TOKYO J. MATH.
VOL. 17, NO. 1, 1994

A Construction of Irreducible Representations of the Algebra of Invariant Differential Operators on a Homogeneous Vector Bundle and Its Applications

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(Communicated by Y. Shimizu)

1. Introduction.

In this paper we construct irreducible representations of the algebra of invariant differential operators on a homogeneous vector bundle, and give a condition for the nontriviality.

Let G be a connected real semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. Let E_τ denote the homogeneous vector bundle over G/K associated to a representation (τ, V_τ) of K (See [10] for the definition), and let $D(E_\tau)$ denote the algebra of G-invariant differential operators on E_τ.

Let (τ^*, V_{τ^*}) be the contragredient representation of τ, and let $\mathfrak{g}$ be the Lie algebra of G. Then for an irreducible $(\mathfrak{g}, K)$-module W (cf. §3), $\Gamma_\tau(W)=\mathrm{Hom}_K(V_{\tau^*}, W)$ gives rise to an irreducible representation of $D(E_\tau)$ unless trivial (Proposition 4.2). Our choice of W is rather special. But this specialization allows us to apply the classification theory of Langlands which tells us whether $\Gamma_\tau(W)$ be trivial or not.

The construction and the non-triviality condition (cf. §4) yield two applications: first, a simple proof of (a part of) theorem of Deitmar ([2, Theorem 6]) which claims that $D(E_\tau)$ is commutative if and only if $\tau|_M$ is multiplicity free (Proposition 6.2); second, a sufficient condition for some kind of restriction of the Poisson transform to be injective modulo the kernel of some surjection (Theorem 7.2). Because we can describe the injectivity condition in terms of Harish-Chandra's C-function (cf. §7), Theorem 7.2 is a weak analogue of the result for a line bundle (cf. [8]) which shows that the Poisson transform is injective if and only if C-function is non-zero.

The author should like to express his gratitude to Professor Y. Shimizu and Professor K. Minemura for their useful suggestions and encouragement. The author also thanks

Received October 26, 1992
Revised May 18, 1993

Dr. N. Shimeno and Professor T. Kawazoe for their helpful advice.

2. The algebra of invariant differential operators.

We recall the structure-theory of the algebra of invariant differential operators on a homogeneous vector bundle over G/K which is developed in [7].

We keep to the notation in the introduction. Let $\mathfrak{k}$ be the Lie algebra of K. Then we fix a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Here $\mathfrak{p}$ is an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{g}$, that is, G/K is reductive.

Let $\mathfrak{g}_c$, $\mathfrak{k}_c$, and $\mathfrak{p}_c$ be the complexification of $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{p}$ respectively. Let $U(\mathfrak{g}_c)$, $U(\mathfrak{k}_c)$ be the universal enveloping algebra of $\mathfrak{g}_c$, $\mathfrak{k}_c$ respectively. The anti-automorphism $(\cdot)^{\mathsf{T}}$ of $U(\mathfrak{g}_c)$ is determined by

$$1^{\mathsf{T}}=1\,,\quad X^{\mathsf{T}}=-X\quad(X\in\mathfrak{g}_c)\,,\quad (XY)^{\mathsf{T}}=Y^{\mathsf{T}}X^{\mathsf{T}}\quad(X,\,Y\in\mathfrak{g}_c)\,.$$

Let τ be an irreducible representation of K on a finite dimensional complex vector space V_τ, and let $d\tau$ be its differential representation of $U(\mathfrak{k}_c)$. We sometimes write for this simply τ. Let $\mathscr{I}$ be the kernel of $d\tau$ in $U(\mathfrak{k}_c)$, and let $\mathscr{I}^{\mathsf{T}}$ denote the image of $\mathscr{I}$ under $(\cdot)^{\mathsf{T}}$. Let $U(\mathfrak{g}_c)^K$ denote the set of K-invariants in $U(\mathfrak{g}_c)$. By Theorem 1.3 of [7] there is an algebla isomorphism μ:

$$D(E_\tau)\simeq U(\mathfrak{g}_c)^K/U(\mathfrak{g}_c)^K\cap U(\mathfrak{g}_c)\mathscr{I}^{\mathsf{T}}\,. \tag{2.1}$$

We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a linear ordering in $\mathfrak{a}^*$. Let M be the centralizer of $\mathfrak{a}$ in K. We denote by $\mathfrak{g}(\alpha;\mathfrak{a})$ the root space of $\mathfrak{a}$ in $\mathfrak{g}$ corresponding to α. Let $\Phi(\mathfrak{g};\mathfrak{a})$ be the set of (restricted) roots and $\Phi^+(\mathfrak{g};\mathfrak{a})$ the set of positive roots. We put $\mathfrak{n}=\sum_{\alpha\in\Phi^+(\mathfrak{g};\mathfrak{a})}\mathfrak{g}(\alpha;\mathfrak{a})$, $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+(\mathfrak{g};\mathfrak{a})}(\dim\mathfrak{g}(\alpha;\mathfrak{a}))\alpha$.

The direct sum decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is called the Iwasawa decomposition. We define a map $\omega: U(\mathfrak{g}_c)^K\to U(\mathfrak{a}_c)U(\mathfrak{k}_c)$ by $D-\omega(D)\in\mathfrak{n}U(\mathfrak{g}_c)$. Identifying $U(\mathfrak{a}_c)U(\mathfrak{k}_c)$ with a tensor algebra $U(\mathfrak{a}_c)\otimes U(\mathfrak{k}_c)$, we get an algebra anti-isomorphism $\omega: U(\mathfrak{g}_c)^K\to U(\mathfrak{a}_c)\otimes U(\mathfrak{k}_c)$. We define $\sharp\in\mathrm{Aut}\,U(\mathfrak{a}_c)$ by $\sharp(H)=H+\rho(H)$ $(H\in\mathfrak{a})$. If we put $\omega_\tau=(\sharp\otimes(\tau\circ(\cdot)^{\mathsf{T}}))\circ\omega$, then $\omega_\tau: U(\mathfrak{g}_c)^K\to U(\mathfrak{a}_c)\otimes\mathrm{End}_M(V_\tau)$ is an algebra homomorphism. Since $\mathrm{Ker}(\omega_\tau)=U(\mathfrak{g}_c)^K\cap U(\mathfrak{g}_c)\mathscr{I}^{\mathsf{T}}$, (2.1) implies that there exists an injective algebra homomorphism $\chi_\tau: D(E_\tau)\to U(\mathfrak{a}_c)\otimes\mathrm{End}_M(V_\tau)$ such that

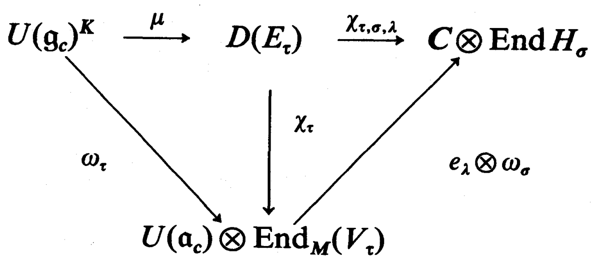
$$\chi_\tau\circ\mu=\omega_\tau\,.$$

We define the evaluation map $e_\lambda: U(\mathfrak{a}_c)\to \boldsymbol{C}$ for $\lambda\in\mathfrak{a}_c^*$ by $e_\lambda(H)=\lambda(H)$ $(H\in\mathfrak{a}_c)$. Let $\hat{M}$ be the set of equivalent classes of irreducible representations of M. If we put $H_\sigma=\mathrm{Hom}_M(V_\sigma,V_\tau)$ for $\sigma\in R_\tau=\{(\sigma,V_\sigma)\in\hat{M}\mid[\tau:\sigma]>0\}$, then we have $V_\tau\simeq\sum_{\sigma\in R_\tau}V_\sigma\otimes H_\sigma$ as an M-module. Thus one has

$$\mathrm{End}_M(V_\tau)\simeq\sum_{\sigma\in R_\tau}\mathrm{End}\,H_\sigma\,.$$

Let ω_σ be the projection from $\mathrm{End}_M(V_\tau)$ onto $\mathrm{End}\,H_\sigma$ corresponding to the above decomposition.

We define the representation $\chi_{\tau,\sigma,\lambda}$ of $D(E_\tau)$ for $\tau\in\hat{K}$, $\sigma\in R_\tau$, $\lambda\in\mathfrak{a}_c^*$ by the following commutative diagram:



Since $\dim H_\sigma=\dim\mathrm{Hom}_M(V_\sigma, V_\tau)<\infty$, the representation $(\chi_{\tau,\sigma,\lambda}, H_\sigma)$ is finite dimensional.

3. The Langlands classification.

We summarize the classification theory of irreducible $(\mathfrak{g}, K)$-modules following [11]. We cite the contents in [11] as follows:

1. Some bars are inserted since the image of the intertwining operator $j_{P_F,\sigma,\mu}$ is in $I_{\bar{P}_F,\sigma,\mu}$. (See the definition right after Lemma 3.9.)

2. The representation space of $\sigma\in\hat{M}$ is denoted by V_σ instead of H_σ.

3. We denote by $(V_{\sigma,\mu+\rho_F})_{K_F}$ the $(\mathfrak{m}_F, K_F)$-module $(V_{\sigma,\mu})_{K_F}$ in [11] (cf. Lemma 3.7). Our notation has no ambiguity about shifting by ρ.

4. The "right" arguments in [11] (e.g., the functional equations and the right regular actions on the functional spaces) are translated to the "left" one by considering the G-isomorphism, $f\mapsto f\circ(\cdot)^{-1}$, from the space of functions on G with right regular action to the one with left regular action.

We omit the proofs, which can be seen in [11, Chapter 5] or [1, Chapter 4].

DEFINITION 3.1. We call W a $(\mathfrak{g}, K)$-module if W is a $\mathfrak{g}$-module (hence a $U(\mathfrak{g}_c)$-module) and a K-module, and satisfies the following three conditions:

1. $k.X.w=\mathrm{Ad}(k)X.k.w$ ($w\in W$, $k\in K$, $X\in\mathfrak{g}$).

2. If $w\in W$ then Kw spans a finite dimensional subspace W_w of W, and K acts on W_w continuously.

3. If $Y\in\mathfrak{k}$, $w\in W$ then $d/dt|_{t=0}\exp(tY)w=Y.w$.

EXAMPLE. Let (π, H) be a Hilbert representation of G. We denote by H_K the space of C^∞, K-finite vectors in H. Then H_K becomes a $(\mathfrak{g}, K)$-module ([11, Lemma 3.3.5]). We call H_K a $(\mathfrak{g}, K)$-module associated to π.

DEFINITION 3.2. A $(\mathfrak{g}, K)$-module W is called irreducible if $\{0\}$ and W are the only $\mathfrak{g}$ and K-invariant subspaces of W.

If W is a K-module and if $\gamma \in \hat{K}$, we denote by $W(\gamma)$ the γ-isotypic component of W.

LEMMA 3.3 ([11, Lemma 3.3.3]). *Let W be a $(\mathfrak{g}, K)$-module. As a K-module $W = \bigoplus_{\gamma \in \hat{K}} W(\gamma)$ (algebraic direct sum).*

DEFINITION 3.4. A $(\mathfrak{g}, K)$-module W is called admmissible if $\dim W(\gamma) < \infty$ for any $\gamma \in \hat{K}$.

LEMMA 3.5 ([11, Corollary 3.4.8]). *If W is an irreducible $(\mathfrak{g}, K)$-module, then W is admissible.*

Let Δ be the system of simple roots in $\Phi(\mathfrak{g}; \mathfrak{a})$. For any subset F of Δ, we denote by (P_F, A_F) the corresponding p-pair, i.e. P_F is a standard parabolic subgroup corresponding to F and A_F is the split component. Let M_F be the reductive compònent of P_F, and N_F the nilradical (See [4, §2.3]). Then we have the Langlands decomposition $P_F = M_F A_F N_F$ with $M_{1F} = M_F A_F$. We put $K_F = K \cap M_{1F}$. Let $\Phi(P_F, A_F)$ be the set of roots of $\mathfrak{a}_F$ on $\mathfrak{n}_F$.

DEFINITION 3.6. A triple (P_F, σ, μ) is called Langlands data if (P_F, A_F) is a p-pair, and (σ, V_σ) is an irreducible unitary representation of M_F such that $(V_\sigma)_{K_F}$, the space of C^∞, K_F-finite vectors in V_σ (cf. Example after Definition 3.1), is a tempered $(\mathfrak{m}_F, K_F)$-module and $\mu \in (\mathfrak{a}_F^*)_c$ satisfies $\mathrm{Re}(\mu, \alpha) > 0$ for all $\alpha \in \Phi(P_F, A_F)$.

We define ${}^\infty H^{P_F,\sigma,\mu}$ to be the set of smooth functions from G to $(V_\sigma)^\infty$, the space of C^∞-vectors in V_σ, such that

$$f(gman) = e^{-(\mu+\rho)H(a)}\sigma(m^{-1})f(g)$$

for $n \in N_F$, $a \in A_F$, $m \in M_F$, $g \in G$, where $H(a) = \log a \in \mathfrak{a}_F$ for $a \in A_F$. Let dk be a normalized Haar measure on K. For $f, g \in {}^\infty H^{P_F,\sigma,\mu}$ we introduce an inner product by $\langle f, g \rangle = \int_K \langle f(k), g(k) \rangle_\sigma dk$. Let $H^{P_F,\sigma,\mu}$ be the Hilbert completion of ${}^\infty H^{P_F,\sigma,\mu}$ with respect to $\langle\ ,\ \rangle$.

By left regular action $\pi_{P_F,\sigma,\mu}(g)f(x) = f(g^{-1}x)$ for $g, x \in G$, $f \in H^{P_F,\sigma,\mu}$, $(\pi_{P_F,\sigma,\mu}, H^{P_F,\sigma,\mu})$ is a Hilbert representation of G. We put $I_{P_F,\sigma,\mu} = (H^{P_F,\sigma,\mu})_K$.

For $(\mathfrak{g}, K)$-modules V and W, we denote by $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ the set of $\mathfrak{g}$ and K-linear maps from V to W.

Let $(V_{\sigma,\mu})_{K_F}$ be the $(\mathfrak{m}_{1F}, K_F)$-module which equals $(V_\sigma)_{K_F}$ as a $(\mathfrak{m}_F, K_F)$-module, on which $\mathfrak{a}_F$ acts by μ. Let $\bar{P}_F$ denote the opposite parabolic subgroup to P_F, that is, $\bar{P}_F = \theta(P_F)$ where θ is the Cartan involution of G fixing K.

LEMMA 3.7 (Frobenius reciprocity). *If V is a $(\mathfrak{g}, K)$-module, then there is a $\mathbf{C}$-linear bijection*:

$$\mathrm{Hom}_{\mathfrak{g},K}(V, I_{\bar{P}_F,\sigma,\mu}) \simeq \mathrm{Hom}_{\mathfrak{m}_{1F},K_F}(V/\bar{\mathfrak{n}}_F V, (V_{\sigma,\mu-\rho_F})_{K_F})\,.$$

PROOF. $T \mapsto \hat{T}$, $\hat{T}(v + \bar{\mathfrak{n}}_F V) = T(v)(e)$ $(v \in V)$ gives the bijection, where e is the identity element in G. Note that $\mathfrak{a}_F$ acts on $(V_{\sigma,\mu})_{K_F}$ by $\mu - \rho_F$ (cf. [11, Lemma 5.2.3]).

We define $(H^{P_F,\sigma,\mu})_\infty$ to be ${}^\infty H^{P_F,\sigma,\mu}$ as a set, and introduce a topology in it by (countable) seminorms:

$$\delta_x(f) = \sup_{k \in K} \delta([\pi_{P_F,\sigma,\mu}(x) f](k))$$

$(x \in U(\mathfrak{g}_c)$, $f \in (H^{P_F,\sigma,\mu})_\infty$, $\delta \in \{p_D\}_{D \in U(\mathfrak{m}_{F,c})})$, where $\{p_D\}_{D \in U(\mathfrak{m}_{F,c})}$ is the usual system of seminorms in $(V_\sigma)^\infty$, i.e.

$$p_D(v) = \|\sigma(D)v\|_{V_\sigma} \qquad (v \in (V_\sigma)^\infty,\ D \in U(\mathfrak{m}_{F,c})) .$$

It is known that $(H^{P_F,\sigma,\mu})_\infty$ becomes a Fréchet space. Let $d\bar{n}$ be an invariant measure on $\bar{N}_F = \theta(N_F)$.

LEMMA 3.8 ([11, Lemma 5.3.1]). *Let (P_F, σ, μ) be a Langlands data. Then*

(1) $\int_{\bar{N}_F} |\langle f(\bar{n}), w\rangle| d\bar{n} < \infty$ *(for $f \in (H^{P_F,\sigma,\mu})_\infty$, $w \in (V_\sigma)_{K_F}$) and the map $(H^{P_F,\sigma,\mu})_\infty \ni f \mapsto \int_{\bar{N}_F} |\langle f(\bar{n}), w\rangle| d\bar{n}$ is continuous.*

(2) *For nonzero $w \in (V_\sigma)_{K_F}$ there exists $f \in I_{P_F,\sigma,\mu}$ such that*

$$\int_{\bar{N}_F} |\langle f(\bar{n}), w\rangle| d\bar{n} \neq 0 .$$

We define a linear map $\beta_{P_F,\sigma,\mu}: I_{P_F,\sigma,\mu} \to (V_\sigma)_{K_F}$ by

$$\langle \beta_{P_F,\sigma,\mu}(f), w\rangle_{H_\sigma} = \int_{\bar{N}_F} \langle f(\bar{n}), w\rangle d\bar{n} \qquad (f \in I_{P_F,\sigma,\mu},\ w \in (V_\sigma)_{K_F}) .$$

$\beta_{P_F,\sigma,\mu}$ is a K_F-homomorphism and

LEMMA 3.9 ([11, Lemma 5.3.3]). $\beta_{P_F,\sigma,\mu}(\bar{\mathfrak{n}}_F I_{P_F,\sigma,\mu}) = 0$.

So $\beta_{P_F,\sigma,\mu}$ induces a map $\alpha_{P_F,\sigma,\mu}: I_{P_F,\sigma,\mu}/\bar{\mathfrak{n}}_F I_{P_F,\sigma,\mu} \to (V_\sigma)_{K_F}$ and $\alpha_{P_F,\sigma,\mu}$ is a $(\mathfrak{m}_{1F}, K_F)$-homomorphism. By Frobenius reciprocity there exists a unique $j_{P_F,\sigma,\mu} \in \mathrm{Hom}_{\mathfrak{g},K}(I_{P_F,\sigma,\mu}, I_{\bar{P}_F,\sigma,\mu})$ such that

$$j_{P_F,\sigma,\mu}(f)(e) = \beta_{P_F,\sigma,\mu}(f) . \tag{3.2}$$

Lemma 3.8 (2) implies that $j_{P_F,\sigma,\mu}$ is not identically zero.

THEOREM 3.10 ([11, Theorem 5.4.1]). *We suppose (P_F, σ, μ) is Langlands data. Put $j_{P_F,\sigma,\mu}(I_{P_F,\sigma,\mu}) = J_{P_F,\sigma,\mu}$.*

(1) *If $f \in I_{P_F,\sigma,\mu}$ satisfies $j_{P_F,\sigma,\mu}(f) \neq 0$ then f generates $I_{P_F,\sigma,\mu}$ as a $(\mathfrak{g}, K)$-module.*

(2) *$J_{P_F,\sigma,\mu}$ is the unique nonzero irreducible $(\mathfrak{g}, K)$-submodule of $I_{\bar{P}_F,\sigma,\mu}$ such that $I_{\bar{P}_F,\sigma,\mu}/J_{P_F,\sigma,\mu}$ is irreducible.*

(3) *If (P_F, σ, μ) and $(P_{F'}, \sigma', \mu')$ are Langlands data and if $J_{P_F,\sigma,\mu} \simeq J_{P_{F'},\sigma',\mu'}$ as a $(\mathfrak{g}, K)$-module, then $F = F'$, $\mu = \mu'$, $\sigma \approx \sigma'$ (unitarily equivalent).*

DEFINITION 3.11. We call $J_{P_F,\sigma,\mu} \subseteq I_{\bar{P}_F,\sigma,\mu}$ Langlands quotient corresponding to (P_F, σ, μ).

THEOREM 3.12 (Langlands ([11, Theorem 5.4.4])). *For any irreducible* $(\mathfrak{g}, K)$-*module* V *there exists Langlands data* (P_F, σ, μ) *such that* $V \simeq J_{P_F,\sigma,\mu}$ *(as a* $(\mathfrak{g}, K)$-*module).*

4. Irreducible representations of $D(E_\tau)$.

We keep to the notation in §3. Let W be an irreducible $(\mathfrak{g}, K)$-module. We fix an irreducible representation (τ, V_τ) of K. Every $z \in U(\mathfrak{g}_c)^K$ acts naturally on $T \in \mathrm{Hom}_K(V_\tau, W)$ by

$$[z.T](v) = z.T(v) \qquad (v \in V_\tau)\,. \tag{4.1}$$

$\mathrm{Hom}_K(V_\tau, W) = \mathrm{Hom}_K(V_\tau, W(\tau))$ is finite dimensional by the admissibility of W. Hence $\mathrm{Hom}_K(V_\tau, W)$ is a finite dimensional $U(\mathfrak{g}_c)^K$-module. Moreover [11, Proposition 3.5.4] claims the following:

PROPOSITION 4.1. *Let* W *be an irreducible* $(\mathfrak{g}, K)$-*module and* $(\tau, V_\tau) \in \hat{K}$. *Then* $\mathrm{Hom}_K(V_\tau, W)$ *is an irreducible* $U(\mathfrak{g}_c)^K$-*module unless trivial.*

We put $\Gamma_\tau(W) = \mathrm{Hom}_K(V_{\tau^*}, W)$ for any irreducible $(\mathfrak{g}, K)$-module W. From Theorem 3.12 it suffices to consider a Langlands quotient $J_{P_F,\sigma,\mu}$ as W. Since $J_{P_F,\sigma,\mu}$ is irreducible, $\Gamma_\tau(J_{P_F,\sigma,\mu})$ is a finite dimensional irreducible $U(\mathfrak{g}_c)^K$-module by Proposition 4.1.

If $z \in \mathcal{I}^{\mathrm{T}}$ then $z^{\mathrm{T}} \in \mathcal{I}$ and

$$\tau^*(z) = {}^t(\tau(z^{\mathrm{T}})) = 0\,.$$

(Here ${}^t(\cdot)$ denotes the transpose map.) It follows that $U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathcal{I}^{\mathrm{T}}$ acts trivially on $\Gamma_\tau(J_{P_F,\sigma,\mu})$. We have proved the next proposition.

PROPOSITION 4.2. $\Gamma_\tau(J_{P_F,\sigma,\mu})$ *is a (finite dimensional) irreducible* $D(E_\tau)$-*module unless trivial.*

Remark 4.3. The functor Γ_τ was originally introduced in [5], [6]. All irreducible $D(E_\tau)$-modules are finite dimensional by [5, Theorem 5.3].

Let χ denote the action of type (4.1). When F is empty, we write P instead of $P_\varnothing$.

LEMMA 4.4.

$$I_{\bar{P},\sigma,\lambda} \simeq I_{P,\sigma^*,-\lambda}$$

as a $(\mathfrak{g}, K)$-*module.*

PROOF. We choose an M-invariant nondegenerate bilinear form $\langle\ ,\ \rangle$ on $V_\sigma \times V_{\sigma^*}$. We define a map $f \mapsto f^*: H^{P,\sigma,\lambda} \to H^{\bar{P},\sigma^*,-\lambda}$ by $f^*(g)v = \langle v, f(g)\rangle$ $(g \in G, v \in V_\sigma)$. Then f^* satisfies the functional equation of $H^{\bar{P},\sigma^*,-\lambda}$. Since this map is a G-isomorphism, the claim follows from $I_{P,\sigma,\lambda} = (H^{P,\sigma,\lambda})_K$.

LEMMA 4.5. *$(\chi, \Gamma_\tau(J_{P,\sigma,\lambda}))$ is a $U(\mathfrak{g}_c)^K$-submodule of $(\chi_{\tau,\sigma,-\lambda}, H_\sigma)$. Hence $(\chi, \Gamma_\tau(J_{P,\sigma,\lambda}))$ is a $D(E_\tau)$-subrepresentation of $(\chi_{\tau,\sigma,-\lambda}, H_\sigma)$.*

PROOF. As a $U(\mathfrak{g}_c)^K$-module,

$$\begin{aligned}\Gamma_\tau(J_{P,\sigma,\lambda}) &\hookrightarrow \Gamma_\tau(I_{\bar{P},\sigma,\lambda}) &&\text{(Theorem 3.10)}\\ &\simeq \Gamma_\tau(I_{P,\sigma^*,-\lambda}) &&\text{(Lemma 4.4)}\\ &\simeq H_\sigma &&\text{([7, Lemma 3.1])}.\end{aligned}$$

Note that $U(\mathfrak{g}_c)^K$ acts on the right hand side by $\chi_{\tau,\sigma,-\lambda}$.

Finally we give a non-triviality condition of $\Gamma_\tau(J_{P_F,\sigma,\mu})$.

THEOREM 4.6. *$\Gamma_\tau(J_{P_F,\sigma,\mu}) \neq \{0\}$ if and only if there exists $f \in I_{P_F,\sigma,\mu}(\tau^*)$ such that*

$$\int_{\bar{N}_F} \langle f(\bar{n}), w\rangle_\sigma d\bar{n} \neq 0 \qquad (\textit{for some } w \in V_\sigma)\,.$$

PROOF. Clearly $\Gamma_\tau(J_{P_F,\sigma,\mu}) \neq \{0\}$ if and only if $J_{P_F,\sigma,\mu}(\tau^*) \neq \{0\}$. Since $J_{P_F,\sigma,\mu} = I_{P_F,\sigma,\mu}/\mathrm{Ker}\, j_{P_F,\sigma,\mu}$ (Theorem 3.10) and the map $j_{P_F,\sigma,\mu}$ is K-linear, $J_{P_F,\sigma,\mu}(\tau^*) \neq \{0\}$ if and only if there exists $f \in I_{P_F,\sigma,\mu}(\tau^*)$ such that $j_{P_F,\sigma,\mu}(f) \neq 0$. We therefore see that

(*) $\Gamma_\tau(J_{P_F,\sigma,\mu}) \neq \{0\}$ if and only if
there exists $f \in I_{P_F,\sigma,\mu}(\tau^*)$ such that $j_{P_F,\sigma,\mu}(f) \neq 0$.

On the other hand, we have from the definition of $\beta_{P_F,\sigma,\mu}$

(**) $\int_{\bar{N}_F} \langle f(\bar{n}), w\rangle_\sigma\, d\bar{n} \neq 0$ (for some $w \in V_\sigma$)
if and only if $\beta_{P_F,\sigma,\mu}(f) \neq 0$.

First we prove "if" part. By (3.2) and (**) there exists $f \in I_{P_F,\sigma,\mu}(\tau^*)$ such that $j_{P_F,\sigma,\mu}(f)(e) \neq 0$. Hence $j_{P_F,\sigma,\mu}(f) \neq 0$ for such f. So (*) completes the proof.

Conversely we assume $\Gamma_\tau(J_{P_F,\sigma,\mu}) \neq \{0\}$. Then (*) implies that there exists $f \in I_{P_F,\sigma,\mu}(\tau^*)$ such that $j_{P_F,\sigma,\mu}(f) \neq 0$. So there exists $g_0 \in G$ such that $j_{P_F,\sigma,\mu}(f)(g_0) \neq 0$. Since $j_{P_F,\sigma,\mu}(f) \in I_{\bar{P}_F,\sigma,\mu}$ and the map $KM_FA_FN_F \to G$, $(k, m, a, n) \mapsto kman$ is surjective (cf. [4]), we have

$$\begin{aligned}[j_{P_F,\sigma,\mu}(f)](g_0) &= [j_{P_F,\sigma,\mu}(f)](\kappa_F(g_0)m_F(g_0)a_F(g_0)n_F(g_0))\\ &= e^{(-\mu+\rho)H(a_F(g_0))}\sigma(m_F(g_0)^{-1})j_{P_F,\sigma,\mu}(f)(\kappa_F(g_0)) \neq 0\end{aligned}$$

$(\kappa_F(g_0) \in K,\ m_F(g_0) \in M_F,\ a_F(g_0) \in A_F,\ n_F(g_0) \in N_F)$.

Hence $j_{P_F,\sigma,\mu}(f)(\kappa_F(g_0)) \neq 0$. We set $f_0 = \pi_{P_F,\sigma,\mu}(\kappa_F(g_0)^{-1})f$. Then f_0 belongs to $I_{P_F,\sigma,\mu}(\tau^*)$ since $\kappa_F(g_0)^{-1}$ is contained in K. Thus we see that

$$\begin{aligned}\beta_{P_F,\sigma,\mu}(f_0)&=j_{P_F,\sigma,\mu}(\pi_{P_F,\sigma,\mu}(\kappa_F(g_0)^{-1})f)(e)\\&=\pi_{\bar{P}_F,\sigma,\mu}(\kappa_F(g_0)^{-1})j_{P_F,\sigma,\mu}(f)(e)\\&=j_{P_F,\sigma,\mu}(f)(\kappa_F(g_0))\neq 0\,.\end{aligned}$$

Now the assertion follows from (**).

COROLLARY 4.7. *$\Gamma_\tau(J_{P,\sigma,\mu})\neq\{0\}$ if and only if there exists $f\in I_{P,\sigma,\mu}(\tau^*)$ such that*

$$\int_{\bar{N}} e^{-(\mu+\rho)H(\bar{n})}\langle\tau^*(\kappa(\bar{n})^{-1})f(e),w\rangle_\sigma d\bar{n}\neq 0 \qquad (\textit{for some } w\in V_\sigma)\,.$$

PROOF. Considering the Iwasawa decomposition: $G\simeq KAN$, $g\mapsto\kappa(g)e^{H(g)}n(g)$ and $f\in I_{P,\sigma,\mu}(\tau^*)$, we have

$$\begin{aligned}f(\bar{n})&=e^{-(\mu+\sigma)H(\bar{n})}f(\kappa(\bar{n}))\\&=e^{-(\mu+\rho)H(\bar{n})}\pi_{P,\sigma,\mu}(\kappa(\bar{n})^{-1})f(e)\\&=e^{-(\mu+\rho)H(\bar{n})}\tau^*(\kappa(\bar{n})^{-1})f(e)\,.\end{aligned}$$

Hence the claim follows from the above theorem.

5. $SL(2, \boldsymbol{R})$.

Let $G=SL(2,\boldsymbol{R})$ throughout this section. Then

$$K=SO(2)=\left\{k_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\,\middle|\,\theta\in\boldsymbol{R}\right\},$$

$$\mathfrak{k}=\boldsymbol{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad \mathfrak{p}=\boldsymbol{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\oplus\boldsymbol{R}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\mathfrak{g}_c=\boldsymbol{C}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\oplus\boldsymbol{C}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}\oplus\boldsymbol{C}\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}.$$

As is well known $\hat{K}=\{\gamma_l\mid l\in\boldsymbol{Z}\}$, where $\gamma_l(k_\theta)=e^{il\theta}$.

Let $\tau\in\hat{K}$ be γ_l for some $l\in\boldsymbol{Z}$. Set

$$h=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

Then we have $U(\mathfrak{k}_c)=\boldsymbol{C}[h]$, $\mathscr{I}=$ideal of $\boldsymbol{C}[h]$ generated by $(h-l)$, $\mathscr{I}^{\top}=$ideal of $\boldsymbol{C}[h]$ generated by $(h+l)$.

We take the following two basis of $\mathfrak{g}_c$:

(1) Eigenvectors for $\mathrm{Ad}(k_\theta)$. Set

$$x=\frac{1}{2}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}, \qquad y=\frac{1}{2}\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}.$$

Then

$$\mathrm{Ad}(k_\theta)x=e^{2i\theta}x\,, \quad \mathrm{Ad}(k_\theta)y=e^{-2i\theta}y\,, \quad \mathrm{Ad}(k_\theta)h=h\,.$$

The elements x, y, h form a standard basis of $\mathfrak{sl}(2, \boldsymbol{C})$, i.e. satisfy the commutation relations:

$$[x, y]=h\,, \quad [h, x]=2x\,, \quad [h, y]=-2y\,.$$

(2) The basis corresponding to $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$. Set

$$X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\in\mathfrak{n}\,, \quad H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\in\mathfrak{a}\,, \quad h_0=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\in\mathfrak{k}\,.$$

Then

$$[H, X]=2X\,, \quad [H, h_0]=-2h_0+4X\,, \quad [h_0, X]=H\,.$$

The theorem of PBW implies that

$$U(\mathfrak{g}_c)^K=\left\{\sum_{\alpha,\beta\in \boldsymbol{N}} c_{\alpha\beta}x^\alpha y^\alpha h^\beta \;\middle|\; c_{\alpha\beta}\in \boldsymbol{C}\right\}.$$

It follows from the commutation relations that xy and h generate $U(\mathfrak{g}_c)^K$.

Next we compute $\chi_{\tau,\sigma,\lambda}$. Let $\mathbf{1}$ denote the trivial representation of M and let ε denote the representation of M such that $\varepsilon(-I)=-id$. Then, since $M=\{\pm I\}$ and $\hat{M}=\{\mathbf{1}, \varepsilon\}$, H_σ and $H_{\tau,\sigma}$ are isomorphic to $\boldsymbol{C}$ if and only if

(a) $l=0 \bmod 2\boldsymbol{Z}$ and $\sigma=\mathbf{1}$ or (b) $l=1 \bmod 2\boldsymbol{Z}$ and $\sigma=\varepsilon$.

Otherwise H_σ and $H_{\tau,\sigma}$ are zero. We assume (a) or (b) below.

From the definition of $\chi_{\tau,\sigma,\lambda}$, for $\Delta=\mu(D)$ ($D\in U(\mathfrak{g}_c)^K$) we have

$$\chi_{\tau,\sigma,\lambda}(\Delta)=(e_\lambda\otimes\omega_\sigma)\circ\omega_\tau(D)\,.$$

It is enough to compute actions of xy and h.

(1) When $D=h$, $\omega_\tau(h)=-i(1\otimes\tau(-h_0))$. Since $\tau(h_0)=il$, we have $\omega_\tau(h)=-l(1\otimes id_{V_\tau})$. Note that $\omega_\sigma(id_{V_\tau})=id_{H_\sigma}$. Thus we obtain that $\chi_{\tau,\sigma,\lambda}(\Delta)=-l\,id_{H_\sigma}$, i.e. h acts by $-l$.

(2) When $D=xy$, the commutation relations and the base change

$$\begin{cases} x=\dfrac{1}{2}H-\dfrac{i}{2}h_0+iX \\ y=\dfrac{1}{2}H+\dfrac{i}{2}h_0-iX \\ h=-ih_0 \end{cases}$$

imply that

$$4xy \equiv H^2-2ih_0+h_0^2-2H \qquad \bmod \mathfrak{n}U(\mathfrak{g}_c)\,.$$

We identify $\mathfrak{a}_c^*$ with C by $\lambda \mapsto \lambda(H)$. Note that $\rho=1$ under this identification. Since $\tau(-h_0)=-il$, we have

$$\omega_\tau(xy)=\frac{1}{4}\{((H+1)^2-2(H+1)-l^2-2l)\otimes id\}\,.$$

So we obtain that

$$\chi_{\tau,\sigma,\lambda}(\Delta)=\frac{1}{4}\{\lambda^2-(l+1)^2\}id_{H_\sigma}\,,$$

i.e. xy acts by $\frac{1}{4}\{\lambda^2-(l+1)^2\}$.

REMARK 5.1. (1) and (2) above show that the Casimir element $\Omega=\frac{1}{2}h^2-h+2xy$ acts by $\frac{1}{2}(\lambda^2-1)$.

We fix a Borel subalgebra $\mathfrak{q}=Ch\oplus Cx$ and we put $C_k=C$. Let $U(\mathfrak{q})$ act on C_k by

$$h.1=k\,, \qquad x.1=0\,.$$

We put

$$V^k=U(\mathfrak{g}_c)\otimes_{U(\mathfrak{q})}C_k \qquad (k\in Z)\,.$$

Note that $\{y^m\otimes 1 \mid m\in N\}$ is a basis of V^k. Let $\mathfrak{g}$ act on V^k by left multiplication on $U(\mathfrak{g}_c)$. It is known that V^k is a $(\mathfrak{g}, K)$-module and irreducible if $k<0$. (See [11, 5.6.2].)

Set $V_{\tau^*}=Cv^*$ (now $\tau^*=\gamma_{-l}$). For $T\in \mathrm{Hom}(V_{\tau^*}, V^k)$, we can write as

$$T(v^*)=\sum_m c_m y^m\otimes 1 \qquad (c_m\in C)\,.$$

Then T belongs to $\Gamma_\tau(V^k)$ if and only if

$$k_\theta \sum c_m y^m\otimes 1=\sum e^{(-2m+k)i\theta}c_m y^m\otimes 1\,.$$

We therefore see that

$$e^{-il\theta}=e^{i(-2m+k)\theta} \tag{5.1}$$

for any $m \in \boldsymbol{N}$ such that $c_m \neq 0$.

Since $h+l \equiv 0 \bmod U(\mathfrak{g}_c)^K \cap U(\mathfrak{g}_c)\mathscr{I}^{\tau}$, each element $h \in U(\mathfrak{g}_c)^K$ acts on $T \in \Gamma_\tau(V^k)$ by $-l$.

On the other hand a direct calculation yields that

$$[h.T](v^*) = \sum (k-2m) c_m y^m \otimes 1 .$$

(Here we have used a relation $hy^m = y^m h - 2my^m$.)

Thus we have shown that

$$-l = k - 2m . \tag{5.2}$$

Since (5.2) implies (5.1), we suppose (5.2) from now on. Then $\Gamma_\tau(V^k) \neq \{0\}$ and

$$m = \frac{l+k}{2} . \tag{5.3}$$

It can be easily seen that

$$xy^{m+1} = y^{m+1}x + (m+1)y^m h - m(m+1)y^m .$$

Hence

$$xy^{m+1} \otimes 1 = (m+1)(k-m)y^m \otimes 1 .$$

By (5.3),

$$(m+1)(k-m) = -\frac{1}{4}(l-k)(l+k+2) .$$

This coincides with the action of xy under $\chi_{\tau,\sigma,\lambda}$ if

$$-(l-k)(l+k+2) = \lambda^2 - (l+1)^2 . \tag{5.4}$$

Proposition 5.2. *Let $G = SL(2, \boldsymbol{R})$.*

(a) *Let $D_{+,0} = V^{-1}$. ($D_{+,0}$ is a $(\mathfrak{g}, K)$-module associated to the limit of discrete series of G.) If l is positive odd then we have the equivalence as a representation of $D(E_{\gamma_l})$:*

$$(\chi, \Gamma_{\gamma_l}(D_{+,0})) \sim (\chi_{\gamma_l,\varepsilon,0}, H_\varepsilon) ,$$

where χ is the action of type (4.1).

(b) *Let $D_k = V^{-k-1}$ for $k > 0$. (D_k is a $(\mathfrak{g}, K)$-module associated to the discrete series of G.) If $l-k$ is positive odd then we have the equivalence as a representation of $D(E_{\gamma_l})$:*

$$(\chi, \Gamma_{\gamma_l}(D_k)) \sim (\chi_{\gamma_l,\varepsilon^l,\pm k}, H_{\varepsilon^l}) .$$

Proof. (a) Put $k = -1$ in the arguments above. Then we see that (5.3) holds if and only if $\lambda = 0$. H_σ is nontrivial if and only if $\sigma = \varepsilon$.

(b) Replace k by $-k-1$ in the arguments above. Then we see that (5.3) holds if and only if $l-k = 2m+1$, (5.4) holds if and only if $\lambda^2 = k^2$. H_σ is equal to $\boldsymbol{C}$ if $\sigma = \varepsilon^l$.

Set $\bar{V}^k = U(\mathfrak{g}_c) \otimes_{U(\bar{\mathfrak{q}})} \bar{C}_k$, when $\bar{\mathfrak{q}} = Ch \oplus Cy$, $\bar{C}_k = C$ (as a linear space). Define the action of $\bar{\mathfrak{q}}$ on $\bar{C}_k$ by $h.1 = k$, $y.1 = 0$. We similarly obtain the following results by considering $\bar{V}^k$ instead of V^k.

PROPOSITION 5.3. *Let $G = SL(2, \boldsymbol{R})$. Put $D_{-,0} = \bar{V}^1$, $D_{-k} = \bar{V}^{k+1}$.*
(a) *If l is negative odd then*

$$(\chi, \Gamma_{\gamma_l}(D_{-,0})) \sim (\chi_{\gamma_l, \varepsilon, 0}, H_\varepsilon) .$$

(b) *If $l+k$ is negative odd ($k>0$) then*

$$(\chi, \Gamma_{\gamma_l}(D_{-k})) \sim (\chi_{\gamma_l, \varepsilon^l, \pm k}, H_{\varepsilon^l}) .$$

6. The commutativity of $D(E_\tau)$.

DEFINITION 6.1. We say that $\tau|_M$ is multiplicity free if $[\tau|_M : \sigma] = 1$ for any $\sigma \in \hat{M}$ such that $[\tau|_M : \sigma] > 0$.

PROPOSITION 6.2 (cf. [2, Theorem 6]). *The following conditions are equivalent:*
(a) *$\tau|_M$ is multiplicity free.*
(b) *$\mathrm{End}_M(V_\tau)$ is commutative.*
(c) *$D(E_\tau)$ is commutative.*

PROOF. First we prove that (a) implies (b). Note that

$$\mathrm{End}_M(V_\tau) \simeq \sum_{\sigma \in R_\tau} \mathrm{End}\, H_\sigma \tag{6.1}$$

as a ring. The assumption implies that

$$H_\sigma = \mathrm{Hom}_M(V_\sigma, V_\tau) = C$$

for any $\sigma \in R_\tau$. Thus the right hand side of (6.1) is equal to a direct sum of C. Hence $\mathrm{End}_M(V_\tau)$ is commutative.

Next we prove that (b) implies (c). From §2 there is an injective algebra homomorphism:

$$D(E_\tau) \hookrightarrow U(\mathfrak{a}_c) \otimes \mathrm{End}_M(V_\tau) .$$

Since $U(\mathfrak{a}_c)$ is commutative, (c) follows from the commutativity of $\mathrm{End}_M(V_\tau)$.

Finally we prove that (c) implies (a). For any $\sigma \in R_\tau$ there exists $\mu \in \mathfrak{a}_c^*$ such that $I_{P,\sigma,\mu}$ is an irreducible $(\mathfrak{g}, K)$-module (cf. [9]). By [11, 4.5.1 and 4.5.2]

$$(I_{P,\sigma,\mu})^{\sim} = I_{P,\sigma^*,-\mu}$$

is also irreducible. $((\cdot)^{\sim}$ means the admissible dual.) Frobenius reciprocity implies that

$$\dim \mathrm{Hom}_K(V_{\tau^*}, I_{P,\sigma^*,-\mu}) = [\tau|_M : \sigma] .$$

This shows that $\Gamma_\tau(I_{P,\sigma^*,-\mu}) \neq \{0\}$. By Proposition 4.2, $\Gamma_\tau(I_{P,\sigma^*,-\mu})$ is a finite dimensional irreducible $D(E_\tau)$-module. Now $D(E_\tau)$ is commutative by the assumption, we can apply Schur's lemma and we have

$$\Gamma_\tau(I_{P,\sigma^*,-\mu})=\boldsymbol{C}\,.$$

This implies that $[\tau|_M:\sigma]=1$ which is desired.

7. The injectivity of the Poisson transform.

We introduce the Poisson transform for vector bundle E_τ following [7]. See [7] for details.

For a finite dimensional representation σ of M on V_σ and $\lambda\in\mathfrak{a}_c^*$, we associate the representation of P defined by

$$P=MAN\ni man\mapsto e^{(-\lambda+\rho)H(a)}\sigma(m) \qquad (m\in M,\, a\in A,\, n\in N)\,. \tag{7.1}$$

We denote by $F_{\sigma,\lambda}$ the homogeneous vector bundle over G/P associated to (7.1). We write simply $F_{\tau,\lambda}$ instead of $F_{\tau|_M,\lambda}$ when $\sigma=\tau|_M$ for $\tau\in\hat{K}$. Let $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ be the space of hyperfunctional sections of E_τ over G/K and of $F_{\sigma,\lambda}$ over G/P respectively. $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ can be regarded as G-modules. We denote by π and $\pi_{\sigma,\lambda}$ the actions of G on $\mathscr{B}(E_\tau)$ and $\mathscr{B}(F_{\sigma,\lambda})$ respectively.

Let $\mathscr{B}(G,(\sigma,\lambda))$ be the space of hyperfunctions from G to V_σ such that

$$f(gman)=e^{(\lambda-\rho)H(a)}\sigma(m^{-1})f(g) \qquad (g\in G,\, m\in M,\, a\in A,\, n\in N)\,,$$

and $\mathscr{B}(G,\tau)$ the space of hyperfunctions from G to V_τ such that

$$f(gk)=\tau(k^{-1})f(g) \qquad (g\in G,\, k\in K)\,.$$

Then there are canonical isomorphisms:

$$\mathscr{B}(E_\tau)\simeq\mathscr{B}(G,\tau)\,, \qquad \mathscr{B}(F_{\sigma,\lambda})\simeq\mathscr{B}(G,(\sigma,\lambda))\,.$$

We identify $\mathscr{B}(E_\tau)$ (resp. $\mathscr{B}(F_{\sigma,\lambda})$) with $\mathscr{B}(G,\tau)$ (resp. $\mathscr{B}(G,(\sigma,\lambda))$) by these isomorphisms. Note that both π and $\pi_{\sigma,\lambda}$ are left regular actions under this identification.

For $\phi\in\mathscr{B}(F_{\tau,\lambda})$ we define the function $\mathscr{P}_{\tau,\lambda}(\phi)$ on G by

$$[\mathscr{P}_{\tau,\lambda}(\phi)](g)=\int_K\tau(k)\phi(gk)dk \qquad (g\in G)\,.$$

Then $\mathscr{P}_{\tau,\lambda}(\phi)$ belongs to $\mathscr{A}(E_\tau)$, the space of analytic sections of E_τ. We call $\mathscr{P}_{\tau,\lambda}:\mathscr{B}(F_{\tau,\lambda})\to\mathscr{A}(E_\tau)$ the Poisson transform for E_τ.

For $\sigma\in R_\tau$, regard $H_\sigma=\mathrm{Hom}_M(V_\sigma,V_\tau)$ as a trivial bundle over G/K. Then we have

$$\mathscr{B}(F_{\tau,\lambda})\simeq\sum_{\sigma\in R_\tau}\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma \qquad \text{(direct sum)}\,.$$

$\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma$ is regarded as a subspace of $\mathscr{B}(F_{\tau,\lambda})$ by the G-isomorphism from

$\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma$ to $\mathscr{B}(F_{\tau,\lambda})$ defined by

$$\sum_i \phi_i\otimes a_i \mapsto \sum_i a_i\phi_i \qquad (a_i\in H_\sigma,\ \phi_i\in\mathscr{B}(F_{\sigma,\lambda}))\,.$$

We define $\mathscr{P}_{\tau,\sigma,\lambda}$ to be the restriction of $\mathscr{P}_{\tau,\lambda}$ to $\mathscr{B}(F_{\sigma,\lambda})\otimes H_\sigma$. Let $H_{\tau,\sigma}=\mathrm{Hom}_M(V_\tau, V_\sigma)$ for $\sigma\in R_\tau$. Then the function ψ_w on G defined by

$$\psi_w(g)=\sum_i e^{(\lambda-\rho)H(g)}b_i\tau(\kappa(g)^{-1})v_i$$

($\lambda\in\mathfrak{a}_c^*$, $w=\sum_i v_i\otimes b_i\in V_\tau\otimes H_{\tau,\sigma}$) belongs to $\mathscr{B}(F_{\sigma,\lambda})$ and the map $\Gamma_{\sigma,\lambda}$ from $V_\tau\otimes H_{\tau,\sigma}$ to $\mathscr{B}(F_{\sigma,\lambda})^\tau$, the space of spherical sections in $\mathscr{B}(F_{\sigma,\lambda})$, $w\mapsto\psi_w$ is isomorphic. We identify $\mathscr{B}(F_{\sigma,\lambda})^\tau$ with $V_\tau\otimes H_{\tau,\sigma}$ by $\Gamma_{\sigma,\lambda}$.

Let (χ, H) be a subrepresentation of $(\chi_{\tau,\sigma,\lambda}, H_\sigma)$. Since σ is irreducible, the bilinear form $H_{\tau,\sigma}\times H_\sigma\to C$, $(b, a)\mapsto\langle b, a\rangle$ defined by

$$\langle b, a\rangle=d(\tau)^{-1}\mathrm{tr}(ba)$$

is nondegenerate. We identify H_σ with $H_{\tau,\sigma}^*$ by this form. We denote by p_χ: $H_{\tau,\sigma}\otimes H\to(D(E_\tau)/\mathrm{Ker}\,\chi)^*$ the transpose of the inclusion,

$$\begin{aligned}\iota:\ D(E_\tau)/\mathrm{Ker}\,\chi&\simeq\chi(D(E_\tau))\subseteq\mathrm{End}\,H\\ &\simeq H\otimes H^*\subseteq H_\sigma\otimes H^*\simeq(H_{\tau,\sigma}\otimes H)^*\,.\end{aligned}$$

Let $\mathscr{A}(E_\tau,\chi)$ denote the space of eigen sections of type χ and $\mathscr{A}(E_\tau,\chi)^\tau$ be the space of spherical sections in $\mathscr{A}(E_\tau,\chi)$. (See [7, §2].) We define a linear map $s_\chi:\mathscr{A}(E_\tau,\chi)\to V_\tau\otimes(D(E_\tau)/\mathrm{Ker}\,\chi)^*$ by

$$\langle s_\chi(u),\ \Delta+\mathrm{Ker}\,\chi\rangle=(\Delta u)(eK) \qquad (u\in\mathscr{A}(E_\tau,\chi),\ \Delta\in D(E_\tau))\,.$$

If we set $H=\Gamma_\tau(J_{P,\sigma,\lambda})$, then (χ, H) is (equivalent to) a subrepresentation of $(\chi_{\tau,\sigma,-\lambda}, H_\sigma)$ of $D(E_\tau)$ by Lemma 4.5. Hence the transpose of this inclusion induces a surjective map from $H_{\tau,\sigma}\otimes H$ to $H^*\otimes H$, which is denoted by p_σ. We denote by ι_1 and ι_2 the inclusions in the defining sequence of ι from $D(E_\tau)/\mathrm{Ker}\,\chi$ to $H\otimes H^*$ and from $H\otimes H^*$ to $(H_{\tau,\sigma}\otimes H)^*$, respectively. Let $\bar{p}_\chi$ be the transpose map of ι_1. Then we have

$$p_\chi={}^t\iota={}^t(\iota_2\circ\iota_1)={}^t(\iota_1)\circ{}^t(\iota_2)=\bar{p}_\chi\circ p_\sigma\,.$$

Thus we have the following commutative diagram:

$$\begin{array}{ccc} H_{\tau,\sigma}\otimes H & \xrightarrow{p_\sigma} & H^*\otimes H \\ \Big\downarrow{\scriptstyle p_\chi} & \swarrow{\scriptstyle \bar{p}_\chi} & \\ (D(E_\tau)/\mathrm{Ker}\,\chi)^*\,. & & \end{array}$$

Note that every map in the diagram is surjective since it is the transpose of an inclusion.

LEMMA 7.1. *Let* $\lambda \in \mathfrak{a}_c^*$, $\sigma \in R_\tau$, $H = \Gamma_\tau(J_{P,\sigma,\lambda})$. *Then* $(\mathrm{Ker}\,\mathscr{P}_{\tau,\sigma,-\lambda}) \cap V_\tau \otimes H_{\tau,\sigma} \otimes H = \mathrm{Ker}(id \otimes p_\sigma)$ *if there exists* $f \in I_{P,\sigma,\lambda}(\tau^*)$ *such that*

$$\int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})} \langle \tau^*(\kappa(\bar{n})^{-1}) f(e), w \rangle_\sigma d\bar{n} \neq 0 \qquad (\textit{for some } w \in V_\sigma)\,.$$

PROOF. Combining the preceding diagram with [7, Lemma 3.4 and Theorem 3.5], we get the following commutative diagram:

$$\begin{array}{ccccc}
\mathscr{B}(F_{\sigma,-\lambda})^\tau \otimes H & \xleftarrow{\Gamma_{\sigma,-\lambda}\otimes id} & V_\tau \otimes H_{\tau,\sigma} \otimes H & \xrightarrow{id\otimes p_\sigma} & V_\tau \otimes H^* \otimes H \\
\Big\downarrow \mathscr{P}_{\tau,\sigma,-\lambda} & & \Big\downarrow id\otimes p_\chi & \swarrow id\otimes \bar{p}_\chi & \\
\mathscr{A}(E_\tau,\chi)^\tau & \xrightarrow{s_\chi} & V_\tau \otimes (D(E_\tau)/\mathrm{Ker}\,\chi)^*\,. & &
\end{array}$$

It follows from Corollary 4.7 that the assumption implies that (χ, H) is nontrivial. Thus (χ, H) is irreducible by Proposition 4.2. It is well known that χ is surjective if (χ, H) is irreducible. Therefore we see that $\bar{p}_\chi$ is bijective. The above diagram shows that $\mathrm{Ker}(id \otimes p_\sigma) = \mathrm{Ker}(id \otimes p_\chi)$ since both $id \otimes p_\sigma$ and $id \otimes p_\chi$ are surjective. Now the lemma follows.

We introduce the notation in [3]. For $\tau \in \hat{K}$ and $\lambda \in \mathfrak{a}_c^*$, we set

$$B_\tau(\bar{P}:P:\lambda) = \int_{\bar{N}} \tau(\kappa(\bar{n})^{-1}) e^{-(\lambda+\rho)H(\bar{n})} d\bar{n}\,.$$

Since $B_\tau(\bar{P}:P:\lambda)$ commutes with the action of M, we put

$$B_\tau^\sigma(\bar{P}:P:\lambda) = B_\tau(\bar{P}:P:\lambda)|_{V_\tau(\sigma)}$$

for $\sigma \in \hat{M}$. $\lambda \mapsto B_\tau^\sigma(\bar{P}:P:\lambda)$ is called Harish-Chandra's C-function. For $v \in V_\tau$ and $T \in \mathrm{Hom}_M(V_\tau, V_\sigma)$, we define $f_{v,T}: G \to V_\sigma$ by

$$f_{v,T}(kan) = e^{-(\lambda+\rho)H(a)} T(\tau(k^{-1})v)$$

$(k \in K, a \in A, n \in N)$. Then $V_\tau \otimes \mathrm{Hom}_M(V_\tau, V_\sigma) \to I_{P,\sigma,\lambda}(\tau)$, $v \otimes T \mapsto f_{v,T}$ gives a K-module isomorphism. We identify $I_{P,\sigma,\lambda}(\tau)$ with $V_\tau \otimes \mathrm{Hom}_M(V_\tau, V_\sigma)$ by this isomorphism. Note that $T(v) = f_{v,T}(e)$ under this identification.

Since $id \otimes p_\sigma$ is surjective, we say that $\mathscr{P}_{\tau,\sigma,-\lambda}$ is injective on $V_\tau \otimes H^* \otimes H$ modulo $\mathrm{Ker}(id \otimes p_\sigma)$ if $id \otimes \bar{p}_\chi$ is injective (hence bijective).

THEOREM 7.2. *Under the assumption of Lemma* 7.1, $\mathscr{P}_{\tau,\sigma,-\lambda}$ *is injective on* $V_\tau \otimes H^* \otimes H$ *modulo* $\mathrm{Ker}(id \otimes p_\sigma)$ *if* $B_{\tau^*}^\sigma(\bar{P}:P:\lambda)$ *is non-zero as an operator.*

PROOF. We have

$$\int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})}\langle \tau^*(\kappa(\bar{n})^{-1})f_{v,T}(e), w\rangle d\bar{n}$$

$$=\int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})}\langle T(\tau^*(\kappa(\bar{n})^{-1})v), w\rangle d\bar{n}$$

$$=\langle T(B_{\tau^*}(\bar{P}:P:\lambda)v), w\rangle .$$

Hence the claim follows from the proof of Lemma 7.1 since T is M-linear and $w\in V_\sigma$.

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