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On the First Coefficients in q of the Kazhdan-Lusztig Polynomials

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§0. Introduction.

The purpose of this article is to find a combinatorial description of the first coefficient in q of the Kazhdan-Lusztig polynomial (Theorem A) by introducing a left subword, which is a special one of subwords (see Def. 1.6). From its description, we show the non negativity of the first coefficient in q of the Kazhdan-Lusztig polynomial for x, w satisfying $l(w)=l(x)+l(x^{-1}w)$ or $l(w)=l(x)+l(wx^{-1})$, where x, w are elements of an arbitrary Coxeter system (W, S) and l is the length function.

In §1 we find a combinatorial description of the first coefficient in q of the Kazhdan-Lusztig polynomial (Theorem A). In particular, for $x, w \in W$ satisfying $l(w)=l(x)+l(x^{-1}w)$ ($l(w)=l(x)+l(wx^{-1})$), the first coefficient in q is equal to $c^{-}(x, w)-g(x^{-1}w)$ (resp. $c^{-}(x, w)-g(wx^{-1})$), where $c^{-}(x, w)$ ($g(w)$) is the number of coatoms (resp. atoms) of the interval $[x, w]$ (resp. $[e, w]$, e is the identity element) in the Bruhat order (see Def. 1.3).

In §2 we give the proof of the non negativity of $c^{-}(x, w)-g(x^{-1}w)$ for $x, w \in W$ satisfying $l(w)=l(x)+l(x^{-1}w)$.

Let us give a brief review of known results. It is conjectured in [KL] that all coefficients of the Kazhdan-Lusztig polynomials are non negative. This is still an open problem, but some of the special cases are verified. For example, this conjecture is correct for finite Weyl groups, affine Weyl groups and dihedral groups. M. Dyer has proved the non negativity of the first coefficients in q of the Kazhdan-Lusztig polynomials for $e, w \in W$ by showing that the first coefficient is equal to $c^{-}(e, w)-g(w)$ in this case and it is non negative ([D]). So, our results include his.

§1. Combinatorial description of the first coefficient.

At first, we shall define the Bruhat order and the Kazhdan-Lusztig polynomials. Throughout this article, (W, S) is an arbitrary Coxeter system, where S denotes a

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privileged set of involutions in W.

DEFINITION 1.1 (Bruhat order). We put $T:=\{wsw^{-1}; s\in S, w\in W\}$. For $y, z\in W$, we denote $y<'z$ if and only if there exists $t\in T$ such that $l(tz)<l(z)$ and $y=tz$, where l is the length function. Then the Bruhat order denoted by $\leq$ is defined as follows. For $x, w\in W$, $x\leq w$ if and only if there exist $x_0, x_1, \cdots, x_r\in W$ satisfying $x=x_0<' x_1<'\cdots<'x_r=w$. We also use the notation $x\lessdot w$ if $x<w$ and $l(x)=l(w)-1$.

The following is well known (cf. [H]). Let $w=s_1s_2\cdots s_m\in W$ be a reduced decomposition (i.e. all $s_i\in S$ and $l(w)=m$). For $x\in W$, $x\leq w$ if and only if there exist $i_1, i_2, \cdots, i_t$ such that $1\leq i_l<i_2<\cdots<i_t\leq m$ and $x=s_{i_1}s_{i_2}\cdots s_{i_t}$. This expression is not reduced in general, i.e. it may be the case that $l(x)<t$. However it is known that one can find $1\leq j_1<j_2<\cdots<j_r\leq m$ such that $x=s_{j_1}s_{j_2}\cdots s_{j_r}$ and $l(x)=r$.

DEFINITION 1.2 (Kazhdan-Lusztig polynomial). For $x, w\in W$, we define the Kazhdan-Lusztig polynomial for x, w denoted by $P_{x,w}=\sum_{i\geq 0}p_i(x, w)q^i\in \mathbf{Z}[q]$ as follows.

$$P_{x,w}=0 \quad \text{if} \quad x\nleq w\,,$$

$$P_{x,w}=1 \quad \text{if} \quad x=w\,.$$

When $x<w$, for fixed $s\in S$ satisfying $l(sw)<l(w)$, we set

$$c:=\begin{cases}0 & \text{if} \quad x<sx\\ 1 & \text{if} \quad sx<x\,.\end{cases}$$

Then, $P_{x,w}$ is defined inductively as follows.

$$P_{x,w}=q^{1-c}P_{sx,sw}+q^cP_{x,sw}-\sum_{sz<z<sw}\mu(z, sw)q^{(l(w)-l(z))/2}P_{x,z}\,,$$

where $\mu(z, sw)$ is the coefficient of $q^{(l(sw)-l(z)-1)/2}$ of $P_{z,sw}$.

REMARK. For the equivalence of this definition with the original definition in [KL], we refer the reader to [H].

We define some notations.

DEFINITION 1.3. For $x, w\in W$, we put

$$[x, w]:=\{y\in W\,;\, x\leq y\leq w\}\,,$$

$$C^-(x, w):=\{y\in[x, w]\,;\, l(y)=l(w)-1\}\,,$$

$$C^+(x, w):=\{y\in[x, w]\,;\, l(y)=l(x)+1\}\,,$$

$$c^-(x, w):=\sharp C^-(x, w)\,,$$

$$c^+(x, w):=\sharp C^+(x, w)\,,$$

$$g(w) := c^+(e, w) .$$

$C^+(x, w)$ is the set of atoms in the interval $[x, w]$ with respect to the Bruhat order, and $C^-(x, w)$ is the set of coatoms in $[x, w]$ i.e. atoms in the dual of $[x, w]$ (cf. [St]).

For a statement ST, we put $\delta(ST) := 1$ if ST is correct and $\delta(ST) := 0$ if ST is incorrect.

We shall begin with showing the next proposition.

PROPOSITION 1.4. *Let $x, w \in W$ with $x \leq w$ and let $s_1 s_2 \cdots s_m$ be a reduced decomposition of w. We let x' denote $s_1 x$ and w' denote $s_1 w$. Then, we have the following.*
(i) *If $l(x') > l(x)$ (i.e. $s_1 x > x$), then*

$$p_1(x, w) = c^-(x, w) - c^+(x, w) + p_1(x, w') - c^-(x, w') + c^+(x, w') .$$

(ii) *If $l(x') < l(x)$ (i.e. $s_1 x < x$), then*

$$p_1(x, w) = c^-(x, w) + p_1(x', w') - c^-(x', w') .$$

Before the proof of this proposition, we will show a lemma.

LEMMA 1.5. *We use the same notations as in Proposition 1.4.*
If $l(x') > l(x)$ (i.e. $s_1 x > x$), then we have
(i) $p_0(x', w') + \#\{z \in W ; x \lessdot z \leq w, z \not\leq w'\} = 1,$
(ii) $c^-(x, w) = 1 + \#\{z \in W ; x \leq z \lessdot w', z < s_1 z\}.$
If $l(x') < l(x)$ (i.e. $s_1 x < x$), then we have the following.
(iii) *For $z \in W$, if $s_1 z < z$, then $x' \leq z$ is equivalent to $x \leq z$.*
(iv) $c^-(x, w) = p_0(x, w') + \#\{z \in W ; x' \leq z \lessdot w', z < s_1 z\}.$

PROOF. (i) From the well known fact that $p_0(x', w') = 1$ if $x' \leq w'$ and $P_{x', w'} = 0$ if $x' \not\leq w'$ (cf. [H]), it is enough to show that

$$\#\{z \in W ; x \lessdot z \leq w, z \not\leq w'\} = \begin{cases} 1 & \text{if } x' \not\leq w' \\ 0 & \text{if } x' \leq w' . \end{cases}$$

Let $z \in \{z \in W ; x \lessdot z \leq w, z \not\leq w'\}$. From $z \leq w = s_1 s_2 \cdots s_m$ and $z \not\leq w' = s_2 s_3 \cdots s_m$, there exist $i_1, i_2, \cdots, i_r$ such that $2 \leq i_1 < i_2 < \cdots < i_r \leq m$, $l(z) = r + 1$ and $z = s_1 s_{i_1} s_{i_2} \cdots s_{i_r}$. Therefore, from $x \lessdot z$, $x = s_{i_1} s_{i_2} \cdots s_{i_r}$ or $x = s_1 s_{i_1} s_{i_2} \cdots \widehat{s_{i_j}} \cdots s_{i_r}$ $(1 \leq j \leq r)$, where $s_{i_{j-1}} \widehat{s_{i_j}} s_{i_{j+1}}$ is $s_{i_{j-1}} s_{i_{j+1}}$. In the second case, we have $s_1 x < x$ and this is a contradiction. Hence, we may consider the first case. Then, we can obtain $z = s_1 x = x'$. So, we have

$$\#\{z \in W ; x \lessdot z \leq w, z \not\leq w'\} = \begin{cases} 1 & \text{if } x' \not\leq w' \\ 0 & \text{if } x' \leq w' . \end{cases}$$

Therefore, we get (i).

(ii) By the definition of $C^-(x, w)$, we have

$$C^-(x, w)=\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 1\le i\le m,\ x\le z\}\ .$$

From $x<s_1x$, we can see $x\le s_2s_3\cdots s_m$. So, we have

$$C^-(x, w)=\{s_2s_3\cdots s_m\}\coprod\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 2\le i\le m,\ x\le z\}\ .$$

We can define a function φ from $\{z\in W\,;\,x\le z\lessdot w',\,z<s_1z\}$ to $\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 2\le i\le m,\ x\le z\}$ by $\varphi(z):=s_1z$. Since φ is obviously a bijection, we obtain

$$c^-(x, w)=1+\#\{z\in W\ ;\ x\le z\lessdot w',\ z<s_1z\}\ .$$

(iii) We suppose that $x'\le z$. From $s_1z<z$, there exists z' such that $z=s_1z'$ and $l(z)=l(z')+1$. So, from $x'\le s_1z'=z$ and $x'<s_1x'$, we have $x'\le z'$. Hence, we see $x=s_1x'\le s_1z'=z$. From $x'<x$, it is trivial that $x'\le z$ if $x\le z$.

(iv) From the definition of $c^-(x, w)$, we have

$$\begin{aligned}
c^-(x, w)&=\#\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 1\le i\le m,\ x\le z\}\\
&=\begin{cases}1 & \text{if}\ \ x\le w'\\ 0 & \text{if}\ \ x\not\le w'\end{cases}\\
&\quad+\#\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 2\le i\le m,\ x\le z\}\\
&=p_0(x, w')\\
&\quad+\#\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 2\le i\le m,\ x\le z\}\ .
\end{aligned}$$

We can define a function ψ from $\{z\in W\,;\,x'\le z\lessdot w',\,z<s_1z\}$ to $\{z=s_1s_2\cdots\widehat{s_i}\cdots s_m\in C^-(e, w)\ ;\ 2\le i\le m,\ x\le z\}$ by $\psi(z):=s_1z$. We can easily see that ψ is a bijection. Hence, we obtain

$$c^-(x, w)=p_0(x, w')+\#\{z\in W\ ;\ x'\le z\lessdot w',\ z<s_1z\}\ . \qquad \square$$

We shall prove Proposition 1.4.

PROOF OF PROPOSITION 1.4. (i) We put $s=s_1$, then $c=0$ from $x<s_1x$. By the definition of Kazhdan-Lusztig polynomials, we have

$$P_{x,w}=qP_{x',w'}+P_{x,w'}-\sum_{s_1z<z<w'}\mu(z, w')q^{(l(w)-l(z))/2}P_{x,z}\ .$$

From $1\le(l(w)-l(z))/2$, we can see

$$\begin{aligned}
p_1(x, w)&=p_0(x', w')+p_1(x, w')-\sum_{s_1z<z<w',\,l(w)-l(z)=2}\mu(z, w')p_0(x, z)\\
&=p_0(x', w')+p_1(x, w')-\sum_{s_1z<z\lessdot w'}p_0(z, w')p_0(x, z)
\end{aligned}$$

$$=p_0(x', w')+p_1(x, w')-\sum_{s_1z<z\lessdot w', x\leq z} 1$$

$$=p_0(x', w')+p_1(x, w')-\#\{z\in W\,;\, x\leq z\lessdot w', s_1z<z\}$$

$$=p_0(x', w')+p_1(x, w')-c^-(x, w')$$

$$+\#\{z\in W\,;\, x\leq z\lessdot w', z<s_1z\}\,.$$

From $c^+(x, w)=c^+(x, w')+\#\{z\in W\,;\, x\lessdot z\leq w, z\nleq w'\}$, we have

$$p_1(x, w)=p_0(x', w')+\#\{z\in W\,;\, x\lessdot z\leq w, z\nleq w'\}$$

$$+\#\{z\in W\,;\, x\leq z\lessdot w', z<s_1z\}-c^+(x, w)$$

$$+p_1(x, w')-c^-(x, w')+c^+(x, w')\,.$$

Hence, by Lemma 1.5, we obtain

$$p_1(x, w)=c^-(x, w)-c^+(x, w)+p_1(x, w')-c^-(x, w')+c^+(x, w')\,.$$

(ii) Similarly, we put $s=s_1$, then $c=1$ from $s_1x<x$. We can easily see that

$$p_1(x, w)=p_1(x', w')+p_0(x, w')-\#\{z\in W\,;\, x\leq z\lessdot w', s_1z<z\}\,.$$

From Lemma 1.5 (iii), we have

$$p_1(x, w)=p_1(x', w')+p_0(x, w')-\#\{z\in W\,;\, x'\leq z\lessdot w', s_1z<z\}$$

$$=p_1(x', w')+p_0(x, w')-c^-(x', w')$$

$$+\#\{z\in W\,;\, x'\leq z\lessdot w', z<s_1z\}\,.$$

Hence, by Lemma 1.5 (iv), we obtain

$$p_1(x, w)=c^-(x, w)+p_1(x', w')-c^-(x', w')\,. \qquad \square$$

For $x, w\in W$ with $x\leq w$, we introduce the left subword and right subword of a reduced decomposition $s_1s_2\cdots s_m$ of w in order to find a combinatorial description of the first coefficient in q of $P_{x,w}$.

DEFINITION 1.6. Let $w=s_1s_2\cdots s_m\in W$ be a reduced decomposition. We put $[m]=\{1, 2, \cdots, m\}$ and define a map f from $[m]$ to S by $f(i)=s_i$. Let $J=(j_1, j_2, \cdots, j_r)$ and $J'=(j'_1, j'_2, \cdots, j'_r)$ be subsequences of $(1, 2, \cdots, m)$. We call $f|_J$ (restriction map) subword of $s_1s_2\cdots s_m$ and we set $|f|_J|=f(j_1)f(j_2)\cdots f(j_r)=s_{j_1}s_{j_2}\cdots s_{j_r}$. We define $J<J'$ if and only if there exists $k\in[r]$ such that $j_1=j'_1, j_2=j'_2, \cdots, j_{k-1}=j'_{k-1}$ and $j_k<j'_k$ (i.e. lexicographic order). For $x\in W$ with $x\leq w$ and $l(x)=r$, if $I=(i_1, i_2, \cdots, i_r)=\min\{J=(j_1, j_2, \cdots, j_r)$ subsequence of $(1, 2, \cdots, m)\,;\, |f|_J|=x\}$, then we call $f|_I$ left subword of $s_1s_2\cdots s_m$.

From now, we identify $|f|_J|$ with $f|_J$ when there is no confusion.

We define a right subword $s_{a_1}s_{a_2}\cdots s_{a_r}$ of $s_1s_2\cdots s_m$ if and only if $s_{a_r}s_{a_{r-1}}\cdots s_{a_1}$ is a left subword of $s_ms_{m-1}\cdots s_1$.

EXAMPLE 1.7. Let $\mathfrak{S}_4$ be the symmetric group of degree 4. For $i\in[3]$, we let $\sigma_i=(i, i+1)$, where (i,j) is the transposition of i and j. Then $f|_{\{1,2\}}$ is a left subword of $\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$ but $f|_{\{1,4\}}$ is not a left subword.

NOTE. From the definition of left subword, we can easily check the following. Let $s_{a_1}s_{a_2}\cdots s_{a_r}$ be a left subword of $s_1s_2\cdots s_m$. For each $p\in[r]$ and $a_{p-1}+1\leq j\leq a_p-1$, we have

$$l(s_{a_p}s_{a_{p+1}}\cdots s_{a_r})<l(s_js_{a_p}s_{a_{p+1}}\cdots s_{a_r}),$$

where $a_0=0$.

Then, $p_1(x, w)$ is described as follows.

THEOREM A. *For $x, w\in W$ with $x\leq w$, let*

$$x=s_1s_2\cdots\widehat{s_{i_1}}s_{i_1+1}\cdots\widehat{s_{i_2}}s_{i_2+1}\cdots\widehat{s_{i_k}}s_{i_k+1}\cdots s_r$$

be the left subword of a reduced decomposition $s_1s_2\cdots s_m$ of w. Then, we have

$$\begin{aligned}p_1(x, w)=&\,c^-(x, w)-g(s_{r+1}s_{r+2}\cdots s_m)\\&-\sum_{j=1}^{k}\delta(s_{i_j}\cdots\widehat{s_{i_{j+1}}}\cdots\widehat{s_{i_{j+2}}}\cdots\widehat{s_{i_k}}\cdots s_r\not\leq s_{i_j+1}s_{i_j+2}\cdots s_m)\,.\end{aligned}$$

REMARK. We can also obtain the similar equation with respect to the right order.

PROOF. We will show this theorem by induction with respect to k. When $k=0$, from Proposition 1.4, we have

$$\begin{aligned}p_1(x, w)&=c^-(x, w)+p_1(s_2s_3\cdots s_r, s_2s_3\cdots s_m)\\&\quad-c^-(s_2s_3\cdots s_r, s_2s_3\cdots s_m)\\&=c^-(x, w)+p_1(e, s_{r+1}s_{r+2}\cdots s_m)\\&\quad-c^-(e, s_{r+1}s_{r+2}\cdots s_m)\\&=c^-(x, w)-c^+(e, s_{r+1}s_{r+2}\cdots s_m)\\&\quad+p_1(e, s_{r+2}s_{r+3}\cdots s_m)\\&\quad-c^-(e, s_{r+2}s_{r+3}\cdots s_m)+c^+(e, s_{r+2}s_{r+3}\cdots s_m)\\&=c^-(x, w)-c^+(e, s_{r+1}s_{r+2}\cdots s_m)\\&\quad+p_1(e, e)-c^-(e, e)+c^+(e, e)\\&=c^-(x, w)-g(s_{r+1}s_{r+2}\cdots s_m)\,.\end{aligned}$$

We suppose that theorem is correct if $k\leq u-1$ $(1\leq u)$. When $k=u$, from Proposition

1.4, we have

$$
\begin{aligned}
p_1(x, w) = {} & c^-(x, w) + p_1(s_2 s_3 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_2 s_3 \cdots s_m) \\
& - c^-(s_2 s_3 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_2 s_3 \cdots s_m) \\
= {} & c^-(x, w) + p_1(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1} s_{i_1+1} \cdots s_m) \\
& - c^-(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1} s_{i_1+1} \cdots s_m) \\
= {} & c^-(x, w) - c^+(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1} s_{i_1+1} \cdots s_m) \\
& + p_1(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1+1} s_{i_1+2} \cdots s_m) \\
& - c^-(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1+1} s_{i_1+2} \cdots s_m) \\
& + c^+(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1+1} s_{i_1+2} \cdots s_m) \\
= {} & c^-(x, w) - c^+(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1} s_{i_1+1} \cdots s_m) \\
& + c^+(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1+1} s_{i_1+2} \cdots s_m) \\
& - g(s_{r+1} s_{r+2} \cdots s_m) \\
& - \sum_{j=2}^{u} \delta(s_{i_j} \cdots \widehat{s_{i_{j+1}}} \cdots \widehat{s_{i_{j+2}}} \cdots \widehat{s_{i_k}} \cdots s_r \not\leq s_{i_j+1} s_{i_j+2} \cdots s_m) .
\end{aligned}
$$

On the other hand, we can easily see that

$$
\begin{aligned}
& c^+(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1} s_{i_1+1} \cdots s_m) \\
& - c^+(s_{i_1+1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_u}} \cdots s_r,\ s_{i_1+1} s_{i_1+2} \cdots s_m) \\
& = \delta(s_{i_1} \cdots \widehat{s_{i_2}} \cdots \widehat{s_{i_3}} \cdots \widehat{s_{i_u}} \cdots s_r \not\leq s_{i_1+1} s_{i_1+2} \cdots s_m) .
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
p_1(x, w) = {} & c^-(x, w) - g(s_{r+1} s_{r+2} \cdots s_m) \\
& - \sum_{j=1}^{u} \delta(s_{i_j} \cdots \widehat{s_{i_j+1}} \cdots \widehat{s_{i_j+2}} \cdots \widehat{s_{i_k}} \cdots s_r \not\leq s_{i_j+1} s_{i_j+2} \cdots s_m) .
\end{aligned}
$$

So, we obtain Theorem A. □

From Theorem A, we can easily get the following corollary.

COROLLARY 1.8. *For $x, w \in W$ with $l(w) = l(x) + l(x^{-1}w)$, we have*

$$p_1(x, w) = c^-(x, w) - g(x^{-1}w) .$$

For $x, w \in W$, it is known that $P_{x,w} = P_{x^{-1},w^{-1}}$ (cf. [Sh]). Hence, from Corollary 1.8, we also see the following.

COROLLARY 1.9. *For $x, w \in W$ with $l(w) = l(x) + l(wx^{-1})$, we have*

$$p_1(x, w) = c^-(x, w) - g(wx^{-1}) .$$

PROOF. From the assumption on x and w, we see $l(w^{-1})=l(x^{-1})+l(xw^{-1})$. So, we have $p_1(x, w)=p_1(x^{-1}, w^{-1})=c^-(x^{-1}, w^{-1})-g(xw^{-1})$. Also, it is easy to check that $[x, w]$ is isomorphic to $[x^{-1}, w^{-1}]$ as the partially ordered set in the Bruhat order (cf. [St]). Hence, we have $p_1(x, w)=c^-(x, w)-g(xw^{-1})$. □

If W has the longest element w_0, then we know $P_{x,w_0}=1$ and $l(w_0)=l(x)+l(x^{-1}w_0)$. Hence, we can immediately obtain an interesting corollary as follows.

COROLLARY 1.10. *Let (W, S) be a Coxeter system with the longest element w_0. For $x\in W$, we have*

$$c^-(x, w_0)=g(x^{-1}w_0).$$

§2. Nonnegativity of the first coefficient.

In this section, we shall show the non negativity of $p_1(x, w)$ for x, w satisfying $l(w)=l(x)+l(x^{-1}w)$ or $l(w)=l(x)+l(wx^{-1})$.

PROPOSITION 2.1. *Let $x, w\in W$.*
(i) *If $l(w)=l(x)+l(x^{-1}w)$, then we have $c^-(x, w)-g(x^{-1}w)\geq 0$.*
(ii) *If $l(w)=l(x)+l(wx^{-1})$, then we have $c^-(x, w)-g(wx^{-1})\geq 0$.*

PROOF. We will show the first statement by an induction with respect to $l(x^{-1}w)$ and $l(x)$.

If $l(x^{-1}w)=0$ or 1, it is easy to check that the statement is correct. We prove the statement in case $l(x^{-1}w)=k+1$ under the assumption that it is correct when $l(x^{-1}w)\leq k$ $(1\leq k)$. Here, we use the induction with respect to $l(x)$.

If $l(x)=0$, we put $w=s_1s_2s_3\cdots s_{k+1}$ (a reduced decomposition), and we consider two cases.

Case (1): $s_1\leq s_2s_3\cdots s_{k+1}$. We have $c^-(e, w)=c^-(s_1, w)\geq g(s_2s_3\cdots s_{k+1})=g(w)$.

Case (2): $s_1\nleq s_2s_3\cdots s_{k+1}$. We have $C^-(e, w)=\{s_2s_3\cdots s_{k+1}\}\coprod C^-(s_1, w)$. Hence, we see $c^-(e, w)=1+c^-(s_1, w)\geq 1+g(s_2s_3\cdots s_{k+1})=g(w)$.

We suppose that $c^-(x, w)-g(x^{-1}w)\geq 0$ if $l(x)\leq r$ $(0\leq r)$, and show the statement in case $l(x)=r+1$ from now. We put $x=s_1s_2\cdots s_{r+1}$ and $x^{-1}w=s'_1s'_2s'_3\cdots s'_{k+1}$ (a reduced decomposition).

Case (a): $s'_1\leq s'_2s'_3\cdots s'_{k+1}$. We have $c^-(x, w)-g(x^{-1}w)=c^-(x, w)-g(s'_2s'_3\cdots s'_{k+1})\geq c^-(xs'_1, w)-g(s'_2s'_3\cdots s'_{k+1})\geq 0$.

Case (b): $s'_1\nleq s'_2s'_3\cdots s'_{k+1}$. We put $J:=S\setminus\{s'_1\}$, $W_J:=$subgroup of W generated by J and $W^J:=\{y\in W\,;\, l(yz)=l(y)+l(z)$ for any $z\in W_J\}$. Then, there exist $w^J\in W^J$ and $w_J\in W_J$ such that $xs'_1=w^Jw_J$. In general, for $J\subset S$ and $w\in W$, there exist $w^J\in W^J$ and $w_J\in W_J$ such that $w=w^Jw_J$ (cf. [H]). Note that $w^J\neq e$ and $w_Js'_2s'_3\cdots s'_{k+1}\in W_J$. Also, we put $w^J=\tilde{s}_1y$ satisfying $l(w^J)=1+l(y)$. Now, $w=\tilde{s}_1yw_Js'_2s'_3\cdots s'_{k+1}$.

Case (b)-(α): $\tilde{s}_1w\in C^-(x, w)$. By the induction hypothesis, $c^-(w_J, w_Js'_2s'_3\cdots s'_{k+1})\geq$

$g(s'_2s'_3\cdots s'_{k+1})$. From the definition of W^J, $w^JC^-(w_J, w_Js'_2s'_3\cdots s'_{k+1})\subset C^-(x, w)$. Of course, we see $\tilde{s}_1w\in C^-(x, w)\setminus w^JC^-(w_J, w_Js'_2s'_3\cdots s'_{k+1})$. Hence, we have $c^-(x, w)\geq 1+g(s'_2s'_3\cdots s'_{k+1})=g(x^{-1}w)$.

Case (b)-(β): $\tilde{s}_1w\notin C^-(x, w)$. Note that $x\not\leq\tilde{s}_1w<w$. From $xs'_1=s_1s_2\cdots s_{r+1}s'_1=w^Jw_J=\tilde{s}_1yw_J$, we get $xs'_1=\tilde{s}_1s_1s_2\cdots\widehat{s_i}\cdots s_{r+1}s'_1$ for some $i\in[r+1]$ or $xs'_1=\tilde{s}_1s_1s_2\cdots s_{r+1}=\tilde{s}_1x$. If $xs'_1=\tilde{s}_1x$, then we see $x\leq\tilde{s}_1w$ and this is a contradiction. Hence, it is sufficient to consider the first case. Note that $x=s_1s_2\cdots s_{r+1}=\tilde{s}_1s_1s_2\cdots\widehat{s_i}\cdots s_{r+1}$ from $s_1s_2\cdots s_{r+1}s'_1=\tilde{s}_1s_1s_2\cdots\widehat{s_i}\cdots s_{r+1}s'_1$. We put $s_1s_2\cdots\widehat{s_i}\cdots s_{r+1}=\tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+1}$ and $s'_t=\tilde{s}_{r+1+t}$ for $t\in[k+1]$. Now, $w=\tilde{s}_1\tilde{s}_2\cdots\tilde{s}_{r+k+2}$ and $x=\tilde{s}_1\tilde{s}_2\cdots\tilde{s}_{r+1}$. By the induction hypothesis, we have $c^-(\tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+1}, \tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+k+2})\geq g(\tilde{s}_{r+2}\tilde{s}_{r+3}\cdots\tilde{s}_{r+k+2})=g(x^{-1}w)$. If there exists $y=\tilde{s}_2\tilde{s}_3\cdots\widehat{\tilde{s}_j}\cdots\tilde{s}_{r+k+2}\in C^-(\tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+1}, \tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+k+2})$ such that $l(\tilde{s}_1y)<l(y)$, then, there exists $q\in\{j+1, j+2, \cdots, r+k+2\}$ satisfying $y=\tilde{s}_1\tilde{s}_2\cdots\widehat{\tilde{s}_j}\cdots\widehat{\tilde{s}_q}\cdots\tilde{s}_{r+k+2}$. Since $\tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+1}\leq y$ and $\tilde{s}_1\tilde{s}_2\cdots\tilde{s}_{r+1}$ is a reduced decomposition of x, we have $\tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+1}\leq\tilde{s}_2\cdots\widehat{\tilde{s}_j}\cdots\widehat{\tilde{s}_q}\cdots\tilde{s}_{r+k+2}$. Hence, we have $x=\tilde{s}_1\tilde{s}_2\cdots\tilde{s}_{r+1}\leq\tilde{s}_1\tilde{s}_2\cdots\widehat{\tilde{s}_j}\cdots\widehat{\tilde{s}_q}\cdots\tilde{s}_{r+k+2}=y=\tilde{s}_2\tilde{s}_3\cdots\widehat{\tilde{s}_j}\cdots\tilde{s}_{r+k+2}<\tilde{s}_1w$. But this contradicts our assumption that $x\not\leq\tilde{s}_1w$. So, $\tilde{s}_1C^-(\tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+1}, \tilde{s}_2\tilde{s}_3\cdots\tilde{s}_{r+k+2})\subset C^-(x, w)$. It follows that $c^-(x, w)\geq g(x^{-1}w)$.

From (i) and $c^-(x, w)=c^-(x^{-1}, w^{-1})$, we can easily verify the second statement. □

Therefore, from Corollary 1.8, Corollary 1.9 and Proposition 2.1, we can obtain the following.

THEOREM B. *Let (W, S) be an arbitrary Coxeter system and x, w be elements of W. If $l(w)=l(x)+l(x^{-1}w)$ or $l(w)=l(x)+l(wx^{-1})$, then the first coefficient in q of the Kazhdan-Lusztig polynomial for x, w is non negative.*

REMARK. Of course, from Proposition 2.1, $p_1(x, w)\geq 0$ if all terms of $\delta(\cdot)$ in Theorem A is equal to 0.

We can find an interesting corollary as follows.

COROLLARY 2.2. *For $w\in W$, if there exist $x\in W$ and $s\in S$ satisfying one of the following conditions, then $p_1(e, w)\geq 1$.*

(i) $l(w)=l(x)+l(x^{-1}w)$, $l(sw)<l(w)$, $g(w)=g(x^{-1}w)$ *and* $x\not\leq sw$.

(ii) $l(w)=l(x)+l(wx^{-1})$, $l(ws)<l(w)$, $g(w)=g(wx^{-1})$ *and* $x\not\leq ws$.

PROOF. In the case of (i), we can easily see $sw\in C^-(e, w)\setminus C^-(x, w)$. Hence, we get $c^-(e, w)>c^-(x, w)\geq g(x^{-1}w)=g(w)$. It follows that $p_1(e, w)=c^-(e, w)-g(w)\geq 1$. Similarly, we can show the statement in case (ii). □

REMARK. Unfortunately, the converse of Corollary 2.2 is false. For example, we may think $w=\sigma_2\sigma_3\sigma_1\sigma_2\in\mathfrak{S}_4$.

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