TOKYO J. MATH.
VOL. 17, NO. 1, 1994

L^p-Approach to Mixed Boundary Value Problems for Second-Order Elliptic Operators

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(Communicated by S. Ōuchi)

§1. Introduction and results.

Let Ω be a bounded domain in Euclidean space $\boldsymbol{R}^N$ with C^∞-boundary Γ and its closure $\bar{\Omega}=\Omega\cup\Gamma$ is an N-dimensional compact C^∞-manifold with boundary. Let

$$A=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N}a^{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_{j=1}^{N}b^j(x)\frac{\partial}{\partial x_j}+c(x)$$

be an elliptic differential operator of second order. Here the coefficients of A satisfy the following conditions:

(1) a^{ij}, b^j and c are real-valued C^∞-functions on $\bar{\Omega}$ for $1\le i, j\le N$.

(2) $a^{ij}(x)=a^{ji}(x)$ for $x\in\bar{\Omega}$, $1\le i, j\le N$.

(3) A is strongly elliptic; there exists a constant $a_0>0$ such that

$$\sum_{i,j=1}^{N}a^{ij}(x)\xi_i\xi_j\ge a_0|\xi|^2, \qquad x\in\Omega, \quad \xi\in\boldsymbol{R}^N.$$

(4) $c(x)\le 0$, $x\in\bar{\Omega}$.

First, we shall give regularity, existence and uniqueness theorems of the solutions of a nonhomogeneous elliptic boundary value problem $(*)$:

$$(*)\qquad \begin{cases} Au=f & \text{in} \quad \Omega\,, \\ Lu=a\dfrac{\partial u}{\partial \nu}+bu\Big|_{\Gamma}=\varphi & \text{on} \quad \Gamma\,. \end{cases}$$

Here

(1) a and b are real-valued C^∞-functions on Γ.

(2) $\partial/\partial\nu$ is the conormal derivative associated with the matrix (a^{ij}):

$$\frac{\partial}{\partial \nu}=\sum_{i,j=1}^{N}a^{ij}n_j\frac{\partial}{\partial x_i}\,,$$

Received October 12, 1992

where $n=(n_1, \cdots, n_N)$ is the unit exterior normal to Γ. We associate with Problem $(*)$ an operator $\mathscr{A}$:

$$\begin{array}{ccc} \mathscr{A}: H^{s,p}(\Omega) & \longrightarrow & H^{s-2,p}(\Omega) \times B^{s-1-1/p,p}_{(a,b)}(\Gamma) \\ u & \longmapsto & (Au, Lu)\,. \end{array}$$

We shall explain function spaces. $H^{s,p}(\Omega)$ ($s \in \boldsymbol{R}$, $1<p<\infty$) denotes the set of all functions in $L^p(\Omega)$ whose derivatives up to order s belong to $L^p(\Omega)$ (for the precise definition, see Section 2). $B^{s-1/p,p}(\Gamma)$ ($s>1/p$, $1<p<\infty$) denotes the set of the boundary values of functions in $H^{s,p}(\Omega)$ with the norm:

$$|\varphi|_{s-1/p,p} = \inf\{\|u\|_{s,p}\,;\, u \in H^{s,p}(\Omega),\ u|_\Gamma = \varphi\}\,, \qquad \varphi \in B^{s-1/p,p}(\Gamma)\,.$$

$B^{s-1-1/p,p}_{(a,b)}(\Gamma)$ denotes the following function space introduced from $B^{s-1/p,p}(\Gamma)$ and $B^{s-1-1/p,p}(\Gamma)$ in connection with the boundary condition L:

$$B^{s-1-1/p,p}_{(a,b)}(\Gamma) = \{\varphi = a\varphi_1 + b\varphi_2\,;\, \varphi_1 \in B^{s-1-1/p,p}(\Gamma),\ \varphi_2 \in B^{s-1/p,p}(\Gamma)\}$$

with the norm:

$$|\varphi|_{a,b;s-1-1/p,p} = \inf\{|\varphi_1|_{s-1-1/p,p} + |\varphi_2|_{s-1/p,p}\,;\, \varphi = a\varphi_1 + b\varphi_2\}\,,$$
$$\varphi \in B^{s-1-1/p,p}_{(a,b)}(\Gamma)\,.$$

Then we have the following:

THEOREM 1. *Let $1<p<\infty$. Assume that the following conditions are satisfied:*

(A.1) $a(x') \geq 0$ *and* $b(x') \geq 0$ *on* Γ.

(A.2) $b(x')>0$ *on* $\Gamma_0 = \{x' \in \Gamma;\, a(x')=0\}$.

(A.3) $c(x)<0$ *in* Ω.

Then, the operator $\mathscr{A}$ is a homeomorphism for all $s \geq 2$.

The proof of Theorem 1 mainly consists of those of the surjectivity and injectivity of $\mathscr{A}$. The injectivity of $\mathscr{A}$ is proved by means of the strong maximum principle and the boundary point lemma due to Protter and Weinberger [9]. On the other hand, we need characterization of the adjoint operator of $\mathscr{A}$ in the proof of the surjectivity of $\mathscr{A}$. Lions and Magenes [6] consider the case when $a(x')>0$ on Γ, in the space $L^p(\Omega)$. In [6], the adjoint operator of $\mathscr{A}$ is characterized by means of Green's formula. But, the technique of Lions and Magenes seems to be difficult to characterize the adjoint operator when $a(x') \geq 0$ on Γ. A method which we use in this paper is that of reducing Problem $(*)$ to the study of a first order pseudo-differential operator T on the boundary.

Taira [13], by using this technique of the reduction to the boundary, investigates Problem $(*)$ under the assumptions (A.1) and (A.2) in the space $L^2(\Omega)$. He makes an argument in which he mainly use an inequality of Gårding type due to Melin [7], to prove the surjectivity of $\mathscr{A}$. But, in the present paper, the results with regard to existence of a parametrix for the operator T due to Hörmander [3] and boundedness of the parametrix due to Bourdaud [4], play a main role in order to prove the surjectivity of $\mathscr{A}$.

Taira [15] studies Problem (*), in the case of the homogeneous boundary condition with the assumptions (A.1) and (A.2), in the space $L^p(\Omega)$. Taira [15] gives the a priori estimate for Problem (*) such that for any solution $u \in H^{2,p}(\Omega)$ of Problem (*) with $f \in L^p(\Omega)$ and $\varphi \in B^{2-1/p,p}(\Gamma)$, we have

$$\|u\|_{2,p} \leq C(\|f\|_p + |\varphi|_{2-1/p,p} + \|u\|_p)$$

with some constant $C>0$. In Theorem 1 with $s=2$, it follows from the continuity of the inverse $\mathscr{A}^{-1}$ of $\mathscr{A}$ that for any solution $u \in H^{2,p}(\Omega)$ of Problem (*) with $f \in L^p(\Omega)$ and $\varphi \in B^{2-1/p,p}_{(a,b)}(\Gamma)$, we have

$$\|u\|_{2,p} \leq C(\|f\|_p + |\varphi|_{a,b;1-1/p,p})$$

with some constant $C>0$. This implies that the space $B^{s-1/p,p}_{(a,b)}(\Gamma)$ which we introduce from both $B^{s-1/p,p}(\Gamma)$ and $B^{s-1-1/p,p}(\Gamma)$ in connection with the boundary condition L, plays an important role. We remark that if $a \equiv 0$ on Γ and $b>0$ on Γ (the Dirichlet condition), our estimate becomes

$$\|u\|_{2,p} \leq C(\|f\|_p + |\varphi|_{2-1/p,p}),$$

and that if $a>0$ on Γ (the Neumann condition or the third condition), our estimate becomes

$$\|u\|_{2,p} \leq C(\|f\|_p + |\varphi|_{1-1/p,p}).$$

Secondly we shall derive some properties of eigenvalues and eigenfunctions for certain unbounded linear operators in the Lebesgue space $L^p(\Omega)$ in connection with a homogeneous boundary value problem (**):

$$(**) \qquad \begin{cases} Au = \lambda u & \text{in } \Omega, \\ Lu = 0 & \text{on } \Gamma. \end{cases}$$

Here $\lambda \in \boldsymbol{C}$. We associate with the homogeneous problem (**) an operator A_p in the space $L^p(\Omega)$:

(a) The domain $\mathscr{D}(A_p)$ of A_p is

$$\mathscr{D}(A_p) = \{u \in H^{2,p}(\Omega) \,;\, Lu = 0 \text{ on } \Gamma\}.$$

(b) $A_p u = Au$, $u \in \mathscr{D}(A_p)$.

In order to characterize the adjoint operator of A_p we introduce an unbounded linear operator $A_{p'}$ in the space $L^{p'}(\Omega)$:

$$\text{(a)} \qquad \mathscr{D}(A_{p'}) = \left\{ v \in H^{2,p'}(\Omega) \,;\, L^* v \equiv a \frac{\partial v}{\partial \nu} + \left\{ b - a\left(\sum_{i=1}^{N} b^i n_i \right) \right\} v \bigg|_{\Gamma} = 0 \right\}.$$

(b) $A_{p'}v = A^*v\,, \qquad v \in \mathscr{D}(A_{p'})\,.$

Here $1/p + 1/p' = 1$ and

$$A^*v(x) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial v}{\partial x_j}(x)\right) - \sum_{i=1}^{N} b^i(x)\frac{\partial v}{\partial x_i}(x)$$
$$+\left(c(x) - \sum_{i=1}^{N} \frac{\partial b^i}{\partial x_i}(x)\right)v(x)\,.$$

Agmon [1] treated the case that $a(x) > 0$ on Γ. When $1 < p < \infty$, he proved the following results (see [1], Theorem 4.4; [2], Theorem 15.1):

(1.1) The operator $A_{p'}$ is the adjoint operator of A_p.

(1.2) The eigenvalues and eigenfunctions of A_p are common to all p.

(1.3) The real parts of the eigenvalues of A_p are bounded above by some constant K.

(1.4) All rays $\arg\lambda = \theta$ different from the negative axis are rays of minimal growth of the resolvent $(A_p - \lambda)^{-1}$.

(1.5) The spectrum of A_p is discrete, the eigenvalues of A_p have finite multiplicities and there are only a finite number of eigenvalues outside any angle: $|\arg\lambda| < \pi - \varepsilon$, $\varepsilon > 0$.

(1.6) The negative axis is a direction of condensation of eigenvalues.

(1.7) If $\{\lambda_j\}$ is the sequence of eigenvalues of A_p, counted according to multiplicity, and if for $\lambda \leq K$, where the constant K is the same as in the assertion (1.3), $N(\lambda)$ is the number of λ_j such that $\lambda \leq \mathrm{Re}\,\lambda_j$, then one obtains that

$$N(\lambda) = \frac{|\Omega|}{2^N \pi^{N/2}\Gamma(N/2+1)}\,\lambda^{N/2} + o(\lambda^{N/2})\,,$$

where $|\Omega|$ denotes the volume of Ω.

(1.8) The generalized eigenfunctions are complete in $L^p(\Omega)$; they are also complete in $\mathscr{D}(A_p)$ in the $\|\cdot\|_{2,p}$ norm for all p.

In this paper, we shall prove

THEOREM 2. *Let* $1 < p < \infty$ *and* $p' = p/(p-1)$. *Assume that the conditions* (A.1) *and* (A.2) *are satisfied, and further that the following condition is satisfied*:

(A.4) $$b(x) - a(x)\left(\sum_{i=1}^{N} b^i(x) n_i\right) \geq 0 \qquad \text{on} \quad \Gamma\,.$$

Then, we have the assertions (1.1)–(1.8).

The assertion (1.1) is proved by means of Green's formula. By virtue of the hypoellipticity (Theorem 3.6) of the mapping $\mathscr{A}$ in Theorem 1, we prove the assertion (1.2). Using integration by parts as in [13], we obtain the assertion (1.3). The assertion (1.4) follows from Theorem 2 of [15]. If the assertion (1.2) is true, we have only to verify the assertion (1.3) and (1.5)–(1.7) in the case $p=2$ and have only to verify the assertion (1.8) in the case $p=2$ by virtue of Theorem of Agmon [1]. Taira [13] investigates this problem in the space $L^2(\Omega)$ in the case that the operator A is the usual Laplacian. He proves the assertions (1.1) and (1.3)–(1.8) (see [13], Theorems 2 and 7.4). In this paper, in order to prove (1.5)–(1.8), we apply the results due to Agmon ([1], [2]) to the operator A_2 as in [13].

The rest of this paper is organized as follows:

Section 2 is devoted to the basic definitions and properties of function spaces and pseudo-differential operators.

Section 3 is devoted to the proof of the injectivity of the operator $\mathscr{A}$.

In Section 4 we prove the surjectivity and continuity of the operator $\mathscr{A}$. In this section, we finish the proof of Theorem 1.

In section 5 we prove Theorem 2.

§2. Function spaces and pseudo-differential operators.

2.1. Function spaces. If $1\le p<\infty$, $L^p(\Omega)$ denotes the space of Lebesgue measurable functions u on Ω such that $|u|^p$ is integrable on Ω. The space $L^p(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_p=\left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

We recall the Fourier transform $\mathscr{F}$ and the inverse Fourier transform $\mathscr{F}^*$:

$$\mathscr{F}f(\xi)=\int_{\boldsymbol{R}^N} e^{-ix\cdot\xi}f(x)dx\,, \qquad f\in L^1(\boldsymbol{R}^N)\,,$$

$$\mathscr{F}^*g(x)=\left(\frac{1}{2\pi}\right)^N\int_{\boldsymbol{R}^N} e^{ix\cdot\xi}g(\xi)d\xi\,, \qquad g\in L^1(\boldsymbol{R}^N)\,,$$

where $x\cdot\xi=x_1\xi_1+\cdots+x_N\xi_N$. $\mathscr{S}(\boldsymbol{R}^N)$ denotes the space of C^∞ functions φ on $\boldsymbol{R}^N$ such that for any non-negative integer j, we have that $\sup\{(1+|x|^2)^{j/2}|\partial^\alpha\varphi(x)|\,;\,x\in\boldsymbol{R}^N, 0\le|\alpha|\le j\}<\infty$, where $\partial^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}$, $\alpha=(\alpha_1,\cdots,\alpha_N)$ and α_i $(1\le i\le N)$ is a non-negative integer. The transforms $\mathscr{F}$ and $\mathscr{F}^*$ can be extended to the transforms on the dual spaces $\mathscr{S}'(\boldsymbol{R}^N)$ of $\mathscr{S}(\boldsymbol{R}^N)$. If $s\in\boldsymbol{R}$, we get a mapping J^s: $\mathscr{S}'(\boldsymbol{R}^N)\to\mathscr{S}'(\boldsymbol{R}^N)$ by the formula:

$$J^s u=\mathscr{F}^*((1+|\xi|^2)^{-s/2}\mathscr{F}u)\,, \qquad u\in\mathscr{S}'(\boldsymbol{R}^N)\,.$$

We define function spaces which are called generalized Sobolev spaces. If $1<p<\infty$, $H^{s,p}(\boldsymbol{R}^N)$ denotes the image of $L^p(\boldsymbol{R}^N)$ under the mapping J^s. The space $H^{s,p}(\boldsymbol{R}^N)$ is a Banach space with norm $\|u\|_{s,p}=\|J^{-s}u\|_p$. We remark that, if s is a non-negative integer, $H^{s,p}(\boldsymbol{R}^N)$ coincides with the usual Sobolev space which is the space of functions $u\in L^p(\boldsymbol{R}^N)$

whose derivatives $\partial^\alpha u$, $|\alpha|\leq s$, belong to the space $L^p(\boldsymbol{R}^N)$, and that for $u\in H^{s,p}(\boldsymbol{R}^N)$ the norm $\|u\|_{s,p}$ is equivalent to the norm

$$\left(\sum_{|\alpha|\leq s}\int_{\boldsymbol{R}^N}|\partial^\alpha u(x)|^p dx\right)^{1/p}.$$

Next, if $1<p<\infty$, $B^{1,p}(\boldsymbol{R}^{N-1})$ denotes the space of functions $\varphi\in L^p(\boldsymbol{R}^{N-1})$ such that

$$\iint_{\boldsymbol{R}^{N-1}\times\boldsymbol{R}^{N-1}}\frac{|\varphi(x+y)-2\varphi(x)+\varphi(x-y)|^p}{|y|^{N-1+p}}dydx<\infty .$$

The space $B^{1,p}(\boldsymbol{R}^{N-1})$ is a Banach space with norm

$$|\varphi|_{1,p}=\left(\int_{\boldsymbol{R}^{N-1}}|\varphi|^p dx+\iint_{\boldsymbol{R}^{N-1}\times\boldsymbol{R}^{N-1}}\frac{|\varphi(x+y)-2\varphi(x)+\varphi(x-y)|^p}{|y|^{N-1+p}}dydx\right)^{1/p}.$$

If $s\in\boldsymbol{R}$, $B^{s,p}(\boldsymbol{R}^{N-1})$ denotes the image of $B^{1,p}(\boldsymbol{R}^{N-1})$ under the mapping J^{s-1} on $\boldsymbol{R}^{N-1}$. The space $B^{s,p}(\boldsymbol{R}^{N-1})$ is a Banach space with norm $|\varphi|_{s,p}=|J^{-s+1}\varphi|_{1,p}$.

Now, we define $H^{s,p}(\Omega)$ and $B^{s,p}(\Gamma)$. $H^{s,p}(\Omega)$ denotes the space of restrictions to Ω of functions in $H^{s,p}(\boldsymbol{R}^N)$. The space $H^{s,p}(\Omega)$ has the norm $\|u\|_{s,p}=\inf\|U\|_{s,p}$ where the infimum is taken over all $U\in H^{s,p}(\boldsymbol{R}^N)$ which equal u in Ω. Let $\{O_j\}$ be any open covering over Γ such that there is a C^∞-homeomorphism Φ_j $(j=1,2,\cdots)$ which maps $\{x'\in\boldsymbol{R}^{N-1};|x'|<1\}$ onto O_j, and let $\{g_j\}$ be any partition of the unity subordinate to the covering $\{O_j\}$, that is, $g_j\in C^\infty(\Gamma)$, $0\leq g_j\leq 1$ on Γ, $\operatorname{supp}g_j\subset O_j$ and $\sum_j g_j\equiv 1$ on Γ. $B^{s,p}(\Gamma)$ denotes the space of functions φ on Γ such that $(g_j\varphi)\circ\Phi_j\in B^{s,p}(\boldsymbol{R}^{N-1})$ for all j. The space $B^{s,p}(\Gamma)$ has the norm $|\varphi|_{s,p}=\sum_j|(g_j\varphi)\circ\Phi_j|_{s,p}$. We remark that since Ω is a bounded domain in $\boldsymbol{R}^N$, the norm $|\varphi|_{s,p}$ defined above is always finite. $H^{s,p}_{\text{loc}}(\Omega)$ denotes the space of distributions $u\in\mathscr{D}'(\Omega)$ such that $\psi u\in H^{s,p}(\boldsymbol{R}^N)$ for all $\psi\in C_0^\infty(\Omega)$. $B^{s,p}_{\text{loc}}(\Omega)$ denotes the space of distributions $\varphi\in\mathscr{D}'(\Omega)$ such that $\psi\varphi\in B^{s,p}(\boldsymbol{R}^N)$ for all $\psi\in C_0^\infty(\Omega)$. We equip the space $H^{s,p}_{\text{loc}}(\Omega)$ with the topology defined by the seminorms $u\mapsto\|\varphi u\|_{s,p}$ as φ range over $C_0^\infty(\Omega)$. It is easy to verify that $H^{s,p}_{\text{loc}}(\Omega)$ is a Fréchet space. Similarly, the space $B^{s,p}_{\text{loc}}(\Omega)$ can be topologized.

In addition, we recall a relation between $H^{s,p}(\Omega)$ and $B^{s,p}(\Gamma)$. It is well known (cf. [11], [12]) that the trace map

$$\begin{array}{rccc}\rho: & H^{s,p}(\Omega) & \longrightarrow & B^{s-1/p,p}(\Gamma)\\ & u & \longmapsto & u|_\Gamma .\end{array}$$

is well-defined, continuous and surjective for $s>1/p$.

Finally we introduce a subspace of $B^{s,p}(\Gamma)$ associated with the boundary condition L. If $a(x')^2+b(x')^2>0$ on Γ, we let

$$B^{s,p}_{(a,b)}(\Gamma)=\{\varphi=a\varphi_1+b\varphi_2\ ;\ \varphi_1\in B^{s,p}(\Gamma),\ \varphi_2\in B^{s+1,p}(\Gamma)\}$$

and

$$|\varphi|_{a,b;s,p}=\inf\{|\varphi_1|_{s,p}+|\varphi_2|_{s+1,p}\ ;\ \varphi=a\varphi_1+b\varphi_2\}\ .$$

Then we can check that the quantity $|\cdot|_{a,b;s,p}$ is a norm in the space $B^{s,p}_{(a,b)}(\Gamma)$ and that the space $B^{s,p}_{(a,b)}(\Gamma)$ is a Banach space with respect to the $|\cdot|_{a,b;s,p}$ norm. We remark that

$$B^{s,p}_{(a,b)}(\Gamma)=B^{s+1,p}(\Gamma) \qquad \text{if} \quad a\equiv 0 \text{ and } b>0 \text{ on } \Gamma\ ,$$

$$B^{s,p}_{(a,b)}(\Gamma)=B^{s,p}(\Gamma) \qquad \text{if} \quad a>0 \text{ on } \Gamma$$

and

$$B^{s+1,p}(\Gamma)\subset B^{s,p}_{(a,b)}(\Gamma)\subset B^{s,p}(\Gamma)\ ,$$

with continuous injections.

2.2. Pseudo-differential operators. Let Ω be an open subset of $\boldsymbol{R}^N$. If $m\in\boldsymbol{R}$ and $0\le\delta<\rho\le 1$, $S^m_{\rho,\delta}(\Omega\times\boldsymbol{R}^N)$ denotes the space of all functions $a\in C^\infty(\Omega\times\boldsymbol{R}^N)$ with the property that, for any compact $K\subset\Omega$ and multi-indices α, β, there exists a constant $C_{K,\alpha,\beta}>0$ such that we have for all $x\in K$ and $\theta\in\boldsymbol{R}^N$

$$|\partial^\alpha_\theta\partial^\beta_x a(x,\theta)|\le C_{K,\alpha,\beta}(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|}\ .$$

We set $S^{-\infty}(\Omega\times\boldsymbol{R}^N)=\bigcap_{m\in\boldsymbol{R}}S^m_{\rho,\delta}(\Omega\times\boldsymbol{R}^N)$. A symbol $a(x,\theta)\in S^m_{1,0}(\Omega\times\boldsymbol{R}^N)$ is said to be classical if there exist C^∞-functions $a_j(x,\theta)$, positively homogeneous of degree $m-j$ in θ for $|\theta|\ge 1$, such that

$$a\sim\sum_{j=0}^{\infty}a_j\ .$$

Here $a\sim\sum_{j=0}^\infty a_j$ means that for any $k>0$, there exists $b_k\in S^{m-k}_{1,0}(\Omega\times\boldsymbol{R}^N)$ such that $a=a_0+a_1+\cdots+a_{k-1}+b_k$. The homogeneous function a_0 of degree m is called the principal part of a. $S^m_{cl}(\Omega\times\boldsymbol{R}^N)$ denotes the set of all classical symbols of order m.

We call an operator A of the following form a pseudo-differential operator of order m on Ω:

$$Au(x)=\iint_{\Omega\times\boldsymbol{R}^N}e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi\ , \qquad u\in C^\infty_0(\Omega)\ ,$$

with some $a\in S^m_{\rho,\delta}(\Omega\times\Omega\times\boldsymbol{R}^N)$. Let $L^m_{\rho,\delta}(\Omega)$ be the set of all pseudo-differential operators of order m on Ω. We set $L^{-\infty}(\Omega)=\bigcap_{m\in\boldsymbol{R}}L^m_{\rho,\delta}(\Omega)$. We remark that if $p(x,\xi)\in S^m_{\rho,\delta}(\Omega\times\boldsymbol{R}^N)$, then the operator $p(x,D)$:

$$p(x,D)u(x)=\frac{1}{(2\pi)^N}\int_{\boldsymbol{R}^N}e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi\ , \qquad u\in C^\infty_0(\Omega)\ ,$$

is a pseudo-differential operator of order m on Ω, that is, $p(x,D)\in L^m_{\rho,\delta}(\Omega)$.

We recall that a continuous linear operator $A: C_0^\infty(\Omega) \to \mathscr{D}'(\Omega)$ is said to be properly supported if the following two conditions are satisfied:

(1) For any compact subset K of Ω, there exists a compact subset K' of Ω such that if $\operatorname{supp} v \subset K$, then we have $\operatorname{supp} Av \subset K'$.

(2) For any compact subset K' of Ω, there exists a compact subset K of Ω such that if $\operatorname{supp} v \cap K = \emptyset$, then we have $\operatorname{supp} Av \cap K' = \emptyset$. If A is properly supported, then it maps $C_0^\infty(\Omega)$ continuously into $\mathscr{E}'(\Omega)$, and further it extends to a continuous linear operator on $C^\infty(\Omega)$ into $\mathscr{D}'(\Omega)$. Let $A \in L^m_{\rho,\delta}(\Omega)$. A properly supported $B \in L^{-m}_{\rho,\delta}(\Omega)$ is called a parametrix for A if the operator B satisfies

$$\begin{cases} AB \equiv I & \bmod\ L^{-\infty}(\Omega)\,, \\ BA \equiv I & \bmod\ L^{-\infty}(\Omega)\,. \end{cases}$$

Finally we cite the following two theorems which will be used in the sequel.

THEOREM 2.1 (Bourdaud [3], Theorem 1). *Let $\Omega \subset \boldsymbol{R}^N$. Every properly supported operator $A \in L^m_{1,\delta}(\Omega)$, $0 \le \delta < 1$, extends to continuous linear operators*

$$A: H^{s,p}_{\mathrm{loc}}(\Omega) \longrightarrow H^{s-m,p}_{\mathrm{loc}}(\Omega)\,,$$

$$A: B^{s,p}_{\mathrm{loc}}(\Omega) \longrightarrow B^{s-m,p}_{\mathrm{loc}}(\Omega)\,,$$

for all $s \in \boldsymbol{R}$.

THEOREM 2.2 (Hörmander [4], Theorem 4.2). *Let $\Omega \subset \boldsymbol{R}^N$, $1-\rho \le \delta < \rho \le 1$ and let $A = p(x, D) \in L^m_{\rho,\delta}(\Omega)$ be properly supported. Assume that, for any compact $K \subset \Omega$ and any multi-indices α, β, there exist constants $C_{K,\alpha,\beta} > 0$, $C_K > 0$ and $\mu \in \boldsymbol{R}$ such that we have for all $x \in K$ and $|\xi| \ge C_K$,*

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \le C_{K,\alpha,\beta} |p(x, \xi)| (1 + |\xi|)^{-\rho|\alpha| + \delta|\beta|}\,,$$

$$|p(x, \xi)|^{-1} \le C_K (1 + |\xi|)^\mu\,.$$

Then, there exists a parametrix $B \in L^\mu_{\rho,\delta}(\Omega)$ for A.

REMARK. Even if Ω is an N-dimensional compact C^∞ manifold without boundary, it is known that Theorem 2.1 is valid, and that Theorem 2.2 is also valid by virtue of the condition that $1-\rho \le \delta < \rho \le 1$.

§3. Proof of Theorem 1. Part 1.

In this section, after we reduce Problem $(*)$ to the boundary, we shall prove the injectivity of the operator $\mathscr{A}$. If $s \ge 2$, it is well known that the Dirichlet problem:

$$\begin{cases} Aw = 0 & \text{in } \Omega\,, \\ w = \varphi & \text{on } \Gamma\,, \end{cases}$$

has a unique solution $w\in H^{s,p}(\Omega)$ for any $\varphi\in B^{s-1/p,p}(\Gamma)$. Letting $w=P\varphi$, the operator $P: B^{s-1/p,p}(\Gamma)\to H^{s,p}(\Omega)$ is called the Poisson operator. Moreover, according to Seely [10] (Theorems 5 and 6), the operator P maps $B^{s-1/p,p}(\Gamma)$ isomorphically onto $N(A, s, p)=\{w\in H^{s,p}(\Omega)\,;\, Aw=0\}$, the trace on Γ of elements in $N(A, s, p)$ can be defined and the inverse of P is the restriction map; $w\mapsto w|_\Gamma$.

Let T be

$$\begin{aligned} T\colon C^\infty(\Gamma) &\longrightarrow C^\infty(\Gamma)\\ \varphi &\longmapsto LP\varphi\,. \end{aligned}$$

Then we have

$$T=a\Pi+b\,,$$

where $\Pi\varphi=(\partial/\partial\nu)(P\varphi)|_\Gamma$. We remark that the operator Π is a classical pseudo-differential operator of first order on the boundary Γ. By Theorem 2.1, the operator T can be extended to a continuous linear operator:

$$T\colon B^{t,p}(\Gamma)\longrightarrow B^{t-1,p}(\Gamma)$$

for all $t\in\boldsymbol{R}$.

Now we reduce Problem (∗) to the study of the operator T on Γ. We have the equivalence of the regularity of $\mathscr{A}$ with that of T as follows (cf. [15], Theorem 3.9):

THEOREM 3.1. *For any $s\geq 2$, the following two assertions are equivalent*:

(1) *The condition that $u\in L^p(\Omega)$, $Au\in H^{s-2,p}(\Omega)$ and $Lu\in B^{s-1-1/p,p}_{(a,b)}(\Gamma)$ implies that $u\in H^{s,p}(\Omega)$.*

(2) *The condition that $\varphi\in B^{-1/p,p}(\Gamma)$ and $T\varphi\in B^{s-1/p,p}(\Gamma)$ implies that $\varphi\in B^{s-1/p,p}(\Gamma)$.*

PROOF. First we show that the assertion (1) implies the assertion (2). We let $\varphi\in B^{-1/p,p}(\Gamma)$ and $T\varphi\in B^{s-1/p,p}(\Gamma)$. And we set $u=P\varphi$. Then, we have that $u\in L^p(\Omega)$, $Au=0$ in Ω and $Lu=LP\varphi=T\varphi\in B^{s-1/p,p}(\Gamma)$. By the assertion (1) we obtain that $P\varphi\in H^{s,p}(\Omega)$. It follows that $\varphi\in B^{s-1/p,p}(\Gamma)$.

Next, we show that the assertion (2) implies the assertion (1). We let $u\in L^p(\Omega)$, $Au\in H^{s-2,p}(\Omega)$ and $Lu=a\varphi_1+b\varphi_2\in B^{s-1-1/p,p}_{(a,b)}(\Gamma)$ where $\varphi_1\in B^{s-1-1/p,p}(\Gamma)$ and $\varphi_2\in B^{s-1/p,p}(\Gamma)$. Combining Theorems 3.1 and 3.3 of [15], we have that the Neumann problem:

$$\begin{cases} Av=Au & \text{in} \quad \Omega\,,\\ \dfrac{\partial v}{\partial\nu}=\varphi_1 & \text{on} \quad \Gamma\,, \end{cases}$$

has a unique solution $v\in H^{s,p}(\Omega)$. Let $w=u-v$. We obtain that $w\in L^p(\Omega)$ and $Aw=0$ in Ω. It follows that there exists $\varphi\in B^{-1/p,p}(\Gamma)$ such that $P\varphi=w$. But we have by the assertion (2) that $\varphi \in B^{s-1/p,p}(\Gamma)$, since $T\varphi = LP\varphi = Lw = Lu - Lv = a\varphi_1 + b\varphi_2 - a(\partial v/\partial \nu) - bv = b(\varphi_2 - v) \in B^{s-1/p,p}(\Gamma)$. Hence we obtain that $u = w + v = P\varphi + v \in H^{s,p}(\Omega)$.

The proof of Theorem 3.1 is now complete.

Next, about properties of the operator Π, we have the following:

THEOREM 3.2. (1) *The operator Π is strongly elliptic, that is, we have, for all $\varphi \in C^\infty(\Gamma)$,*

$$\text{(3.1)} \qquad \operatorname{Re} \int_\Gamma \Pi\varphi \cdot \bar{\varphi} d\sigma \geq c_1 |\varphi|^2_{1/2,2} - c_2 |\varphi|^2_{-1/2,2},$$

with some constants $c_1 > 0$ and $c_2 > 0$. Here $d\sigma$ is the surface element on Γ.

(2) *Let $p_1(x', \xi') + \sqrt{-1}\, q_1(x', \xi')$ be the principal symbol of the operator Π. Then, we have*

$$\text{(3.2)} \qquad p_1(x', \xi') \geq c_0 |\xi'|$$

on the bundle $T^*(\Gamma) \setminus \{0\}$ of non-zero cotangent vectors with some constant $c_0 > 0$. Here $|\xi'|$ is the length of ξ' with respect to the Riemannian metric of Γ induced by the natural metric of $\boldsymbol{R}^N$.

PROOF. (1) By the divergence theorem, for $u \in C^\infty(\bar{\Omega})$, we have

$$\begin{aligned} -\operatorname{Re} \iint_\Omega Au \cdot \bar{u} dx = &\iint_\Omega \sum_{i,j=1}^N a^{ij}(x) \frac{\partial u}{\partial x_i}(x) \cdot \overline{\frac{\partial u}{\partial x_j}(x)} dx \\ &- \operatorname{Re} \int_\Gamma \frac{\partial u}{\partial \nu}(x') \cdot \overline{u(x')} d\sigma \\ &+ \iint_\Omega \left(\frac{1}{2} \sum_{j=1}^N \frac{\partial b^j}{\partial x_j}(x) - c(x) \right) |u(x)|^2 dx \\ &- \frac{1}{2} \int_\Gamma \sum_{j=1}^N b^j(x') \cdot n_j |u(x')|^2 d\sigma. \end{aligned}$$

Taking $u = P\varphi$ where $\varphi \in C^\infty(\Gamma)$, we have

$$\begin{aligned} \operatorname{Re} \int_\Gamma \frac{\partial}{\partial \nu}(P\varphi) \cdot \bar{\varphi} d\sigma = &\iint_\Omega \sum_{i,j=1}^N a^{ij} \cdot \frac{\partial}{\partial x_i}(P\varphi) \cdot \overline{\frac{\partial}{\partial x_j}(P\varphi)} dx \\ &+ \iint_\Omega \left(\frac{1}{2} \sum_{j=1}^N \frac{\partial b^j}{\partial x_j}(x) - c(x) \right) |P\varphi|^2 dx \\ &- \frac{1}{2} \int_\Gamma \sum_{j=1}^N b^j(x') \cdot n_j |\varphi|^2 d\sigma. \end{aligned}$$

Letting

$$K=\max\left\{0,\ \sup_{\Omega}\left(c(x)-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial b^j}{\partial x_j}(x)\right)\right\}$$

and

$$c_3=\max\left\{1,\ \sup_{\Gamma}\sum_{j=1}^{N} b^j(x')n_j\right\},$$

we have

$$\begin{aligned}\mathrm{Re}\int_{\Gamma}\Pi\varphi\cdot\bar{\varphi}d\sigma &\geq a_0\iint_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}(P\varphi)\right|^2 dx-K\iint_{\Omega}|P\varphi|^2dx-\frac{c_3}{2}\int_{\Gamma}|\varphi|^2d\sigma\\ &=a_0\|P\varphi\|_{1,2}^2-(a_0+K)\|P\varphi\|_{0,2}^2-\frac{c_3}{2}\int_{\Gamma}|\varphi|^2d\sigma\,.\end{aligned}$$

From Theorem 3.16 of Mizohata [8], we obtain that for any $\varepsilon>0$, there exists a constant $C(\varepsilon)>0$ such that

$$\int_{\Gamma}|\varphi(x')|^2d\sigma\leq\varepsilon\|P\varphi\|_{1,2}^2+C(\varepsilon)\|P\varphi\|_{0,2}^2\,.$$

Hence, taking $\varepsilon=a_0/c_3$, we have

$$\mathrm{Re}\int_{\Gamma}\Pi\varphi\cdot\bar{\varphi}d\sigma\geq\frac{a_0}{2}\|P\varphi\|_{1,2}^2-\left(a_0+K+\frac{c_3C(\varepsilon)}{2}\right)\|P\varphi\|_{0,2}^2\,.$$

Since the operator P maps $B^{t-1/2,2}(\Gamma)$ isomorphically onto $N(A, t, 2)$ for all $t\in\boldsymbol{R}$, we have

$$\mathrm{Re}\int_{\Gamma}\Pi\varphi\cdot\bar{\varphi}d\sigma\geq c_1|\varphi|_{1/2,2}^2-c_2|\varphi|_{-1/2,2}^2$$

with some constants c_1 and $c_2>0$.

(2) It is known (cf. [5]) that the inequality (3.1) implies the strong ellipticity (3.2) of the operator Π.

The proof of Theorem 3.2 is now complete.

Using Theorem 3.2, we shall prove the hypoellipticity of the operator T (see [15], Lemma 4.1):

THEOREM 3.3. *Let* $1<p<\infty$. *Assume that the conditions* (A.1) *and* (A.2) *are satisfied. Then, for any distribution* φ *on* Γ, *the condition that* $T\varphi\in B^{s,p}(\Gamma)$ *implies that* $\varphi\in B^{s,p}(\Gamma)$ *for any* $s\in\boldsymbol{R}$. *And further we have, for* $t<s$,

$$|\varphi|_{s,p}\leq C_{s,t}(|T\varphi|_{s,p}+|\varphi|_{t,p})$$

with some constant $C_{s,t}>0$.

In order to prove Theorem 3.3, we introduce some lemmas. The first lemma is (cf. [15], Lemma 4.2)

LEMMA 3.4. *Let* $t(x', \xi')$ *be a symbol of the operator* $T=a\Pi+b$. *Assume that the conditions* (A.1) *and* (A.2) *are satisfied. Then, for any* $x'\in\Gamma$, *there exists a neighborhood* $U(x')$ *of* x' *such that, for any compact subset* $K\subset U(x')$ *and any multi-indices* α, β, *we have for* $x'\in K$ *and* $|\xi'|\geq C_k$

$$|\partial_{\xi'}^{\alpha}\partial_{x'}^{\beta}t(x', \xi')|\leq C_{K,\alpha,\beta}|t(x', \xi')|(1+|\xi'|)^{-|\alpha|+(1/2)|\beta|}, \tag{3.3}$$

$$|t(x', \xi')|^{-1}\leq C_k \tag{3.4}$$

with some constants $C_{K,\alpha,\beta}$ *and* $C_K>0$.

PROOF. First we prove the inequality (3.4). By Theorem 3.2, the operator Π has the symbol

$$p_1(x', \xi')+\sqrt{-1}q_1(x', \xi')+p_0(x', \xi')+\sqrt{-1}q_0(x', \xi')+\text{terms of order } \leq -1,$$

where $p_j, q_j\in S_{1,0}^j(\Gamma)$ $(j=0, 1)$ and $p_1(x', \xi')\geq c_0|\xi'|$ on $T^*(\Gamma)\setminus\{0\}$. It follows that

$$\begin{aligned}t(x', \xi')=&a(x')[p_1(x', \xi')+\sqrt{-1}q_1(x', \xi')]\\&+\{b(x')+a(x')[p_0(x', \xi')+\sqrt{-1}q_0(x', \xi')]\}\\&+\text{terms of order } \leq -1.\end{aligned}$$

If $a(x')>0$, we can take $|\xi'|$ enough large such that

$$\begin{aligned}|t(x', \xi')|&\geq|a(x')[p_1(x', \xi')+p_0(x', \xi')]+b(x')|\\&\geq|a(x')[p_1(x', \xi')+p_0(x', \xi')]|-b(x')\\&\geq\frac{c_0}{2}a(x')|\xi'|-b(x')\\&\geq\frac{c_0}{4}a(x')|\xi'|+1.\end{aligned}$$

If $a(x')=0$, for $|\xi'|$ enough large, we have

$$|t(x', \xi')|\geq b(x'). \tag{3.5}$$

Noting that the set $\Gamma_0=\{x'\in\Gamma; a(x')=0\}$ is compact, it follows that

$$|t(x', \xi')|\geq C\{a(x')|\xi'|+1\} \qquad \text{for enough large} \quad |\xi'|.$$

Here and in the sequel, the letter C denotes a generic positive constant. This proves the inequality (3.4).

Secondly we verify the inequality (3.3). When $|\alpha|=1$ and $|\beta|=0$, we can take $|\xi'|$ enough large such that

$$|\partial_{\xi'}^{\alpha} t(x', \xi')| \leq C(a(x') + |\xi'|^{-1})$$
$$\leq C(1+|\xi'|)^{-1}(a(x')|\xi'|+1).$$

By the inequality (3.5), we have

$$|\partial_{\xi'}^{\alpha} t(x', \xi')| \leq C\, t(x', \xi')|(1+|\xi'|)^{-1}.$$

This proves the inequality (3.3) with $|\alpha|=1$ and $|\beta|=0$.

When $|\alpha|=0$ and $|\beta|=1$, we need the following lemma to prove the inequality (3.3).

LEMMA 3.5. *Let f be a non-negative C^2-function of $\boldsymbol{R}$ such that*

$$\sup_{x \in \boldsymbol{R}} |f''(x)| \leq c_4,$$

with some constant $c_4>0$. Then, we have

$$|f'(x)| \leq \sqrt{2c_4}\sqrt{f(x)} \qquad on \quad \boldsymbol{R}.$$

Now we continue the proof of Lemma 3.4. For $|\xi'|$ enough large, we have

$$|\partial_{x'}^{\beta} t(x', \xi')| \leq C(|\partial_{x'}^{\beta} a(x')||\xi'| + a(x')|\xi'|+1).$$

From Lemma 3.5 and the inequality (3.5), it follows that

$$|\partial_{x'}^{\beta} t(x', \xi')| \leq C(\sqrt{a(x')}|\xi'| + a(x')|\xi'|+1)$$
$$\leq C\{|\xi'|^{1/2}(a(x')|\xi'|+1)^{1/2} + (a(x')|\xi'|+1)\}$$
$$\leq C|t(x', \xi')|(|\xi'|^{1/2}|t(x', \xi')|^{-1/2}+1)$$
$$\leq C|t(x', \xi')|(1+|\xi'|)^{1/2}.$$

This proves the inequality (3.3) with $|\alpha|=0$ and $|\beta|=1$.

For general case that $|\alpha|+|\beta|=k$ $(k=2, 3, \cdots)$, we can similarly prove the inequality (3.3). The proof of Lemma 3.4 is complete.

Using Lemma 3.4, we prove Theorem 3.3.

PROOF OF THEOREM 3.3. First we remark that we can have a finite number of local charts $\{(U_j, \chi_j)\}_{j=1}^m$ of the boundary Γ such that the estimates (3.3) and (3.4) hold in each of them. One can check that the operator T satisfies the conditions of Theorem 2.2 with $\mu=0$, $\rho=1$ and $\delta=1/2$. So, by Theorem 2.2 there exists a parametrix $S \in L_{1,1/2}^0(U_j)$ of T. Let $\{\varphi_j\}_{j=1}^m$ be a partition of unity subordinate to the covering $\{U_j\}_{j=1}^m$, and let $\psi_j \in C_0^\infty(U_j)$ be such that $\psi_j=1$ on supp φ_j. Then, the operator T is decomposed as follows:

$$T\varphi=\sum_{j=1}^{m}\varphi_j T\varphi=\sum_{j=1}^{m}\varphi_j\{T\psi_j\varphi+T(1-\psi_j)\varphi\}$$
$$=\sum_{j=1}^{m}\varphi_j T\psi_j\varphi+\sum_{j=1}^{m}\varphi_j T(1-\psi_j)\varphi\ .$$

The second term of the right-hand side belongs to $L^{-\infty}(\Gamma)$ since $\varphi_j(1-\psi_j)=0$. The proof of Theorem 3.3 is reduced to those of the following assertions:

(1) If $\psi_j\varphi\in B^{t,p}(U_j)$ and $T\psi_j\varphi\in B^{s,p}(U_j)$, then we have $\psi_j\varphi\in B^{s,p}(U_j)$.

(2) $|\psi_j\varphi|_{s,p}\leq C_{s,t}(|T\psi_j\varphi|_{s,p}^2+|\psi_j\varphi|_{t,p}^2)$ with some constant $C_{s,t}>0$.

We shall prove the assertion (1). Since the operator T has the parametrix S, we obtain that

$$ST\psi_j\varphi=\psi_j\varphi+R\psi_j\varphi\ ,$$

where $R\in L^{-\infty}(U_j)$. Theorem 2.1 tells us that the operator S maps $B^{s,p}_{\mathrm{loc}}(\Gamma)$ continuously into itself for all $s\in\boldsymbol{R}$. Hence it follows that $\psi_j\varphi\in B^{s,p}(U_j)$. And further we have the inequality (2). The proof of Theorem 3.3 is complete.

Combining Theorem 3.1 and Theorem 3.3, we have the following:

THEOREM 3.6. *Let* $1<p<\infty$. *Assume that the conditions* (A.1) *and* (A.2) *are satisfied. Then, for any* $s\geq 2$ *the condition that* $u\in L^p(\Omega)$, $Au\in H^{s-2,p}(\Omega)$ *and* $Lu\in B^{s-1-1/p,p}_{(a,b)}(\Gamma)$ *implies that* $u\in H^{s,p}(\Omega)$.

Now, we shall show the injectivity of the operator $\mathscr{A}$. For any $s\geq 2$, let $u\in H^{s,p}(\Omega)$, and suppose that

(3.6) $$Au=0 \quad \text{in} \quad \Omega$$

and

(3.7) $$Lu=0 \quad \text{on} \quad \Gamma\ .$$

Then, we shall show that $u\equiv 0$ in Ω. First we obtain by Theorem 3.6 that $u\in H^{t,p}(\Omega)$ for all $t\geq 2$. Hence, using the Sobolev imbedding theorem, we have $u\in C^\infty(\bar{\Omega})$. We suppose that we do not have $u(x)=0$ for all $x\in\bar{\Omega}$. Then, by replacing u by $-u$ if $u(x)\leq 0$ in $\bar{\Omega}$, we have only to consider the two cases:

(i) $\max_{\bar{\Omega}} u=u(x_0)>0$ at some $x_0\in\Omega$.

(ii) $\max_{\bar{\Omega}} u=u(x_1')>0$ at some $x_1'\in\Gamma$.

If $\max_{\bar{\Omega}} u=u(x_0)>0$, it follows from the strong maximum principle (cf. [9], Chap. 2, Sec. 3, Theorem 6) that $u(x)\equiv u(x_0)$, $x\in\Omega$. By (3.6) and the condition (A.3), we obtain that $0=Au(x)=c(x)u(x)<0$. This leads to a contradiction.

If $\max_{\bar\Omega} u = u(x'_1) > 0$, it follows from the boundary point lemma (cf. [9], Chap. 2, Sec. 3, Theorem 8) that $(\partial u/\partial \nu)(x'_1) > 0$. By (3.7) we have that $0 = Lu(x'_1) = a(x'_1)(\partial u/\partial \nu)(x'_1) + b(x'_1)u(x_1)$. On the other hand, if $a(x_1) > 0$, then $a(x'_1)(\partial u/\partial \nu)(x'_1) + b(x'_1)u(x'_1) > 0$. If $a(x'_1) = 0$ then, $a(x'_1)(\partial u/\partial \nu)(x'_1) + b(x'_1)u(x'_1) = b(x'_1)u(x'_1) > 0$. These assertions also lead to a contradiction. Hence, we have $u \equiv 0$ in $\bar\Omega$.

We have finished the proof of the injectivity of the operator $\mathscr{A}$.

§4. Proof of Theorem 1. Part 2.

In this section we shall prove the surjectivity and continuity of the operator $\mathscr{A}$. We consider the surjectivity of the pseudo-differential operator T to which we reduce Problem $(*)$ on the boundary in Section 3. We recall the operator T:

$$\begin{aligned} T\colon B^{s-1/p,p}(\Gamma) &\longrightarrow B^{s-1-1/p,p}(\Gamma)\,, \qquad s \geq 2\,, \\ \varphi \quad &\longmapsto \quad a\Pi\varphi + b\varphi\,, \end{aligned}$$

where $\Pi\varphi = (\partial/\partial\nu)(P\varphi)|_\Gamma$. We introduce an operator $\mathscr{T}$ in $B^{s-1/p,p}(\Gamma)$ $(s \geq 2)$ associated with T as follows:

(a) The domain $\mathscr{D}(\mathscr{T})$ of $\mathscr{T}$ is the space

$$\mathscr{D}(\mathscr{T}) = \{\varphi \in B^{s-1/p,p}(\Gamma)\ ;\ T\varphi \in B^{s-1/p,p}(\Gamma)\}\,.$$

(b) $\mathscr{T}\varphi = T\varphi$, $\varphi \in \mathscr{D}(\mathscr{T})$.

We remark that the operator $\mathscr{T}$ is densely defined, and closed in $B^{s-1/p,p}(\Gamma)$.

Next, instead of Problem $(*)$, we consider the following problem:

$$(*)_\lambda \qquad \begin{cases} (A-\lambda)u = f & \text{in } \Omega\,, \\ Lu = \varphi & \text{on } \Gamma\,, \end{cases}$$

where $\lambda \geq 0$. We associate with Problem $(*)_\lambda$ a linear operator $\mathscr{A}_\lambda$:

$$\begin{aligned} \mathscr{A}_\lambda\colon H^{s,p}(\Omega) &\longrightarrow H^{s-2,p}(\Omega) \times B^{s-1-1/p,p}_{(a,b)}(\Gamma) \\ u \quad &\longmapsto \quad ((A-\lambda)u,\ Lu)\,. \end{aligned}$$

We remark that the operator $\mathscr{A}_\lambda$ coincides with $\mathscr{A}$ when $\lambda = 0$. We reduce Problem $(*)_\lambda$ to the study of a pseudo-differential operator on the boundary similarly as in the case of Problem $(*)$. The Dirichlet problem:

$$\begin{cases} (A-\lambda)w = 0 & \text{in } \Omega\,, \\ w = \varphi & \text{on } \Gamma\,, \end{cases}$$

has a unique solution $w \in H^{t,p}(\Omega)$ for any $\varphi \in B^{t-1/p,p}(\Gamma)$ when $t \in \boldsymbol{R}$ and $\lambda \geq 0$. The Poisson operator $P(\lambda)$:

$$P(\lambda)\colon B^{t-1/p,p}(\Gamma) \longrightarrow H^{t,p}(\Omega)\,,$$

is given by the formula that $P(\lambda)\varphi = w$. Further the operator $P(\lambda)$ maps $B^{t-1/p,p}(\Gamma)$ isomorphically onto $N(A, t, p) = \{u \in H^{t,p}(\Omega)\,;\,(A-\lambda)u = 0 \text{ in } \Omega\}$ $(t \in \boldsymbol{R})$ and its inverse is the trace operator on Γ. We let

$$\begin{aligned} T(\lambda): C^\infty(\Gamma) &\longrightarrow C^\infty(\Gamma) \\ \varphi &\longmapsto LP(\lambda)\varphi\,, \qquad \lambda \geq 0\,. \end{aligned}$$

By Theorem 2.1, the operator $T(\lambda)$ is extended to a continuous linear operator $T(\lambda): B^{t-1/p,p}(\Gamma) \to B^{t-1-1/p,p}(\Gamma)$ $(t \in \boldsymbol{R})$. As in the case of the operator T, we introduce a linear operator $\mathcal{T}(\lambda)$ in $B^{s-1/p,p}(\Gamma)$ $(s \geq 2)$ as follows:

(a) The domain $\mathcal{D}(\mathcal{T}(\lambda))$ of $\mathcal{T}(\lambda)$ is the space

$$\mathcal{D}(\mathcal{T}(\lambda)) = \{\varphi \in B^{s-1/p,p}(\Gamma)\,;\, T(\lambda) \in B^{s-1/p,p}(\Gamma)\}\,.$$

(b) $\mathcal{T}(\lambda)\varphi = T(\lambda)\varphi$, $\varphi \in \mathcal{D}(\mathcal{T}(\lambda))$.

We remark that the operator $\mathcal{T}(\lambda)$ coincides with the operator $\mathcal{T}$ when $\lambda = 0$, and that the operator $\mathcal{T}(\lambda)$ is densely defined, closed in $B^{s-1/p,p}(\Gamma)$.

Further, introducing an auxiliary variable y of the unit circle

$$S = \boldsymbol{R}/2\pi\boldsymbol{Z}\,,$$

we consider the following problem $(\tilde{*})$substituted for Problem $(*)$:

$$(\tilde{*}) \qquad \begin{cases} \tilde{A}\tilde{u} \equiv (A + \partial^2/\partial y^2)\tilde{u} = \tilde{f} & \text{in} \quad \Omega \times S\,, \\ L\tilde{u} \equiv a\dfrac{\partial \tilde{u}}{\partial v} + b\tilde{u}\Big|_{\Gamma \times S} = \tilde{\varphi} & \text{on} \quad \Gamma \times S\,. \end{cases}$$

Similarly, we reduce Problem $(\tilde{*})$ to the boundary $\Gamma \times S$. The Dirichlet problem

$$\begin{cases} \tilde{A}\tilde{w} = 0 & \text{in} \quad \Omega \times S\,, \\ \tilde{w} = \tilde{\varphi} & \text{on} \quad \Gamma \times S\,, \end{cases}$$

has a unique solution $\tilde{w} \in H^{t,p}(\Omega \times S)$ for any $\tilde{\varphi} \in B^{t-1/p,p}(\Gamma \times S)$ $(t \in \boldsymbol{R})$. The operator $\tilde{P}: B^{t-1/p,p}(\Gamma \times S) \to H^{t,p}(\Omega \times S)$ is defined by the formula $\tilde{P}\tilde{\varphi} = \tilde{w}$. Further the operator $\tilde{P}$ maps $B^{t-1/p,p}(\Gamma \times S)$ isomorphically onto $N(\tilde{A}, t, p) = \{\tilde{u} \in H^{t,p}(\Omega \times S)\,;\, \tilde{A}\tilde{u} = 0$ in $\Omega \times S\}$ $(t \in \boldsymbol{R})$ and its inverse is the trace operator on $\Gamma \times S$. We let

$$\begin{aligned} \tilde{T}: C^\infty(\Gamma \times S) &\longrightarrow C^\infty(\Gamma \times S) \\ \tilde{\varphi} &\longmapsto L\tilde{P}\tilde{\varphi}\,. \end{aligned}$$

Then we can write

$$\tilde{T} = a\tilde{\Pi} + b\,,$$

where $\tilde{\Pi}\tilde{\varphi} = (\partial/\partial v)(\tilde{P}\tilde{\varphi})|_{\Gamma \times S}$. Then, the operator $\tilde{\Pi}$ is a classical pseudo-differential operator of first order on $\Gamma \times S$. The symbol of the operator $\tilde{\Pi}$ is given by the formula

$$[\tilde{p}_1(x', \xi', y, \eta)+\sqrt{-1}\,\tilde{q}_1(x', \xi', y, \eta)]$$
$$+[\tilde{p}_0(x', \xi', y, \eta)+\sqrt{-1}\,\tilde{q}_0(x', \xi', y, \eta)]$$
$$+\text{terms of order } \leq -1$$

where $\tilde{p}_1(x', \xi', y, \eta)\geq \tilde{c}_0(|\xi'|^2+\eta^2)^{1/2}$ on $T^*(\Gamma\times S)\setminus\{0\}$ with some constant $\tilde{c}_0>0$. Hence, we find that the operator $\tilde{T}=a\tilde{\Pi}+b$ is a classical pseudo-differential operator of first order on $\Gamma\times S$ and that the symbol $\tilde{t}(x', \xi', y, \eta)$ of $\tilde{T}$ is given by the formula

$$a(x')[\tilde{p}_1(x', \xi', y, \eta)+\sqrt{-1}\,\tilde{q}_1(x', \xi', y, \eta)]$$
$$+\{b(x')+a(x')[\tilde{p}_0(x', \xi', y, \eta)+\sqrt{-1}\,\tilde{q}_0(x', \xi', y, \eta)]\}$$
$$+\text{terms of order } \leq -1\,.$$

Further, by Theorem 2.1, the operator $\tilde{T}$ is extended to a continuous linear operator $\tilde{T}$: $B^{t-1/p,p}(\Gamma\times S)\to B^{t-1-1/p,p}(\Gamma\times S)$ $(t\in\boldsymbol{R})$. As in the case of T, we find that for $(x', y)\in\Gamma$ and multi-indices α, β there exist constants $\tilde{C}>0$ and $\tilde{C}'>0$ such that

$$|\partial^{\alpha}_{\xi',\eta}\partial^{\beta}_{x',y}\tilde{t}(x', \xi', y, \eta)|$$
$$\leq \tilde{C}|\tilde{t}(x', \xi', y, \eta)|(1+(|\xi'|^2+\eta^2)^{1/2})^{-|\alpha|+(1/2)|\beta|}\,,$$
$$|\tilde{t}(x', \xi', y, \eta)|^{-1}\leq \tilde{C}$$

for $(|\xi'|^2+\eta^2)^{1/2}\geq\tilde{C}'$. By noting Remark of Theorems 2.1 and 2.2, it follows from Theorems 2.1 and 2.2 that there exists a parametrix for $\tilde{T}$ in $L^0_{1,1/2}(\Gamma\times S)$, and that the parametrix is bounded in $B^{s,p}(\Gamma\times S)$ for all $s\in\boldsymbol{R}$. Hence it follows that the condition that $\tilde{\varphi}\in\mathscr{D}'(\Gamma\times S)$ and $\tilde{T}\tilde{\varphi}\in B^{s,p}(\Gamma\times S)$ implies that $\tilde{\varphi}\in B^{s,p}(\Gamma\times S)$ for all $s\in\boldsymbol{R}$, and that

$$|\tilde{\varphi}|_{s,p}\leq\tilde{C}_{s,t}(|\tilde{T}\tilde{\varphi}|_{s,p}+|\tilde{\varphi}|_{t,p})\,,\qquad t<s$$

with some constant $\tilde{C}_{s,t}>0$.

As in the case of the operator T, we introduce a linear operator $\tilde{\mathscr{T}}$ in $B^{s-1/p,p}(\Gamma\times S)$ $(s\geq 2)$ as follows:

(a) The domain $\mathscr{D}(\tilde{\mathscr{T}})$ of $\tilde{\mathscr{T}}$ is the space

$$\mathscr{D}(\tilde{\mathscr{T}})=\{\tilde{\varphi}\in B^{s-1/p,p}(\Gamma\times S)\ ;\ \tilde{T}\tilde{\varphi}\in B^{s-1/p,p}(\Gamma\times S)\}\,.$$

(b) $\tilde{\mathscr{T}}\tilde{\varphi}=\tilde{T}\tilde{\varphi}$, $\tilde{\varphi}\in\mathscr{D}(\tilde{\mathscr{T}})$.

We remark that the operator $\tilde{\mathscr{T}}$ is densely defined, closed in $B^{s-1/p,p}(\Gamma\times S)$.

Here, we recall that a densely defined, closed linear operator F from a Banach space X to itself is called a Fredholm operator if the following three conditions are satisfied:

(1) The null space $\mathscr{N}(F)$ of F has finite dimension.
(2) The range $\mathscr{R}(F)$ of F is closed in X.
(3) The range $\mathscr{R}(F)$ has finite codimension in X.

In this case, we introduce the index of F by the following formula:

$$\operatorname{ind} F = \dim \mathscr{N}(F) - \operatorname{codim} \mathscr{R}(F) .$$

Now, we have obtained that the operator $\mathscr{A}$ maps $H^{s,p}(\Omega)$ injectively into $H^{s-2,p}(\Omega) \times B^{s-1-1/p,p}_{(a,b)}(\Gamma)$, that is, we have $\dim \mathscr{N}(\mathscr{A}) = 0$. This implies that $\dim \mathscr{N}(\mathscr{T}) = 0$ from the definition of $\mathscr{T}$. Hence, the surjectivity of $\mathscr{A}$ follows from the following four assertions:

PROPOSITION 4.1. *$\operatorname{ind} \tilde{\mathscr{T}}$ is finite.*

PROPOSITION 4.2. *If $\operatorname{ind} \tilde{\mathscr{T}}$ is finite, then there exists a finite subset $K \subset \boldsymbol{Z}$ such that $\operatorname{ind} \mathscr{T}(\lambda') = 0$ for all $\lambda' = l^2$ with $l \in \boldsymbol{Z} \setminus K$.*

PROPOSITION 4.3. *$\operatorname{ind} \mathscr{T}(\lambda) = \operatorname{ind} \mathscr{T}(\mu)$ for all $\lambda, \mu \geq 0$.*

PROPOSITION 4.4. *If the operator $\mathscr{T}$ in $B^{s-1/p,p}(\Gamma)$ with domain $\mathscr{D}(\mathscr{T})$ is surjective, then the map $\mathscr{A}: H^{s,p}(\Omega) \to H^{s-2,p}(\Omega) \times B^{s-1-1/p,p}_{(a,b)}(\Gamma)$ is surjective.*

In fact, combining Propositions 4.1 and 4.2, we have that $\operatorname{ind} \mathscr{T}(\lambda') = 0$ for all $\lambda' = l^2$ satisfying $l \in \boldsymbol{Z} \setminus K$. Hence, it follows from Proposition 4.3 that $\operatorname{ind} \mathscr{T} = 0$. Since we have already obtained that $\dim \mathscr{N}(\mathscr{T}) = 0$, we have that $\operatorname{codim} \mathscr{R}(\mathscr{T}) = 0$, that is, the operator $\mathscr{T}$ is surjective. By Proposition 4.4, we find that the operator $\mathscr{A}$ is surjective.

We shall prove the above propositions. First we prove Proposition 4.1. We use the following result in the proof of Proposition 4.1.

LEMMA 4.5 (Peetre). *Let X, Y be Banach spaces, $X \subset Z$ a compact injection and F a closed linear operator from X into Y with domain $\mathscr{D}(F)$. Then, the following two assertions are equivalent:*

(i) *$\dim \mathscr{N}(F) < \infty$ and $\mathscr{R}(F)$ is closed in Y.*

(ii) *There exists a constant $C > 0$ such that*

$$\|x\|_X \leq C(\|Fx\|_Y + \|x\|_Z), \qquad x \in \mathscr{D}(F).$$

Applying Lemma 4.5 to $F = \tilde{\mathscr{T}}$, $X = Y = B^{s-1/p,p}(\Gamma \times S)$, $Z = B^{t,p}(\Gamma \times S)$, $t < s - 1/p$, we have $\dim \mathscr{N}(\tilde{\mathscr{T}}) < \infty$ and $\boldsymbol{R}(\tilde{\mathscr{T}})$ is closed in $B^{s-1/p,p}(\Gamma \times S)$.

We consider the adjoint operator $\tilde{\mathscr{T}}^*$ of $\tilde{\mathscr{T}}$. Let $\tilde{t}(x', \xi', y, \eta)^*$ be a symbol of $\tilde{T}^*$. Using a well known asymptotic expansion formula with regard to symbols of adjoint operators, we have

$$\tilde{t}(x', \xi', y, \eta)^* \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial^{\alpha}_{\xi', \eta} D^{\beta}_{x', y} (\overline{\tilde{t}(x', \xi', y, \eta)}),$$

where $D_{x',y} = -\sqrt{-1}\,\partial_{x',y}$. By an argument as in the case of $\tilde{T}$, we find that

$$|\tilde{\psi}|_{-s+1/p,p'} \leq \tilde{C}^*(|\tilde{T}\tilde{\psi}|_{-s+1/p,p'} + |\tilde{\psi}|_{t,p'}), \qquad \tilde{\psi} \in \mathscr{D}(\tilde{\mathscr{T}}^*)$$

with some constant $\tilde{C}^* > 0$, where $p' = p/(p-1)$ and $t < -s + 1/p$. By applying Lemma

4.5 to $F=\tilde{\mathscr{T}}^*$, $X=Y=B^{-s+1/p,p'}(\Gamma\times S)$, $Z=B^{t,p'}(\Gamma\times S)$, $t<-s+1/p$, and by the closed range theorem, it follows that

$$\operatorname{codim}\mathscr{R}(\tilde{\mathscr{T}})=\dim\mathscr{N}(\tilde{\mathscr{T}}^*)<\infty .$$

Hence we have

$$\operatorname{ind}\tilde{\mathscr{T}}=\dim\mathscr{N}(\tilde{\mathscr{T}})-\operatorname{codim}\mathscr{R}(\tilde{\mathscr{T}})<\infty ,$$

which implies that the proof of Proposition 4.1 is complete.

We can prove Propositions 4.2 and 4.3 similarly as in Section 8.4 of [14] and Corollary 5.3 of [13], respectively. Next, we prove Proposition 4.4.

Proof of Proposition 4.4. Let $f\in H^{s-2,p}(\Omega)$, $\varphi\in B^{s-1-1/p,p}_{(a,b)}(\Gamma)$ where $\varphi=a\varphi_1+b\varphi_2$ with $\varphi_1\in B^{s-1/p,p}(\Gamma)$ and $\varphi_2\in B^{s-1-1/p,p}(\Gamma)$. Combining Theorems 3.1 and 3.3 of [15], we have that the Neumann problem:

$$\begin{cases} Av=f & \text{in } \Omega , \\ \dfrac{\partial v}{\partial \nu}=\varphi_1 & \text{on } \Gamma , \end{cases}$$

has a unique solution $v\in H^{s,p}(\Omega)$. Since the operator $\mathscr{T}$ is surjective, there exists $\psi\in B^{s-1/p,p}(\Gamma)$ such that $\mathscr{T}\psi=b(v|_\Gamma-\varphi_2)$. We let $w=P\psi$. Then it follows that

$$\begin{cases} w\in H^{s,p}(\Omega) , & \\ Aw=0 & \text{in } \Omega , \\ Lw=b(v|_\Gamma-\varphi_2) & \text{on } \Gamma . \end{cases}$$

Let $u=v-w$. We have

$$\begin{cases} u\in H^{s,p}(\Omega) , & \\ Au=f & \text{in } \Omega , \\ Lu=a\varphi_1+b\varphi_2 & \text{on } \Gamma . \end{cases}$$

This proves that the mapping $\mathscr{A}$ is surjective from $H^{s,p}(\Omega)$ onto $H^{s-2,p}(\Omega)\times B^{s-1-1/p,p}_{(a,b)}(\Gamma)$. The proof of Proposition 4.4 is complete.

Finally, we can trivially prove that the operator $\mathscr{A}$ is continuous. Hence, by the closed graph theorem, we have that the inverse of $\mathscr{A}$ is continuous. We have obtained that the operator $\mathscr{A}$ is homeomorphic.

§5. Proof of Theorem 2.

(i) This section is devoted to the proof of Theorem 2. We start with the proof of the assertion (1.1). We use the method due to Taira ([13], Theorem 7.3). By Green's formula, we have that for $u, v \in C^2(\bar{\Omega})$

$$\iint_\Omega (Au \cdot \bar{v} - u \cdot \overline{A^* v})dx = -\int_\Gamma u\left\{\overline{\frac{\partial v}{\partial \nu}} - \left(\sum_{i=1}^N b^i n_i\right)\bar{v}\right\}d\sigma + \int_\Gamma \frac{\partial u}{\partial \nu} \cdot \bar{v} d\sigma .$$

Here $d\sigma$ is the surface element on Γ. If u and v satisfy the boundary conditions

$$a\frac{\partial u}{\partial \nu} + bu = 0 \qquad \text{on} \quad \Gamma ,$$

$$a\frac{\partial v}{\partial \nu} + \left\{b - a\left(\sum_{i=1}^N b^i n_i\right)\right\}v = 0 \qquad \text{on} \quad \Gamma ,$$

respectively, these conditions are represented as follows:

$$\begin{pmatrix} \dfrac{\partial u}{\partial \nu} & u \\ \overline{\dfrac{\partial v}{\partial \nu}} - \left(\sum_{i=1}^N b^i n_i\right)\bar{v} & \bar{v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{on} \quad \Gamma .$$

Since $(a, b) \neq (0, 0)$ on Γ, we have

$$\det \begin{vmatrix} \dfrac{\partial u}{\partial \nu} & u \\ \overline{\dfrac{\partial v}{\partial \nu}} - \left(\sum_{i=1}^N b^i n_i\right)\bar{v} & \bar{v} \end{vmatrix} = 0 \qquad \text{on} \quad \Gamma .$$

That is,

$$\frac{\partial u}{\partial \nu}\bar{v} - u\left\{\overline{\frac{\partial v}{\partial \nu}} - \left(\sum_{i=1}^N b^i n_i\right)\bar{v}\right\} = 0 \qquad \text{on} \quad \Gamma .$$

Hence we have

$$\langle Au, \bar{v}\rangle = \langle u, \overline{A^* v}\rangle \qquad \text{for} \quad u \in C^2(\bar{\Omega}) \cap \mathcal{D}(A_p), \quad v \in C^2(\bar{\Omega}) \cap \mathcal{D}(A_{p'}) .$$

Here $\langle\ ,\ \rangle$ denotes the pairing of $L^p(\Omega) \times L^{p'}(\Omega)$. Since the spaces $C^2(\bar{\Omega}) \cap \mathcal{D}(A_p)$ and $C^2(\bar{\Omega}) \cap \mathcal{D}(A_{p'})$ are dense in $\mathcal{D}(A_p)$ with norm $\|\cdot\|_{2,p}$ and $\mathcal{D}(A_{p'})$ with norm $\|\cdot\|_{2,p'}$, respectively, we obtain

$$\langle Au, \bar{v}\rangle = \langle u, \overline{A^* v}\rangle \qquad \text{for} \quad u \in \mathcal{D}(A_p) \quad \text{and} \quad v \in \mathcal{D}(A_{p'}) .$$

This proves that $A_{p'} \subset A_p^*$. Here A_p^* denotes the adjoint operator of A_p.

Next we shall show that $A_{p'} = A_p^*$. By virtue of the condition (A.4), we can apply Theorem 1 with $s = 2$ to $A_{p'}$ as well as A_p, so that it follows that there exists a constant

$\lambda<0$ such that

$$(\lambda+A_{p'}) : \mathscr{D}(A_{p'}) \longrightarrow L^{p'}(\Omega)\,,$$
$$(\lambda+A_p) : \mathscr{D}(A_p) \longmapsto L^p(\Omega)\,,$$

are both bijective. Therefore, for any $v\in\mathscr{D}(A_p^*)$, there exists $v_0\in\mathscr{D}(A_{p'})$ such that

$$(\lambda+A_{p'})v_0=(\lambda+A_p^*)v\,.$$

For any $u\in\mathscr{D}(A_p)$, we have

$$\langle(\lambda+A_p)u, \overline{v-v_0}\rangle=\langle u, \overline{(\lambda+A_{p'})v_0-(\lambda+A_p^*)v}\rangle=0\,.$$

Since the operator $\lambda+A_p$ is bijective, this implies that $v=v_0$. Hence we obtain that $A_{p'}=A_p^*$. The proof of the assertion (1.1) is complete.

Secondly we shall prove the assertion (1.2). Let $1<p<\infty$, and let λ_p be one of eigenvalues of A_p, and u_p an eigenfunction for the eigenvalue λ_p. We can write $A_pu_p=\lambda_pu_p$. This implies that

$$\begin{cases} u_p\in H^{2,p}(\Omega)\,, & \\ (A-\lambda_p)u_p=0 & \text{in} \quad \Omega\,, \\ Lu_p=0 & \text{on} \quad \Gamma\,. \end{cases}$$

By Theorem 3.6 we have $u_p\in H^{s,p}(\Omega)$ for all $s\geq 2$. From the Sobolev imbedding theorem, it follows that $u_p\in C^\infty(\bar\Omega)$. This implies that $u_p\in\mathscr{D}(A_q)$ for all $1<q<\infty$. This proves that the eigenfunctions are common to all operators A_p $(1<p<\infty)$, and further this implies that the eigenvalue λ_p of A_p is an eigenvalue of A_q. Hence, it follows that the eigenvalues are also common to all operators A_p $(1<p<\infty)$. We have finished the proof of the assertion (1.2).

Thirdly we shall prove the assertion (1.3). In view of the assertion (1.2), we shall show the assertion (1.3) under the situation that $p=2$. We shall prove that

$$\mathrm{Re}(A_2u, u)\leq K\|u\|_2^2\,, \qquad u\in\mathscr{D}(A_2)\,,$$

with some constant K. Here (,) is the inner product in $L^2(\Omega)$.

By the divergence theorem, for $u\in C^\infty(\bar\Omega)\cap\mathscr{D}(A_2)$, we have

$$\begin{aligned} -2\mathrm{Re}(A_2u, u)&=-\{(A_2u,u)+\overline{(A_2u, u)}\} \\ &=\iint_\Omega \sum_{i,j=1}^N a^{ij}(x)\frac{\partial u}{\partial x_i}(x)\cdot\overline{\frac{\partial u}{\partial x_j}}(x)dx+\iint_\Omega \sum_{i,j=1}^N a^{ij}(x)\overline{\frac{\partial u}{\partial x_i}}(x)\cdot\frac{\partial u}{\partial x_j}(x)dx \\ &\quad -\int_\Gamma\left\{\frac{\partial u}{\partial \nu}(x')\cdot\overline{u(x')}+\overline{\frac{\partial u}{\partial \nu}}(x')\cdot u(x')\right\}d\sigma+\iint_\Omega \sum_{j=1}^N \frac{\partial b^j}{\partial x_j}(x)|u(x)|^2dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{\Gamma}\sum_{j=1}^{N} b^j(x')n_j|u(x')|^2 d\sigma - 2\iint_{\Omega} c(x)|u(x)|^2 dx \\
&\geq -\int_{\Gamma}\left\{\frac{\partial u}{\partial \nu}(x')\cdot\overline{u(x')}+\overline{\frac{\partial u}{\partial \nu}(x')}\cdot u(x')+\sum_{j=1}^{N} b^j(x')n_j|u(x')|^2\right\}d\sigma \\
&\quad -2\iint_{\Omega}\left\{c(x)-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial b^j}{\partial x_j}(x)\right\}|u(x)|^2 dx .
\end{aligned}
$$

We estimate the first term of the right-hand side as follows.

$$
\begin{aligned}
&-\int_{\Gamma}\left\{\frac{\partial u}{\partial \nu}(x')\cdot\overline{u(x')}+\overline{\frac{\partial u}{\partial \nu}(x')}\cdot u(x')+\sum_{j=1}^{N} b^j(x')n_j|u(x')|^2\right\}d\sigma \\
&= -\int_{\Gamma\setminus\Gamma_0}\left\{\frac{\partial u}{\partial \nu}(x')\cdot\overline{u(x')}+\overline{\frac{\partial u}{\partial \nu}(x')}\cdot u(x')+\sum_{j=1}^{N} b^j(x')n_j|u(x')|^2\right\}d\sigma \\
&= \int_{\Gamma\setminus\Gamma_0}\left\{\frac{2b(x')}{a(x')}-\sum_{j=1}^{N} b^j(x')n_j\right\}|u(x')|^2 d\sigma \geq 0 ,
\end{aligned}
$$

because the conditions (A.1) and (A.4) are satisfied.

Next, we set

$$
K=\sup_{\Omega}\left(c(x)-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial b^j}{\partial x_j}(x)\right)<\infty .
$$

And, we obtain that

$$
-2\mathrm{Re}(A_2 u, u)\geq -2K\|u\|_2^2 , \qquad u\in C^\infty(\bar{\Omega})\cap\mathscr{D}(A_2) ,
$$

that is,

$$
\mathrm{Re}(A_2 u, u)\leq K\|u\|_2^2 , \qquad u\in C^\infty(\bar{\Omega})\cap\mathscr{D}(A_2) .
$$

Since the space $C^\infty(\bar{\Omega})\cap\mathscr{D}(A_2)$ is dense in $\mathscr{D}(A_2)$ in the $\|\cdot\|_{2,2}$ norm, it follows that

$$
\mathrm{Re}(A_2 u, u)\leq K\|u\|_2^2 , \qquad u\in\mathscr{D}(A_2) .
$$

This proves the assertion (1.3).

(ii) From the assertion (1.2), we shall prove the assertions (1.4)–(1.8) when $p=2$. We need the following theorem to show these assertions.

THEOREM 5.1. *Let* $1<p<\infty$. *Assume that the conditions* (A.1) *and* (A.2) *are satisfied. Then, we have that for any* $\varepsilon>0$, *there exists a constant* $r(\varepsilon)>0$ *such that the resolvent set* $\rho(A_p)$ *contains the set* $\Sigma(\varepsilon)=\{\lambda=r^2e^{i\theta}\,;\, r\geq r(\varepsilon),\ -\pi+\varepsilon\leq\theta\leq\pi-\varepsilon\}$, *and that the resolvent* $(A_p-\lambda)^{-1}$ *satisfies the estimate*:

$$\|(A_p-\lambda)^{-1}\| \leq \frac{C(\varepsilon)}{|\lambda|}, \qquad \lambda \in \Sigma(\varepsilon),$$

with some constant $C(\varepsilon)>0$ *depending on* ε.

We can similarly prove Theorem 5.1 as in Theorem 6.1 of [15].

First, the assertion (1.4) immediately follows by Theorem 5.1. Secondly, by Theorem 3.6 we obtain that $\mathscr{D}((A_2)^k) \subset H^{2k,2}(\Omega)$, $k=1, 2, \cdots$. Hence the assertions (1.5) and (1.7) follow from the assertion (1.4), by applying Theorem 15.1 of Agmon [2]. The assertion (1.6) immediately follows by the assertion (1.5). Finally we shall prove the assertion (1.8). Since we have the assertion (1.4), it follows from Theorem 16.5 of Agmon [2] that the generalized eigenfunctions are complete in $L^2(\Omega)$. And since, by Theorem 1, Theorem of Agmon [1] holds in the case that $a(x')\geq 0$ on Γ, the assertion (1.8) follows from Theorem of Agmon [1].

The proof of Theorem 2 is now complete.

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