresponding to the group G, then G/K is the classical hyperbolic space, i.e. the unit ball in $\boldsymbol{F}^n$ (denoted by $B(\boldsymbol{F}^n)$) and it's Martin boundary K/M is the unit sphere in $\boldsymbol{F}^n$ (denoted by $S(\boldsymbol{F}^n)$). The action of G on $B(\boldsymbol{F}^n)$ and $S(\boldsymbol{F}^n)$ is concretely given as follows: for $x={}^t(x_1,\cdots,x_n)\in\boldsymbol{F}^n$ and $g=(g_{pq})_{0\le p,q\le n}\in G$, we define

$$x'=gx\,,$$

where $x'={}^t(x'_1,\cdots,x'_n)$, with

$$x'_p=(g_{p0}+\sum_{q=1}^{n}g_{pq}x_q)(g_{00}+\sum_{q=1}^{n}g_{0q}x_q)^{-1}\,,\qquad 1\le p\le n\,.$$

And the identifications $G/K\cong B(\boldsymbol{F}^n)$ and $K/M\cong S(\boldsymbol{F}^n)$ are given by

$$G/K\cong B(\boldsymbol{F}^n)\ ;\ gK\mapsto gO\,,$$

$$K/M\cong S(\boldsymbol{F}^n)\ ;\ kM\mapsto ke_1\,,$$

where O is the origin of $\boldsymbol{F}^n$ and $e_1={}^t(1,0,\cdots 0)\in S(\boldsymbol{F}^n)$.

We now denote by D the Laplace-Beltrami operater on $G/K\cong B(\boldsymbol{F}^n)$. The Poisson kernel P: $G/K\times K/M\to\boldsymbol{R}$ is given as follows:

$$P(gK,kM)=\left(\frac{1-|x|^2}{|1-{}^t\bar{x}\cdot b|^2}\right)^{\rho}\,,$$

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where $x=gK$, $b=kM$ and $\rho=d-1+((n-1)/2)d$, $d=\dim_{\boldsymbol{R}}\boldsymbol{F}$. As is well known, for any complex number s, we have

$$DP^s=4\rho^2 s(s-1)P^s .$$

In particular, we have the following differential equations for P

$$\begin{cases} DP=0 \\ DP^2=8\rho^2P^2 . \end{cases}$$

Now we consider the following problem.

PROBLEM 1. Suppose that a real valued C^2-class function $F\neq 0$ on G/K satisfies the following conditions:

(a) $DF=0$

(b) $DF^2=8\rho^2F^2$.

Then there exist $c\in\boldsymbol{R}$ and $kM\in K/M$ such that $F(gK)=cP(gK, kM)$.

This problem is solved in the affirmative for $\boldsymbol{F}=\boldsymbol{R}$ in [1]. For $\boldsymbol{F}=\boldsymbol{C}$ also, a solution was given in [5] (the proof seems to have some gap, but we understand Kawazoe has another proof to be published soon). The purpose of this paper is to give a quick proof in the real case using the spherical functions on K/M. Also included is some preliminary results about this problem, which might be useful to solve the problem in other cases.

1. Some lemmas on the function F.

We shall use the following notations for $x={}^t(x_1, \cdots, x_n)\in B(\boldsymbol{F}^n)$.

$$\boldsymbol{F}=\boldsymbol{R}: \quad x_1=x_{11}, \cdots, x_n=x_{n1} ,$$

$$\boldsymbol{F}=\boldsymbol{C}: \quad x_1=x_{11}+ix_{12}, \cdots, x_n=x_{n1}+ix_{n2} ,$$

$$\boldsymbol{F}=\boldsymbol{H}: \quad x_1=x_{11}+ix_{12}+jx_{13}+kx_{14}, \cdots ,$$
$$x_n=x_{n1}+ix_{n2}+jx_{n3}+kx_{n4} ,$$

where $x_{p\nu}$ $(1\leq p\leq n,\ 1\leq\nu\leq d)\in\boldsymbol{R}$.

To return to Problem 1, we have

$$P(gx, b)=P(x, g^{-1}b)P(gO, b)$$

for $x\in B(\boldsymbol{F}^n)$, $b\in S(\boldsymbol{F}^n)$ and $g\in G$. Since D is G-invariant, without loss of generality, we may assume that $F(0)=1$. On the other hand, from (a), (b) and $F(0)=1$, we see that

$$\sum_{\substack{1\leq p\leq n \\ 1\leq\mu\leq d}} \left(\frac{\partial F}{\partial x_{p\mu}}(0)\right)^2=4\rho^2 .$$

So there exists $k\in K$ such that

$$\frac{\partial F_k}{\partial x_{11}}(0)=2\rho\,, \qquad \frac{\partial F_k}{\partial x_{p\mu}}(0)=0 \quad ((p,\mu)\neq(1,1))\,,$$

where $F_k(x)=F(kx)$. Thus we can conclude that Problem 1 is equivalent to the following problem.

PROBLEM 2. Suppose that a real valued C^2-class function F on $B(\boldsymbol{F}^n)$ satisfies the following conditions:

(a) $DF=0$

(b) $DF^2=8\rho^2F^2$

(c) $F(0)=1$

(d) $(\partial F/\partial x_{11})(0)=2\rho$, $(\partial F/\partial x_{p\mu})(0)=0$ $((p,\mu)\neq(1,1))$.

Then $F(x)=P(x,e_1)$ for $x\in B(\boldsymbol{F}^n)$.

We remark that F is analytic, because D is elliptic.

The following lemma holds independent of the choice of $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$.

LEMMA 1. *Let us suppose that F satisfies the assumptions of Problem 2. Then we have*

(1) *For $l\geq 2$, $(\partial^l F/\partial x_{11}^l)(0)$ is expressed as a polynomial of $(\partial^m F/\partial x_{11}^m)(0)$ $(1\leq m\leq l-1)$ and $(\partial^m F/\partial x_{r\lambda}\partial x_{11}^{m-1})(0)$ $((r,\lambda)\neq(1,1),\ 1\leq m\leq l-1)$, whose coefficients are independent of the choice of F.*

(2) *$(\partial^l F/\partial x_{r\lambda}\partial x_{11}^{l-1})(0)=0$ for $(r,\lambda)\neq(1,1)$ and $l\geq 2$. In particular, $(\partial^l F/\partial x_{11}^l)(0)=(\partial^l P/\partial x_{11}^l)(0)$ $(l\geq 0)$ and moreover $F(x_{11},0)=P(x_{11},0)$, where $P(x)=P(x,e_1)$.*

PROOF. We use the following notations

$\boldsymbol{F}=\boldsymbol{R}:\ x_p x_q=A_{pq1}(x)$,

$\boldsymbol{F}=\boldsymbol{C}:\ x_p\bar{x}_q=A_{pq1}(x)+iA_{pq2}(x)$,

$\boldsymbol{F}=\boldsymbol{H}:\ x_p\bar{x}_q=A_{pq1}(x)+iA_{pq2}(x)+jA_{pq3}(x)+kA_{pq4}(x)$,

where $A_{pq\mu}(x)$ $(1\leq p, q\leq n,\ 1\leq\mu\leq d)\in\boldsymbol{R}$. Then the Laplace-Beltrami operator D has the following expression:

$$\begin{aligned} D=&(1-|x|^2)\sum_{p=1}^{n}(1-|x_p|^2)\sum_{\mu=1}^{d}\frac{\partial^2}{\partial x_{p\mu}^2}\\ &+(1-|x|^2)\sum_{\substack{p\neq q\\ 1\leq\mu,\nu\leq d}} s(p,\mu,q,\nu)A_{pqm(p,\mu,q,\nu)}\frac{\partial^2}{\partial x_{p\mu}\partial x_{q\nu}}\\ &+2(d-2)(1-|x|^2)\sum_{\substack{1\leq p\leq n\\ 1\leq\mu\leq d}} x_{p\mu}\frac{\partial}{\partial x_{p\mu}}\,, \end{aligned}$$

where $m(p,\mu,q,\nu)=1,\cdots,d$ and $s(p,\mu,q,\nu)=1,\ -1$. Since F satisfies (a) and (b), we obtain

$$8\rho^2F^2=2(1-|x|^2)\sum_{p=1}^{n}(1-|x_p|^2)\sum_{\mu=1}^{d}\left(\frac{\partial F}{\partial x_{p\mu}}\right)^2$$
$$+2(1-|x|^2)\sum_{\substack{p\neq q\\ 1\leq\mu,\nu\leq d}} s(p,\mu,q,\nu)A_{pqm(p,\mu,q,\nu)}\frac{\partial F}{\partial x_{p\mu}}\frac{\partial F}{\partial x_{q\nu}}. \tag{1.1}$$

Applying the differential operator $\partial^{l-1}/\partial x_{11}^{l-1}$ to (1.1), we put $x=0$. Then we see that

$$8\rho^2\sum_{k=0}^{l-1}\binom{l-1}{k}\frac{\partial^{l-1-k}F}{\partial x_{11}^{l-1-k}}(0)\frac{\partial^k F}{\partial x_{11}^k}(0)$$
$$=2\sum_{\substack{1\leq p\leq n\\ 1\leq\mu\leq d}}\sum_{\alpha+\beta+\gamma=l-1}\frac{(l-1)!}{\alpha!\beta!\gamma!}\left[\frac{\partial^\alpha}{\partial x_{11}^\alpha}(1-|x|^2)\frac{\partial^\beta}{\partial x_{11}^\beta}(1-|x_p|^2)\right]_{x=0}$$
$$\cdot\left(\sum_{q=0}^{\gamma}\binom{\gamma}{q}\frac{\partial^{\gamma+1-q}F}{\partial x_{p\mu}\partial x_{11}^{\gamma-q}}(0)\frac{\partial^{q+1}F}{\partial x_{p\mu}\partial x_{11}^q}(0)\right).$$

This implies our first assertion (1).

We now prove (2) by the induction with respect to l. Applying the differential operator $\partial/\partial x_{r\lambda}$ $((r,\lambda)\neq(1,1))$ to (1.1), we put $x=0$. Then we have

$$16\rho^2F(0)\frac{\partial F}{\partial x_{r\lambda}}(0)=4\sum_{p,\mu}\frac{\partial F}{\partial x_{p\mu}}(0)\frac{\partial^2 F}{\partial x_{r\lambda}\partial x_{p\mu}}(0)=4\frac{\partial^2 F}{\partial x_{r\lambda}\partial x_{11}}(0).$$

Therefore

$$\frac{\partial^2 F}{\partial x_{r\lambda}\partial x_{11}}(0)=0 \qquad \text{for} \quad (r,\lambda)\neq(1,1).$$

Let $l\geq 3$ and we suppose that

$$\frac{\partial^m F}{\partial x_{r\lambda}\partial x_{11}^{m-1}}(0)=0 \qquad \text{for} \quad (r,\lambda)\neq(1,1),\quad 2\leq m\leq l-1.$$

Applying the differential operator $\partial^{l-1}/\partial x_{r\lambda}\partial x_{11}^{l-2}$ $((r,\lambda)\neq(1,1))$ to (1.1), we put $x=0$. Then we have

$$\frac{\partial^l F}{\partial x_{r\lambda}\partial x_{11}^{l-1}}(0)=0.$$

Thus our assertion (2) is true. Finally, it follows from the analyticity of F that $F(x_{11},0)=P(x_{11},0)$. □

Let $\hat{K}$ denote the set of equivalence classes of finite dimensional unitary irreducible representations of K. For $(\tau,V_\tau)\in\hat{K}$ (V_τ is the representation space of τ), we denote by V_τ^M the space of M-fixed vectors in V_τ. And we set $\bar{A}_+=\{a_t \mid t\geq 0\}$ with

$$a_t=\begin{pmatrix}\cosh t & \sinh t & 0\\ \sinh t & \cosh t & \\ 0 & & I_{n-1}\end{pmatrix}.$$

Then we have the Cartan decomposition $G=K\bar{A}_+K$.

LEMMA 2. *Suppose that F satisfies the assumptions of Problem* 2.

(1) (Helgason) *The following equality holds for $t\geq 0$*:

$$\int_K F^2(ka_tO)dk=\int_K P^2(ka_tO, e_1)dk\,,$$

where dk is the normalized Haar measure on K.

(2) (Helgason) *For $(\tau, V_\tau)\in\hat{K}$ which satisfies* $\dim V_\tau^M=1$, *there exists a constant $\alpha_\tau\in C$ such that*

$$\int_K \overline{\varphi_\tau(k)}F(ka_tO)dk=\alpha_\tau\int_K \overline{\varphi_\tau(k)}P(ka_tO, e_1)dk \qquad (t\geq 0)\,,$$

where $\varphi_\tau(k)=\overline{(\tau(k)e_M, e_M)}$ and e_M is the normalized vector in V_τ^M.

(3) *For* $0\leq r<1$,

$$\sum_{\tau\in\hat{K}_M}\alpha_\tau d_\tau\int_K \overline{\varphi_\tau(k)}P(rke_1, e_1)dk$$
$$=\sum_{\tau\in\hat{K}_M} d_\tau\int_K \overline{\varphi_\tau(k)}P(rke_1, e_1)dk\,,$$

where $\hat{K}_M$ is the set of elements $(\tau, V_\tau)\in\hat{K}$ which satisfy $\dim V_\tau^M=1$ *and* $d_\tau=\dim V_\tau$. *And the series' of both sides absolutely converge.*

PROOF. (1) and (2) are proved in [3], [4] and [5].

Proof of (3). We set

$$G(x)=\int_M F(mx)dm\,,$$

where dm is the normalized Haar measure on M. Since the function $k\mapsto G(ka_tO)$ is M bi-invariant and

$$\int_K \overline{\varphi_\tau(k)}F(ka_tO)dk=\int_K \overline{\varphi_\tau(k)}G(ka_tO)dk\,,$$

we have the following Fourier expansion for $k\mapsto G(ka_tO)$:

$$G(ka_tO)=\sum_{\tau\in\hat{K}_M} d_\tau\left[\int_K \overline{\varphi_\tau(k')}G(k'a_tO)dk'\right]\varphi_\tau(k)$$

$$= \sum_{\tau \in \hat{K}_M} d_\tau \left[\int_K \overline{\varphi_\tau(k')} F(k' a_t O) dk' \right] \varphi_\tau(k)$$

$$= \sum_{\tau \in \hat{K}_M} \alpha_\tau d_\tau \left[\int_K \overline{\varphi_\tau(k')} P(k' a_t O, e_1) dk' \right] \varphi_\tau(k) .$$

In particular,

$$G(a_t O) = \sum_{\tau \in \hat{K}_M} \alpha_\tau d_\tau \int_K \overline{\varphi_\tau(k)} P(k a_t O, e_1) dk .$$

Similarly, we see that

$$P(a_t O, e_1) = \sum_{\tau \in \hat{K}_M} d_\tau \int_K \overline{\varphi_\tau(k)} P(k a_t O, e_1) dk .$$

On the other hand, it follows from Lemma 1 and $a_t O = \tanh t e_1$ that $G(a_t O) = P(a_t O, e_1)$. Thus we can obtain our assertion. □

2. A new proof for the real case of Problem 2.

Let $\boldsymbol{F} = \boldsymbol{R}$. We define the functions c_l by

$$P(rb, e_1) = \sum_{l=0}^{\infty} c_l(b) r^l , \qquad 0 \leq r < 1 , \quad b \in S(\boldsymbol{R}^n) .$$

Then we see that

$$\begin{aligned} c_l(b) &= \frac{1}{l!} \left[\frac{\partial^l}{\partial r^l} P(rb, e_1) \right]_{r=0} \\ &= \frac{1}{l!} \left[\frac{\partial^l}{\partial r^l} H(rb_1, r) \right]_{r=0} \\ &= \frac{1}{l!} \sum_{q=0}^{[l/2]} \binom{l}{2q} \frac{\partial^l H}{\partial \xi^{l-2q} \partial \eta^{2q}} (0) b_1^{l-2q} , \end{aligned}$$

where

$$H(\xi, \eta) = \left(\frac{1 - \eta^2}{|1 - \xi|^2} \right)^\rho .$$

By the way, for $(\tau, V_\tau) \in \hat{K}$ which satisfies $\dim V_\tau^M = 1$, there uniquely exists a non-negative integer p such that

$$\varphi_\tau(k) = C_p^{(n-2)/2}(b_1) / C_p^{(n-2)/2}(1) , \qquad b = k e_1 .$$

And we have

$$\int_{S(\boldsymbol{R}^n)} C_p^{(n-2)/2}(b_1)c_m(b)d\sigma_n(b)=0 \qquad (m<p)\,,$$

where $d\sigma_n$ is the normalized element of surface area on $S(\boldsymbol{R}^n)$. So we obtain that

$$\int_K \varphi_\tau(k)P(rke_1, e_1)dk=[C_p^{(n-2)/2}(1)]^{-1}\sum_{l=p}^{\infty} r^l \int_{S(\boldsymbol{R}^n)} C_p^{(n-2)/2}(b_1)c_l(b)d\sigma_n(b)\,.$$

We remark that

$$\int_{S(\boldsymbol{R}^n)} C_p^{(n-2)/2}(b_1)c_p(b)d\sigma_n(b)\neq 0\,.$$

Putting $\alpha_\tau=\alpha_p$ and $d_\tau=d_p$, Lemma 2 (3) can be stated as follows.

$$\sum_{p=0}^{\infty}\alpha_p d_p[C_p^{(n-2)/2}(1)]^{-1}\sum_{l=p}^{\infty} r^l \int_{S(\boldsymbol{R}^n)} C_p^{(n-2)/2}(b_1)c_l(b)d\sigma_n(b)$$

$$=\sum_{p=0}^{\infty} d_p[C_p^{(n-2)/2}(1)]^{-1}\sum_{l=p}^{\infty} r^l \int_{S(\boldsymbol{R}^n)} C_p^{(n-2)/2}(b_1)c_l(b)d\sigma_n(b)\,.$$

Comparing the coefficients of r^l $(l\geq 0)$ in both sides, we can conclude that $\alpha_p=1$ for all $p\geq 0$. Thus we have

$$\int_K \varphi_\tau(k)F(ka_tO)dk=\int_K \varphi_\tau(k)P(ka_tO, e_1)dk\,,$$

for $(\tau, V_\tau)\in\hat{K}$ which satisfies $\dim V_\tau^M=1$. On the other hand, by Lemma 2 (1),

$$\int_K F^2(ka_tO)dk=\int_K P^2(ka_tO, e_1)dk\,.$$

These facts imply that $F(ka_tO)=P(ka_tO, e_1)$ for $k\in K$ and $t\geq 0$, that is to say $F(x)=P(x, e_1)$. This completes the proof for the real case of Problem 2.

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