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Generating Function for the Spherical Functions on a Gelfand Pair of Exceptional Type

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Let X be a non-compact Riemannian symmetric space of rank 1. Then it is known that $X=G/K$, where G is a connected simple Lie group with finite center, K is a maximal compact subgroup of G, and if $G=KAN$ is an Iwasawa decomposition, we have $\dim A=1$. From the classification theory, it is known that X is either one of the classical hyperbolic spaces corresponding to the groups $SO_0(1, n)$, $SU(1, n)$ and $Sp(1, n)$ over the fields $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ or the exceptional space corresponding to the exceptional simple group $F_{4(-20)}$, the so-called Cayley hyperbolic plane. Let M be the centralizer of A in K, then the Martin boundary K/M of $X=G/K$ is not a symmetric space, except for the case of real numbers. But, as is well known, (K, M) is a Gelfand pair, i.e. the convolution algebra of functions on K bi-invariant by M is commutative. A theory of the corresponding spherical functions is given in an exposé of Takahashi [5] for the classical case, while the exceptional case is treated in [6].

In the case of real hyperbolic spaces, the space K/M is the usual unit sphere S^{n-1} in $\boldsymbol{R}^n$, and we have the classical theory of spherical harmonics; the zonal spherical functions are given essentially by the Gegenbauer polynomials $C_p^{(n-2)/2}$ and we have the classical generating function expansion:

$$(1-2tx+t^2)^{-(n-2)/2}=\sum_{p=0}^{\infty} C_p^{(n-2)/2}(x)t^p\,, \qquad -1\leq x\leq 1\,, \quad -1<t<1\,,$$

which can be considered also as giving a generating function for the zonal spherical functions of the space $SO(n)/SO(n-1)$. In the papers [7], [8], we have shown that similar constructions are possible also in the other classical cases. The purpose of the present paper is to give a generating function in the exceptional case.

In what follows, we will follow the notations of [6].

The compact group K acts transitively on $\{F_2^u+F_3^v\,;\, u, v\in \boldsymbol{O}, |u|^2+|v|^2=1\}\cong S^{15}$ and the isotropy group of F_2^1 is the subgroup $M\subset L$, that is $K/M\cong S^{15}$, and it's identification is given by $kM\mapsto kF_2^1=F_2^u+F_3^v$. See §4 iv) in [6].

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A zonal spherical function φ of K/M depends only on $Re(u)$ and $|u|$, and there uniquely exists a pair of nonnegative integers (p, q) such that

$$\varphi(kF_2^1)=c_{pq}C_p^3\left(\frac{Re(u)}{|u|}\right)|u|^p {}_2F_1(-q, p+q+7; p+4; |u|^2),$$

where

$$c_{pq}=\frac{(-1)^q(p+4)_q}{(4)_q}[C_p^3(1)]^{-1}.$$

See the formula (12), (13) of §6 in [6]. From now on, we denote φ by φ_{pq}. When we denote $\{f*\varphi_{pq} \mid f\in L^2(K/M)\}$ by H_{pq}, H_{pq} is K-irreducible and moreover $H_k^{(16)}\cong \bigoplus_{p+2q=k}H_{pq}$, $L^2(K/M)=\bigoplus_{p,q=0}^{\infty}H_{pq}$, where $H_k^{(16)}$ is the space of restrictions to S^{15} of harmonic polynomials on $\boldsymbol{R}^{16}$ which are homogeneous of degree k. The main theorem is

THEOREM 1. *If $k\in K$, $w\in \boldsymbol{O}$, $|w|=1$ and $0\le r<1$, then*

$$\int_{SO(7)}[1-2r\,Re(\alpha(u)w)+r^2]^{-7}d\alpha=\sum_{p,q=0}^{\infty}\gamma_{pq}\varphi_{pq}(kF_2^1)C_p^3(Re(w))r^{p+2q}, \tag{1}$$

where $d\alpha$ is the normalized Haar measure on $SO(7)$ and

$$KF_2^1=F_2^u+F_3^v,$$

$$\gamma_{pq}=\frac{2p+6}{\Gamma(p+q+4)}\frac{(7)_{p+q}(4)_q}{q!}.$$

The series on the right hand side converges absolutely and uniformly for k, w and $r\le\rho$ for each $\rho<1$.

PROOF. The function f defined by

$$f(\xi)=\|\xi-e_1\|^{-14}, \quad \xi=(\xi_1, \xi_2)\in\boldsymbol{O}^2, \quad e_1=(1,0)\in\boldsymbol{O}^2,$$

is harmonic on $|\xi|<1$. Therefore we have the following expansion which converges uniformly on every compact subset of $|\xi|<1$,

$$f(\xi)=\sum_{\nu=0}^{\infty}h_\nu(\xi),$$

where the function h_ν is a harmonic polynomial on $\boldsymbol{R}^{16}$ which is homogeneous of degree ν.

We define the $\boldsymbol{R}$-linear map a as follows:

$$a: F_2^{u'}+F_3^{v'}\mapsto(u', v')\in\boldsymbol{O}^2.$$

For $k\in K$ and $0\le r<1$, if we put $\xi=ra(kF_2^1)$, then we obtain that

$$f(ra(kF_2^1))=\sum_{\nu=0}^{\infty}r^\nu h_\nu(a(kF_2^1)).$$

The function $k \mapsto f(ra(kF_2^1))$ is M bi-invariant because $e_1 = a(mF_2^1)$ for all $m \in M$ and $\|a(lkF_2^1)\| = \|a(kF_2^1)\|$ for all $l \in L$ (see Lemma 2 of §3 in [6]). So the functions $k \mapsto h_\nu(a(kF_2^1))$ are also M bi-invariant. Thus there exist constants $a_{pq} \in \boldsymbol{R}$ such that

$$h_\nu(a(kF_2^1)) = \sum_{p+2q=\nu} a_{pq}\varphi_{pq}(kF_2^1)\,.$$

So we see that

$$f(ra(kF_2^1)) = \sum_{\nu=0}^{\infty} r^\nu \sum_{p+2q=\nu} a_{pq}\varphi_{pq}(kF_2^1)\,.$$

From the function equations for φ_{pq}, for $k' \in L$

$$\int_M f(ra(k'mkF_2^1))dm = \sum_{\nu=0}^{\infty} r^\nu \sum_{p+2q=\nu} a_{pq}\varphi_{pq}(k'F_2^1)\varphi_{pq}(kF_2^1)\,, \tag{2}$$

where dm is the normalized Haar measure on M. We now define $k' \in L$ by

$$(\alpha_1, \alpha_2, \alpha_3) = k' \in D_4\,, \quad \text{where} \quad \alpha_1(u) = wu\,,\ \alpha_2(u) = uw \ \text{and} \ \alpha_3(u) = \bar{w}u\bar{w}$$

(see Lemma 2 of §3 in [6]). If we write $m = (\tilde{\alpha}, \alpha, \kappa\tilde{\alpha})$ ($\alpha \in SO(7)$, $\tilde{\alpha} \in SO(8)$), then

$$k'mkF_2^1 = F_2^{\alpha(u)w} + F_3^{\bar{w}(\kappa\tilde{\alpha}(v))\bar{w}}\,.$$

That is to say

$$a(k'mkF_2^1) = (\alpha(u)w,\ \bar{w}(\kappa\tilde{\alpha}(v))\bar{w})\,.$$

So we see that

$$f(ra(k'mkF_2^1)) = [1 - 2r\,Re(\alpha(u)w) + r^2]^{-7}\,.$$

From (2), we can conclude that

$$\int_{SO(7)} [1 - 2r\,Re(\alpha(u)w) + r^2]^{-7} d\alpha$$

$$= \sum_{\nu=0}^{\infty} r^\nu \sum_{p+2q=\nu} a_{pq}[C_p^3(1)]^{-1} C_p^3(Re(w))\varphi_{pq}(kF_2^1)\,. \tag{3}$$

We now determine the constants a_{pq} using the following formula. See [3].

If $\mu > \lambda > 0$, then we have

$$C_\nu^\mu(t) = \sum_{q=0}^{[\nu/2]} \gamma_q^{(\nu)}(\mu, \lambda) C_{\nu-2q}^\lambda(t)\,,$$

where

$$\gamma_q^{(\nu)}(\mu, \lambda) = \frac{(\lambda+\nu-2q)}{(\lambda+\nu-q)} \frac{(\mu)_{\nu-q}}{(\lambda)_{\nu-q}} \frac{(\mu-\lambda)_q}{q!}\,.$$

Putting $k=e$, i.e. $u=1$, in (3), we obtain that

$$(1-2r\,Re(w)+r^2)^{-7}=\sum_{v=0}^{\infty} r^v \sum_{p+2q=v} a_{pq}[C_p^3(1)]^{-1}C_p^3(Re(w)).$$

So we see that

$$\sum_{p+2q=v} a_{pq}[C_p^3(1)]^{-1}C_p^3(Re(w))=C_v^7(Re(w)),$$

and moreover

$$a_{pq}[C_p^3(1)]^{-1}=\frac{2p+6}{\Gamma(p+q+4)}\frac{(7)_{p+q}(4)_q}{q!}.$$

Finally, it follows from the definitions of φ_{pq} that $|\varphi_{pq}(kF_2^1)|\le\varphi_{pq}(F_2^1)=1$ for $k\in K$, which implies the last assertion. □

The formula (1) means that the zonal spherical functions φ_{pq} appear as the coefficients in the expansion of the left hand side of (1) by the powers of r and by the spherical functions of $S^7\cong\{u\in \boldsymbol{O}\ ;\ |u|=1\}$. So we can consider that (1) gives a generating function for the functions φ_{pq}. This interpretation for generating function is similar to the classical cases. See [7], [8].

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