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Equivariant Harmonic Maps Associated to Large Group Actions

Hajime URAKAWA and Keisuke UENO

Tôhoku University

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§1. Introduction.

For compact Riemannian manifolds (M, g) and (N, h), harmonic maps between them are critical points of the energy functional

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2 dv_g ,$$

on the space of all smooth maps ϕ of M into N. Namely, for any smooth variation of ϕ, ϕ_t, $-\varepsilon<t<\varepsilon$, with $\phi_0=\phi$, it holds that

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t)=0 .$$

A remarkable existence result, in compact case, is due to Eells-Sampson [7]. They showed that if the target manifold (N, h) has non-positive curvature, then there exists a minimizing harmonic map in its homotopy class for any smooth map of (M, g) into (N, h).

On the other hand, the notion of harmonic maps is well-defined for non-compact Riemannian manifolds, and existence problem is also interesting. Recently Li-Tam [10] showed the existence of a harmonic map from (M, g) into (N, h), provided that these manifolds are complete and have some curvature conditions. As an application, they investigated the boundary value problem of harmonic map between the hyperbolic spaces and showed the existence of one which is equal to a given C^3 boundary map with non zero energy density (see also Akutagawa [1]).

But so far results of explicit construction of harmonic maps between non-compact Riemannian manifolds are very few, except works by Baird [3] and Kasue and Washio [9]. Baird [3] reduced the harmonic map equation to an ordinary differential equation defined on $(0, \infty)$, and investigated the behavior of a solution at the origin. Kasue and

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Washio [9] studied the existence of global solution for some warped product. However they did not prove the existence of equivariant harmonic map between the hyperbolic spaces. Our purpose of this paper is extending the results in [15] to non-compact cohomogeneity one Riemannian manifolds, and construct harmonic maps of several non-compact Riemannian manifolds including the standard Euclidean space $(\boldsymbol{R}^m, g_{can})$, the hyperbolic space $(\boldsymbol{RH}^m, g_{can})$, the standard complex Euclidean space $(\boldsymbol{C}^m, g_{can})$ and the complex hyperbolic space $(\boldsymbol{CH}^m, g_{can})$. These manifolds have large Lie group actions which are so-called cohomogeneity one Riemannian manifolds. In this paper, making use of this group actions, we give a set-up to reduce the harmonic map equation to an ordinary differential equation, and construct new harmonic maps as stated in Theorem 6.5 and Corollary 6.6.

§2. Cohomogeneity one Riemannian manifolds.

We shall review the notion of cohomogeneity one Riemannian manifolds due to Hsiang-Lawson [8] (see also Berard [4]). Let (M, g) be a Riemannian manifold and K a compact Lie group which acts isometrically and effectively on (M, g). If there exists a point of $x \in M$ such that $\dim(Kx) = \dim M - 1$, we call that the group K acts cohomogeneity one on (M, g). Then the orbit space $K\backslash M$ of K on M is one of the following seven cases:

$$(1)\ [0, l]\,, \quad (2)\ \text{a circle } [0, l]/\{0, l\}\,, \quad (3)\ [0, l)\,, \quad (4)\ [0, \infty)\,,$$
$$(5)\ (0, l)\,, \quad (6)\ (0, \infty) \quad \text{or} \quad (7)\ (-\infty, \infty)\,.$$

The first two cases occur when M is compact (see [15] for detail). The last five ones are for non-compact and the cases (4), (7) are complete. We treat here only with the case $[0, l)$ or $[0, l]$, where $l < \infty$ or $l = \infty$. In these cases we can give the fine Riemannian structure on (M, g).

Let $c(t)$, $0 \le t \le l \le \infty$, be the geodesic of (M, g) representing the orbit space $K\backslash M$. Let J_t be the isotropy subgroup of K at $c(t)$. Then it is known (cf. [4]) that the subgroup J_t are the same one, denoted by J, for all $0 < t < l$. The Lie algebra $\mathfrak{k}$ of K is decomposed as

$$\mathfrak{k} = \mathfrak{j} \oplus \mathfrak{m}\,,$$

which is orthogonal with respect to the $\mathrm{Ad}(K)$-invariant inner product $\langle\, ,\, \rangle$ on $\mathfrak{k}$, $\mathfrak{j}$ is the Lie algebra of J and $\mathfrak{m}$ is an $\mathrm{Ad}(J)$-invariant subspace of $\mathfrak{k}$.

The mapping $K/J \times (0, l) \ni (kJ, t) \mapsto kc(t) \in M$ is smooth and its image, denoted by M', is open and dense in M. The metric g on M can be expressed on M' as follows:

$$g = dt^2 + g_t\,. \tag{2.1}$$

Here g_t is the K-invariant metric on the orbit $Kc(t)$ through $c(t)$, $0 < t < l$, which is given by

$$g_t(X_{c(t)}, Y_{c(t)}) = \alpha_t(X, Y) \qquad X, Y \in \mathfrak{m}\,, \tag{2.2}$$

where X_p, the tangent vector at $p \in M$, is defined by

$$X_p = \frac{d}{dt}\bigg|_{t=0} \exp tX \cdot p ,$$

for all $X \in \mathfrak{m}$. In what follows, we assume the inner product α_t on $\mathfrak{m}$ is of the form

$$\alpha_t(X_i, X_j) = f_i(t)^2 \delta_{ij} , \qquad 1 \leq i, j \leq m-1 ,$$

where $m = \dim M$ and $\{X_i\}_{i=1}^{m-1}$ is an orthonormal basis of $(\mathfrak{m}, \langle \, , \rangle)$.

§3. The Euler-Lagrange equation.

3.1. Setting. Let (M, g) (resp. $(N, \bar{g})$) be Riemannian manifold admitting cohomogeneity one actions of compact Lie groups K (resp. G). We denote the orbit space $K\backslash M = [0, l]$ (resp. $G\backslash N = [0, \bar{l}]$), where l (resp. $\bar{l}$) is finite or infinite and fix the geodesic $c(t)$, $0 \leq t \leq l$ of (M, g) (resp. $\bar{c}(r)$, $0 \leq r \leq \bar{l}$ of $(N, \bar{g})$) which represents the orbit space $K\backslash M$ (resp. $G\backslash N$). Also we denote by J_t, H_r the isotropy subgroups of K, G at $c(t)$, $\bar{c}(r)$, which coincide with J, H for $0 < t < l$, and $0 < r < \bar{l}$, respectively.

Now let $A : K \rightarrow G$ be a Lie group homomorphism satisfying $A(J) \subset H$. A mapping $\phi : M \rightarrow N$ is A-equivariant if $\phi(k \cdot x) = A(k) \cdot \phi(x)$, for $k \in K$ and $x \in M$. We assume that for the A-equivariant map $\phi : M \rightarrow N$, there exists a C^∞ function $r : [0, l] \rightarrow [0, \bar{l}]$ such that

$$\phi(c(t)) = \bar{c}(r(t)) \qquad t \in [0, l] .$$

Then the A-equivariant map ϕ can be recovered as

$$\phi(k \cdot c(t)) = A(k) \cdot \bar{c}(r(t)) \qquad k \in K, t \in [0, l] ,$$

and ϕ is smooth on M'. Moreover, the necessary and sufficient conditions for an A-equivariant map $\phi : M \rightarrow N$ to be continuous on the whole space M are that the function $r : [0, l] \rightarrow [0, \bar{l}]$ satisfies

$$\text{(3.1)} \qquad r(0) = 0 , \qquad 0 < r(t) < \bar{l} \ \ (0 < t < l) ,$$

where $r(0) = 0$ means $\lim_{t \to 0} r(t) = 0$, and the homomorphism A satisfies

$$\text{(3.2)} \qquad A(J_t) \subset H_{r(t)}$$

for any $t \in [0, l]$. A standard regularity theorem says that any continuous weakly harmonic map is smooth everywhere and harmonic (cf. [5] p. 397).

3.2. The reduction of the Euler-Lagrange equation.

DEFINITION 3.3. A smooth map $\phi : M \rightarrow N$ is said to be harmonic if for any smooth variation $\phi_t : M \rightarrow N$, $-\varepsilon < t < \varepsilon$ with $\phi_0 = \phi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t)=0 .$$

By the first variation formula, it is equivalent that the tension field $\tau(\phi)$ defined by

$$\tau(\phi)=\sum_{i=1}^{m} (\tilde{\nabla}_{e_i}\phi_* e_i-\phi_*\nabla_{e_i}e_i) ,$$

vanishes on M. By virtue of the A-equivariance of ϕ the tension field $\tau(\phi)$ vanishes on the open dense subset M' of M if and only if $\tau(\phi)$ vanishes on the geodesic $c(t)$, $0<t<l$. To state our result on the calculation of $\tau(\phi)$ at $c(t)$, $0<t<l$, we prepare the description of the metrics g and $\bar{g}$ on M, N respectively.

Let $\mathfrak{m}$, $\mathfrak{n}$ be the orthogonal complements of $\mathfrak{j}$, $\mathfrak{h}$ in $\mathfrak{k}$, $\mathfrak{g}$ with respect to the $\mathrm{Ad}(K)$-invariant, $\mathrm{Ad}(G)$-invariant inner products $\langle\ ,\ \rangle$ on $\mathfrak{k}$, $\mathfrak{g}$, respectively. Then

$$g=dt^2+g_t , \qquad \bar{g}=dr^2+\bar{g}_r ,$$

and the g_t, $\bar{g}_r$ are metrics on the orbits $Kc(t)$, $G\bar{c}(r)$ corresponding to the inner products on $\mathfrak{m}$, $\mathfrak{n}$ which are of the form

$$\alpha_t(X_i, X_j)=f_i(t)^2\delta_{ij} , \qquad \beta_t(Y_a, Y_b)=h_a(t)^2\delta_{ab} .$$

Here $\{X_j\}_{j=1}^{m-1}$, $\{Y_a\}_{a=1}^{n-1}$ are orthonormal bases of $(\mathfrak{m}, \langle\ ,\ \rangle)$, $(\mathfrak{n}, \langle\ ,\ \rangle)$, respectively. Then the tension field $\tau(\phi)$ at $c(t)$ is given as the following theorem.

THEOREM 3.4. *Assume that the function $r(t)$ satisfies* (3.1) *and the inner products α_t and β_r satisfy*

$$\alpha_t(X, [Y, X]_{\mathfrak{m}})+\alpha_t([Y, X]_{\mathfrak{m}}, X)=0 , \qquad \forall X, Y\in\mathfrak{m} ,$$

$$\beta_t(X', [Y', X']_{\mathfrak{n}})+\beta_t([Y', X']_{\mathfrak{n}}, X')=0 , \qquad \forall X', Y'\in\mathfrak{n} .$$

Here for $Z\in\mathfrak{k}$ (resp. $Z'\in\mathfrak{g}$), $Z_{\mathfrak{m}}$ (resp. $Z'_{\mathfrak{n}}$) are the $\mathfrak{m}$ (resp. $\mathfrak{n}$)-component corresponding to the decomposition $\mathfrak{k}=\mathfrak{j}\oplus\mathfrak{m}$ (resp. $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}$).

Then the tension field $\tau(\phi)$ of an A-equivariant map $\phi : (M, g)\to(N, \bar{g})$ which is smooth on M' and satisfies $\phi(c(t))=\bar{c}(r(t))$, $0<t<l$, is

$$\begin{aligned}\tau(\phi)(c(t))=\Bigg\{\ddot{r}(t)+\left(\sum_{j=1}^{m-1} f_j(t)^{-1}\dot{f}_j(t)\right)\dot{r}(t)\\ -\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f_j(t)^{-2}h_i(r)^{-3}h_i'(r)\beta_r(Y_i, A(X_j))^2\Bigg\}\dot{\bar{c}}(r(t))\\ +\sum_{j=1}^{m-1} f_j(t)^{-2}[V_j, U_j]_{\bar{c}(r(t))} .\end{aligned}$$

In particular, $\tau(\phi)$ vanishes at $c(t)$, $0<t<l$, if and only if

$$(3.5) \qquad \ddot{r}(t)+\left(\sum_{j=1}^{m-1} f_j(t)^{-1}\dot{f}_j(t)\right)\dot{r}(t)-\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f_j(t)^{-2}h_i(r)^{-3}h_i'(r)\beta_r(Y_i, A(X_j))^2=0\,,$$

and

$$(3.6) \qquad C:=\sum_{j=1}^{m-1} f_j(t)^{-2}[V_j, U_j]_{\bar{c}(r(t))}=0\,, \qquad \textit{in} \quad \mathfrak{g}\,.$$

For proof, see [15, Theorem 2.2].

Here $\dot{\bar{c}}(r)$ is the tangent vector of the curve $r\mapsto\bar{c}(r)$, $A(X)\in\mathfrak{g}$, $X\in\mathfrak{k}$, is defined by $A(X)=(d/dt)|_{t=0}A(\exp tX)$, and $A(X_j)\in\mathfrak{g}$, $1\le j\le m-1$, is decomposed as

$$A(X_j)=U_j+V_j\,, \qquad U_j\in\mathfrak{n},\ V_j\in\mathfrak{h}\,,$$

according to the decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}$. We denote

$$\dot{f}_j(t)=\frac{df_j}{dt}(t) \quad \text{and} \quad h_i'(r)=\frac{dh_i}{dr}(r)\,.$$

We finally note that the total energy of A-equivariant harmonic map ϕ, $E(\phi)$, is given by

$$E(\phi)=\mathrm{Vol}(K, \langle\,,\rangle)\int_0^l e(\phi)(c(t))D(t)dt\,,$$

where the energy density functional $e(\phi)(c(t))$ is

$$(3.7) \qquad e(\phi)(c(t))=\frac{1}{2}\left\{\dot{r}(t)^2+\sum_{j=1}^{m-1} f_j(t)^{-2}\beta_{r(t)}(A(X_j)_{\mathfrak{n}}, A(X_j)_{\mathfrak{n}})\right\},$$

and

$$D(t)=\prod_{j=1}^{m-1} f_j(t)\,.$$

Then the equation (3.5) is the Euler-Lagrange equation of $E(\phi)$.

§4. Cohomogeneity one Riemannian manifolds.

4.1. Non-compact cases. In this subsection, we give examples of non-compact complete manifolds which admit cohomogeneity one group actions. For these examples, we shall consider existence of equivariant harmonic maps in §6.

EXAMPLE 4.1. The orthogonal group action $(SO(n), \boldsymbol{R}^n)$. The special orthogonal group $K=SO(n)$ acts on the Euclidean space $M=\boldsymbol{R}^n$ by the usual manner as

$$\alpha\cdot x:=\alpha x\,, \qquad \alpha=(\alpha_{ij})\,,\ x={}^t(x_1\,,\cdots, x_n)\in\boldsymbol{R}^n\,,$$

where $\cdot$ denotes the matrix multiplication. Consider a Riemannian metric g on $\boldsymbol{R}^n$ invariant under the action of K. This is the simplest example of cohomogeneity one action whose orbit space is the half-line $[0, \infty)$. Let $c(t)$, $0<t<\infty$, be a geodesic of $(\boldsymbol{R}^n, g)$ emanating the origin $\mathbf{0}$ of $\boldsymbol{R}^n$ with the arc length parameter t. Then $c(t)$ is of the form

$$c(t)={}^t(0, \cdots, 0, t), \qquad 0<t<\infty .$$

The isotropy subgroup J_t of K at $c(t)$ is K itself if $t=0$, and for $t>0$,

$$J_t=J=SO(n-1)=\left\{\begin{pmatrix}\alpha' & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix};\ \alpha'\in SO(n-1)\right\}.$$

Ad(K)-invariant inner product on $\mathfrak{k}$ is given by

$$\langle X, Y\rangle=-\tfrac{1}{2}\operatorname{tr}(XY) \qquad X, Y\in\mathfrak{k},$$

and the orthogonal complement $\mathfrak{m}$ of $\mathfrak{j}$ in $\mathfrak{k}$ with respect to $\langle\,,\,\rangle$ is given by

$$\mathfrak{m}=\left\{X(x_1, \cdots, x_{n-1}):=\begin{pmatrix}\mathbf{0} & -{}^tX\\ X & 0\end{pmatrix};\ X=(x_1, \cdots, x_{n-1}),\ x_i\in\boldsymbol{R}\right\}.$$

The orthonormal basis $\{X_j\}_{j=1}^{n-1}$ is given by $X_j=X(0, \cdots, -1, \cdots, 0)$ where j-th entry is -1 for $1\le j\le n-1$. Then the Riemannian metric g is of the form on $M'=\boldsymbol{R}^n-\{\mathbf{0}\}$,

$$g=dt^2+g_t,$$

where g_t is a K-invariant metric on the orbit $Kc(t)$, $0<t<\infty$, given by

$$g_t(X_{c(t)}, Y_{c(t)})=\alpha_t(X, Y) \qquad X, Y\in\mathfrak{m},$$

corresponding to the inner product α_t on $\mathfrak{m}$ which is of the form

$$\alpha_t(X, Y)=f(t)^2\langle X, Y\rangle \qquad X, Y\in\mathfrak{m},$$

that is, $\alpha_t(X_i, X_j)=f(t)^2\delta_{ij}$, $1\le i, j\le n-1$. Necessary and sufficient conditions for g to be extended to a smooth metric on M are that $f(-t)=-f(t)$ and $\dot{f}(t)=1$.

The obtained Riemannian manifold $(\boldsymbol{R}^n, g)$ is a rotationally symmetric space. In particular, in the case $f(t)=t$, we obtain the standard Euclidean space $(\boldsymbol{R}^n, g_{can})$, and in the case $f(t)=\sinh t$, then we obtain the hyperbolic space $(\boldsymbol{RH}^n, g_{can})$ of constant negative curvature -1.

EXAMPLE 4.2. The special unitary group action $(SU(n), \boldsymbol{C}^n)$. The special unitary group $K=SU(n)$ acts on the complex Euclidean space $M=\boldsymbol{C}^n$ by the usual matrix multiplication. Let us take a Riemannian metric g on $\boldsymbol{C}^n$ invariant under the action of K. Let $c(t)$, $0<t<\infty$, be a geodesic of $(\boldsymbol{C}^n, g)$ emanating the origin $\mathbf{0}$ of the same form in Example 4.1 with the arc length parameter t. The isotropy subgroup J_t of K at $c(t)$ is K itself if $t=0$, and for $t>0$,

$$J_t = J = SU(n-1) = \left\{ \begin{pmatrix} \alpha' & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} ;\ \alpha' \in SU(n-1) \right\}.$$

Ad(K)-invariant inner product on $\mathfrak{k}$ is given by

$$\langle X, Y \rangle = -\tfrac{1}{2} \operatorname{tr}(XY) \qquad X, Y \in \mathfrak{k},$$

and the orthogonal complement $\mathfrak{m}$ of $\mathfrak{j}$ in $\mathfrak{k}$ with respect to $\langle\,,\,\rangle$ is decomposed into the direct sum of two Ad(K)-invariant subspaces

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where

$$\mathfrak{m}_1 := \{ Z(0; z_1, \cdots, z_{n-1}) ;\ z_j \in \boldsymbol{C} \},$$
$$\mathfrak{m}_2 := \{ Z(\theta; 0, \cdots, 0) ;\ \theta \in \boldsymbol{R} \}.$$

Here we put

$$Z(\theta; z_1, \cdots, z_{n-1}) := \begin{pmatrix} -i\theta & & & -\bar{z}_1 \\ & \ddots & & \vdots \\ & & -i\theta & -\bar{z}_{n-1} \\ z_1 & \cdots & z_{n-1} & i(n-1)\theta \end{pmatrix}, \qquad \theta \in \boldsymbol{R}, \quad z_j \in \boldsymbol{C}.$$

The orthonormal basis of ($\mathfrak{m}$, $\langle\,,\,\rangle$) is given by

$$\begin{cases} X_j = Z(0; 0, \cdots, 0, 1, 0, \cdots, 0), \\ X_{n-1+j} = Z(0; 0, \cdots, 0, i, 0, \cdots, 0) \qquad \text{for} \quad 1 \le j \le n-1, \\ X_{2n-1} = Z\left(\sqrt{\dfrac{2}{(n-1)n}} ; 0, \cdots, 0 \right). \end{cases}$$

Then the Riemannian metric g is of the form

$$g = dt^2 + g_t,$$

on $M' = \boldsymbol{C}^n - \{\mathbf{0}\}$. Here g_t is a K-invariant metric on the orbit $Kc(t)$ corresponding to the inner product

$$\alpha_t(X, Y) = f_1(t)^2 \langle X', Y' \rangle + f_2(t)^2 \langle X'', Y'' \rangle,$$

for $X = X' + X''$, $Y = Y' + Y''$, where $X', Y' \in \mathfrak{m}_1$, $X'', Y'' \in \mathfrak{m}_2$. That is,

$$\begin{cases} \alpha_t(X_i, X_j) = f_1(t)^2 \delta_{ij}, & 1 \le i, j \le 2n-2, \\ \alpha_t(X_{2n-1}, X_j) = 0, & 1 \le j \le 2n-2, \\ \alpha_t(X_{2n-1}, X_{2n-1}) = f_2(t)^2. \end{cases}$$

Necessary and sufficient conditions for g to be extended to a smooth metric on M are that the functions $f_i(t)$, $i=1, 2$, satisfy

$$\begin{cases} f_i(-t)=-f_i(t) & i=1,2\,, \\ \dot{f}_1(0)=1 \quad \text{and} & \sqrt{\dfrac{n}{2(n-1)}}\dot{f}_2(t)=1\,. \end{cases}$$

The obtained Riemannian manifold $(\boldsymbol{C}^n, g)$ is a unitary rotationally symmetric space. In particular, in the case $f_1(t)=f_2(t)=t$, we obtain the standard complex Euclidean space $(\boldsymbol{R}^n, g_{can})$, and in the case $f_1(t)=\sinh t$, and $f_2(t)=\sqrt{(n-1)/2n}\sinh 2t$, then we obtain the complex hyperbolic space $(\boldsymbol{CH}^n, g_{can})$ of curvature $-4\leq K_{g_{can}}\leq -1$.

4.2. Compact cases. There are several compact Riemannian manifolds admitting a cohomogeneity one group action. All cohomogeneity one actions on the unit sphere (S^n, g_{can}) are classified by Hsiang-Lawson [8], Takagi-Takahashi [12], and Asoh [2], and classification of all cohomogeneity one actions on the complex projective space $(\boldsymbol{CP}^n, g_{can})$ is done by Uchida [13]. Here we only note the simplest actions on the unit sphere and the complex projective space.

EXAMPLE 4.3. The unit sphere S^n. The special orthogonal group

$$K=SO(n)=\left\{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix};\ \alpha\in SO(n)\right\}\subset SO(n+1)$$

acts cohomogeneity one on the unit sphere $S^n\subset \boldsymbol{R}^{n+1}$ by the usual matrix multiplication. We choose a geodesic

$$c(t)={}^t(0\,,\cdots,0\,,\ -\sin t,\ \cos t)=z(t)\cdot{}^t(0\,,\cdots,0\,,\ 1)\,,$$

where

$$z(t)=\begin{pmatrix} I_{n-1} & & \\ & \cos t & -\sin t \\ & \sin t & \cos t \end{pmatrix}.$$

Then the isotropy subgroup J_t of K at $c(t)$ is K itself if $t=0$, π, and for $t>0$,

$$J_t=J=\left\{\begin{pmatrix} y & 0 \\ 0 & I_2 \end{pmatrix};\ y\in SO(n-1)\right\},$$

where I_m is the identity matrix of degree m. The orbit space $K\backslash S^n$ is the closed interval $[0, \pi]$. Ad(K)-invariant inner product on $\mathfrak{k}$ is given by

$$\langle X, Y\rangle=\tfrac{1}{2}\operatorname{tr}(XY) \qquad X, Y\in\mathfrak{k}\,,$$

and the orthogonal complement $\mathfrak{m}$ of $\mathfrak{j}$ in $\mathfrak{k}$ with respect to $\langle\,,\,\rangle$ is given by

$$\mathfrak{m}=\left\{\begin{pmatrix} \mathbf{0} & -{}^tX & \mathbf{0} \\ X & 0 & 0 \\ \mathbf{0} & 0 & 0 \end{pmatrix}\ ;\ X=(x_1,\cdots,x_{n-1}),\ x_i\in\boldsymbol{R}\right\}.$$

The orthonormal basis $\{X_j\}_{j=1}^{n-1}$ of $(\mathfrak{m}, \langle\ ,\ \rangle)$ is given by $X_j=X(0,\cdots,-1,\cdots,0)$ where j-th entry is -1 for $1\le j\le n-1$. A K-invariant Riemannian metric g on S^n is of the form

$$g=dt^2+g_t\,,$$

where g_t is a K-invariant metric on the orbit $Kc(t)$, $0<t<\infty$, corresponding to the inner product

$$\alpha_t(X, Y)=f(t)^2\langle X, Y\rangle \qquad X, Y\in\mathfrak{m}\,,$$

that is, $\alpha_t(X_i, X_j)=f(t)^2\delta_{ij}$, $1\le i, j\le n-1$. Necessary and sufficient conditions for g to be extended to a smooth metric on M are that $f(t)$ satisfies

$$\begin{cases} f(-t)=-f(t), & \dot f(0)=1 \text{ and} \\ f(\pi-t)=-f(\pi+t), & \dot f(\pi)=-1\,. \end{cases}$$

The obtained Riemannian manifold (S^n, g) is a spherical rotationally symmetric space. The standard unit sphere (S^n, g_{can}) corresponds to $f(t)=\sin t$.

EXAMPLE 4.4. The complex projective space $\boldsymbol{CP}^n$. Let us denote the complex projective space

$$\boldsymbol{CP}^n=\{[{}^t(z_1,\cdots,z_{n+1})]\ ;\ z_i\in\boldsymbol{C}\}\,,$$

as a coset space $SU(n+1)/S(U(n)\times U(1))$. The group

$$K=SU(n)=\left\{\begin{pmatrix} \alpha' & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}\ ;\ \alpha'\in SU(n)\right\}\subset SU(n+1)$$

acts on $\boldsymbol{CP}^n$ cohomogeneity one as matrix multiplication. Take a geodesic

$$c(t)=[{}^t(0,\cdots,0,\ \sin t\,,\ \cos t)]\in\boldsymbol{CP}^n\,,$$

with respect to the Fubini-Study metric. Then the isotropy subgroup J_t of K at $c(t)$ is K itself if $t=0$, and for $t=\pi/2$,

$$J_{\pi/2}=\left\{\begin{pmatrix} \alpha' & & \\ & e^{i\theta} & \\ & & 1 \end{pmatrix}\ ;\ \alpha'\in U(n-1),\ \theta\in\boldsymbol{R},\ (\det\alpha')e^{i\theta}=1\right\}$$

which coincides with $S(U(n-1)\times U(1))$. Also for $0<t<\pi/2$

$$J_t=J=\left\{\begin{pmatrix} \alpha' & & \\ & 1 & \\ & & 1 \end{pmatrix}\ ;\ \alpha'\in SU(n-1)\right\}.$$

Their orbits are $Kc(0)=c(0)$ and $Kc(\pi/2)=\boldsymbol{C}\boldsymbol{P}^{n-1}$. We take an inner product on the Lie algebra $\mathfrak{k}$ of K by

$$\langle X, Y\rangle = -\tfrac{1}{2}\operatorname{tr}(XY) \qquad X, Y\in\mathfrak{k}.$$

The orthogonal complement $\mathfrak{m}$ of $\mathfrak{j}$ in $\mathfrak{k}$ with respect to $\langle\ ,\ \rangle$ is also decomposed into the direct sum of two $\mathrm{Ad}(K)$-invariant subspaces

$$\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2,$$

where

$$\mathfrak{m}_1 := \{Z'(0; z_1, \cdots, z_{n-1})\,;\, z_j\in\boldsymbol{C}\},$$
$$\mathfrak{m}_2 := \{Z'(\theta; 0, \cdots, 0)\,;\, \theta\in\boldsymbol{R}\}.$$

Here we put

$$Z'(\theta; z_1, \cdots, z_{n-1}) := (Z(\theta; z_1, \cdots, z_{n-1})\ \mathbf{0})\in\mathfrak{m}.$$

We take an orthonormal basis $\{X_j\}_{j=1}^{2n-1}$ of $(\mathfrak{m}, \langle\ ,\ \rangle)$ as

$$\begin{cases} X_j=Z'(0; 0, \cdots, 0, 1, 0, \cdots, 0), \\ X_{n-1+j}=Z'(0; 0, \cdots, 0, i, 0, \cdots, 0) \qquad \text{for} \quad 1\le j\le n-1, \\ X_{2n-1}=Z'\left(\sqrt{\dfrac{2}{(n-1)n}}; 0, \cdots, 0\right). \end{cases}$$

Then any K-invariant Riemannian metric g on $\boldsymbol{C}\boldsymbol{P}^n$ is of the form

$$g=dt^2+g_t$$

on $M'=\boldsymbol{C}\boldsymbol{P}^n-\{Kc(0)\cup Kc(\pi/2)\}$, where g_t is a K-invariant metric on the orbit $Kc(t)$, $0<t<\pi/2$, corresponding to the inner product

$$\alpha_t(X, Y)=f_1(t)^2\langle X', Y'\rangle+f_2(t)^2\langle X'', Y''\rangle,$$

for $X=X'+X''$, $Y=Y'+Y''$, where $X', Y'\in\mathfrak{m}_1$, $X'', Y''\in\mathfrak{m}_2$. That is,

$$\begin{cases} \alpha_t(X_i, X_j)=f_1(t)^2\delta_{ij}, & 1\le i, j\le 2n-2, \\ \alpha_t(X_{2n-1}, X_j)=0, & 1\le j\le 2n-2, \\ \alpha_t(X_{2n-1}, X_{2n-1})=f_2(t)^2. \end{cases}$$

Necessary and sufficient conditions for g to be extended to a smooth metric on M are that the functions $f_i(t)$, $i=1, 2$, satisfy the following conditions.

$$f_1(-t)=-f_1(t),\ \dot{f}_1(0)=1,\ f_1\left(\frac{\pi}{2}-t\right)=-f_1\left(\frac{\pi}{2}+t\right),\ \dot{f}_1\left(\frac{\pi}{2}\right)=-1;$$

$$f_2(-t)=-f_2(t),\ \sqrt{\frac{n}{2(n-1)}}\dot{f}_2(0)=1,\ f_2\left(\frac{\pi}{2}-t\right)=-f_2\left(\frac{\pi}{2}+t\right),\ \sqrt{\frac{n}{2(n-1)}}\dot{f}_2\left(\frac{\pi}{2}\right)=-1\,.$$

The obtained Riemannian manifold $(\boldsymbol{CP}^n, g)$ is a projective rotationally symmetric space. In particular, in the case $f_1(t)=\sin t$ and $f_2(t)=\sqrt{(n-1)/2n}\sin 2t$, we obtain the complex projective space $\boldsymbol{CP}^n$ with the Fubini-Study metric g_{can}, with sectional curvature $1\leq K_{g_{can}}\leq 4$.

§5. Calculation of harmonic maps.

Under the above preparations, we want to yield harmonic map equations

(1) of a (spherical) rotationally symmetric space $(\boldsymbol{R}^m, g)$ (resp. (S^m, g)) into $(\boldsymbol{R}^n, g)$.
(2) of a (spherical) rotationally symmetric space $(\boldsymbol{R}^m, g)$ (resp. (S^m, g)) into a spherical rotationally symmetric space (S^n, g).
(3) of a unitary (projective) rotationally symmetric space $(\boldsymbol{C}^m, g)$ (resp. $(\boldsymbol{CP}^m, g)$) into $(\boldsymbol{C}^n, g)$.
(4) of a unitary (projective) rotationally symmetric space $(\boldsymbol{C}^m, g)$ (resp. $(\boldsymbol{CP}^m, g)$) into $(\boldsymbol{CP}^n, g)$.

5.1. To construct harmonic maps of the above types (1) and (2), we should prepare a homomorphism A of $SO(m)$ into $SO(n)$ satisfying $A(SO(m-1))\subset SO(n-1)$. To do this, let us take an n-dimensional irreducible orthogonal representation (V, π) of $SO(m)$ which is spherical with respect to $J=SO(m-1)=\left\{\begin{pmatrix}\alpha & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix};\ \alpha\in SO(m-1)\right\}$. That is, there exists a unit vector v_n in V with respect to inner product $(\ ,\ )$ such that $\pi(j)v_n=v_n$ for all $j\in J=SO(m-1)$. Taking an orthonormal basis $\{v_i\}_{i=1}^n$ of $(V, (\ ,\ ))$, we can define a homomorphism $A: K=SO(m)\to G=SO(n)$ by

$$A(k)=(\pi_{ij}(k))\,,\quad \pi_{ij}(k)=(\pi(k)v_j, v_i)\,,\qquad k\in K=SO(m)\,.$$

Then A sends J into H, where

$$J=\left\{\begin{pmatrix}\alpha & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix};\ \alpha\in SO(m-1)\right\}\quad\text{and}\quad H=\left\{\begin{pmatrix}\beta & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix};\ \beta\in SO(n-1)\right\}.$$

Let us denote by $\mathfrak{m}$, $\mathfrak{n}$, the orthogonal complements of $\mathfrak{j}$, $\mathfrak{h}$ in the Lie algebras $\mathfrak{k}$, $\mathfrak{g}$ with respect to the inner products as in Example 4.1 or Example 4.3 respectively, and set

$$A(X_j)=U_j+V_j$$

for an orthonormal basis $\{X_j\}_{j=1}^{m-1}$ of $(\mathfrak{m}, \langle\ ,\ \rangle)$, where $U_j\in\mathfrak{n}$ and $V_j\in\mathfrak{h}$. Then it holds that

$$C=\sum_{j=1}^{m-1}[U_j, V_j]=0\,.$$

Indeed, $A(j)$, $j\in J$, acts on $\mathfrak{n}$ by

$$A(j)\cdot W=A(j)\circ W\circ A(j)^{-1}\,,\qquad j\in J=SO(m-1)\,,\quad W\in\mathfrak{n},$$

where the circle means the composition of endomorphisms. It is clear that $A(j)\cdot C=C$ for all $j\in J$. This implies $A(j)\in SO(n-2)$ for all $j\in J$ unless $C=0$. That is, the irreducible orthogonal representation (V,π) of $SO(m)$ admits two dimensional subspace of V of which elements are fixed by the action of $SO(m-1)$. This contradicts a theorem of E. Cartan about a spherical irreducible representation of $(SO(m), SO(m-1))$. Thus $C=0$.

The inner product $\langle\,,\,\rangle$ in $\mathfrak{m}$, $\mathfrak{n}$ satisfy that

$$\langle X,[Y,X]_{\mathfrak{m}}\rangle+\langle[Y,X]_{\mathfrak{m}},X\rangle=0\qquad \forall X,Y\in\mathfrak{m}\,,$$

$$\langle X',[Y',X']_{\mathfrak{n}}\rangle+\langle[Y',X']_{\mathfrak{n}},X'\rangle=0\qquad \forall X',Y'\in\mathfrak{n}$$

because of $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{j}$ and $[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{h}$.

Case 1. $(S^m,g)\to(S^n,\bar{g})$. In this case, we treat with harmonic maps of a spherical rotationally symmetric space (S^m,g) into another one $(S^n,\bar{g})$. Take the above homomorphism $A:K=SO(m)\to G=SO(n)$ corresponding to the orthogonal irreducible representation (V_k,π_k) of $SO(m)$, spherical with respect to $SO(m-1)$. The representation (V_k,π_k) corresponds to the space of all the k-th order harmonic polynomials on the Euclidean space $\boldsymbol{R}^m$. Note that the restriction to the unit sphere S^{m-1} of the k-th harmonic polynomials are the eigenfunctions of the Laplacian on the standard sphere (S^{m-1},g_{can}) with eigenvalue $\lambda_k=k(k+m-2)$ and $n=\dim V_k=(k+m-3)(2k+m-2)/(k!(m-2)!)$. Then

$$\alpha_t(X,X')=f(t)^2\langle X,X'\rangle\,,\qquad X,X'\in\mathfrak{m}\,,$$

$$\beta_r(Y,Y')=h(r)^2\langle Y,Y'\rangle\,,\qquad Y,Y'\in\mathfrak{n}\,,$$

where the functions $f(t)$, $h(r)$ satisfy

$$f(-t)=-f(t)\,,\quad \dot{f}(0)=1\,,\quad f(\pi-t)=-f(\pi+t)\,,\quad \dot{f}(\pi)=-1\,,$$

$$h(-r)=-h(r)\,,\quad h'(0)=1\,,\quad h(\pi-r)=-h(\pi+r)\,,\quad h'(\pi)=-1\,.$$

On the other hand, we get

$$\sum_{j=1}^{m-1}\sum_{i=1}^{n-1}\langle A(X_j),Y_i\rangle=\sum_{j=1}^{m-1}\|A(X_j)_n\|^2=\lambda_k=k(k+m-2)\,.$$

Then the ordinary differential equation of equivariant harmonic map is given by

$$\ddot{r}(t)+(m-1)\frac{\dot{f}(t)}{f(t)}\dot{r}(t)-k(k+m-2)\frac{h(r(t))h'(r(t))}{f(t)^2}=0\,,\qquad 0<t<\pi\,. \tag{5.1}$$

Here recall $\dot{r}(t)=dr/dt$, $h'(r)=dh/dr$, and $h'(r(t))$ is the substitution of $r(t)$ into $h'(r)$.

Case 2. $(\boldsymbol{R}^m, g) \to (S^n, \bar{g})$. In this case, we can take the same homomorphism $A: K = SO(m) \to G = SO(n)$ and α_t, β_r have the same form. The different point is only the function $f(t)$ is defined on $(0, \infty)$ and satisfies

$$f(-t) = -f(t)\,, \qquad \dot{f}(0) = 1\,.$$

The obtained ordinary differential equation of equivariant harmonic map is

$$\ddot{r}(t) + (m-1)\frac{\dot{f}(t)}{f(t)}\dot{r}(t) - k(k+m-2)\frac{h(r(t))h'(r(t))}{f(t)^2} = 0\,, \qquad 0 < t < \infty\,. \tag{5.2}$$

Case 3. $(\boldsymbol{R}^m, g) \to (\boldsymbol{R}^n, \bar{g})$. In this case, we also can take the same homomorphism $A: K = SO(m) \to G = SO(n)$ and α_t, β_r as in Case 1. The both functions $f(t)$ and $h(r)$ are defined on $0 < t < \infty$ and satisfy

$$f(-t) = -f(t)\,, \qquad \dot{f}(0) = 1\,,$$

$$h(-r) = -h(r)\,, \qquad h'(0) = 1\,.$$

The obtained ordinary differential equation of equivariant harmonic map is the same as (5.2).

Also, in these three cases, the energy density functional is given by

$$e(\phi)(c(t)) = \frac{1}{2}\left\{\dot{r}(t)^2 + k(k+m-2)\frac{h(r(t))^2}{f(t)^2}\right\}$$

from (3.7). The equations (5.1) and (5.2) are the Euler-Lagrange equation of

$$E(\phi) = \mathrm{Vol}(K, \langle\,,\,\rangle)\int_0^l e(\phi)(c(t))D(t)dt\,,$$

where $l = \pi$ or ∞ and $D(t)$ is given by

$$D(t) = f(t)^{m-1}\,.$$

5.2. To construct harmonic maps of the above types (3) and (4), we should prepare a homomorphism A of $SU(m)$ into $SU(n)$ satisfying $A(SU(m-1)) \subset SU(n-1)$. To do this, let us take an n-dimensional irreducible unitary representation (V, π) of $SU(m)$ which is spherical with respect to $J = SU(m-1) = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix};\ \alpha \in SU(m-1)\right\}$. That is, (V, π) is an n-dimensional irreducible representation space of $SU(m)$ which has an orthonormal basis $\{v_i\}_{i=1}^n$ with $\pi(j)v_n = v_n$ for all $j \in J = SU(m-1)$. Then we can define a homomorphism $A: U(m) \to U(n)$ by

$$A(k) = (\pi_{ij}(k))\,, \qquad \pi_{ij}(k) = (\pi(k)v_j, v_i)\,, \quad k \in K = SU(m)\,.$$

Then it is easy to see $A(SU(m)) \subset SU(n)$ and

$$A(J)\subset H=\left\{\begin{pmatrix}\beta & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix};\ \beta\in SO(n-1)\right\},$$

since $\pi(j)v_n=v_n$ for all $j\in SU(m-1)$.

All the above irreducible representations of $SU(m)$ are given as the following way.

Firstly, remember that all irreducible unitary representations of $SU(m)$ are exhausted by the ones $V_{l_1\lambda_1+\cdots+l_{m-1}\lambda_{m-1}}$ with highest weight $V_{l_1\lambda_1+\cdots+l_{m-1}\lambda_{m-1}}$ with respect to the order $\lambda_1\geq\cdots\geq\lambda_{m-1}$. Here each λ_j is a linear map of

$$T=\left\{\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_m\end{pmatrix};\ \varepsilon_i\in \boldsymbol{C},\ |\varepsilon_i|=1,\ \prod_{i=1}^{m}\varepsilon_i=1\right\}$$

assigning its element to $\varepsilon_i\in\boldsymbol{C}$, and each l_j runs over all non-negative integers.

Then the following lemma due to Weyl is essential.

LEMMA 5.1. *Any irreducible unitary representation of $V_{l_1\lambda_1+\cdots+l_{m-1}\lambda_{m-1}}$ can be decomposed into irreducible unitary representation of $SU(m-1)$ regarding as $SU(m-1)$-modules, that is,*

$$V=\sum V'_{l'_1\lambda_1+\cdots+l'_{m-2}\lambda_{m-2}},$$

where $V'_{l'_1\lambda_1+\cdots+l'_{m-2}\lambda_{m-2}}$ is an irreducible unitary representation of $SU(m-1)$. All non-negative integers l'_i $(1\leq i\leq m-1)$, in the sum, run over the ones satisfying the condition

$$l_1\geq l'_1+l_{m-1}\geq l_2\geq l'_2+l_{m-1}\geq\cdots\geq l_{m-2}\geq l'_{m-2}+l_{m-1}.$$

Moreover all V''s in the sum appear once.

PROPOSITION 5.2. *Any irreducible unitary representation of $SU(m)$ which are spherical with respect to $SU(m-1)$ are exhausted by $V=V_{l_1\lambda_1+\cdots+l_{m-1}\lambda_{m-1}}$, where both l_1 and l_2 run over all non-negative integers satisfying the condition $l_1\geq l_2\geq 0$. Moreover, all such representations are decomposed into irreducible unitary ones of $SU(m-1)$ as*

$$V_{l_1\lambda_1+\cdots+l_2\lambda_{m-1}}=\sum V'_{l'_1\lambda_1},$$

where l'_1 runs over all non-negative integers such that $l_1\geq l'_1+l_2$.

Therefore due to this proposition,

(1) the fixed point set of $V=V_{l_1\lambda_1+l_2\lambda_2+\cdots+l_{m-1}\lambda_{m-1}}$ under the action of $SU(m-1)$ is one dimensional and

(2) all irreducible unitary representations of $SU(m)$ which are spherical with respect to $SU(m-1)$ are exhausted by $V=V_{2k\lambda_1+k\lambda_2+\cdots+k\lambda_{m-1}}$, $k\geq 0$, which are of degree k in both variables $z=(z_1,\cdots,z_m)$ and $\bar z=(\bar z_1,\cdots,\bar z_m)$. The space $H^{k,k}(\boldsymbol{C}^m)$ is naturally identified with the eigenspace of the Laplacian of $(\boldsymbol{CP}^{m-1}, g_{can})$ with the eigenvalue $\lambda_k=4k(k+m-1)$. Here g_{can} is the Fubini-Study metric on $\boldsymbol{CP}^{m-1}$ with sectional curvature $1\leq K\leq 4$.

Now let us denote by $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$, $\mathfrak{n}=\mathfrak{n}_1+\mathfrak{n}_2$, the orthogonal complements of $\mathfrak{j}$, $\mathfrak{h}$ in the Lie algebras $\mathfrak{k}$, $\mathfrak{g}$ with respect to the inner products as in Example 4.2 or Example 4.4 respectively. Then for the homomorphism A obtained by Proposition 5.2, set

$$A(X_j)=U_j+V_j\,,$$

for an orthonormal basis $\{X_j\}_{j=1}^{2m-1}$ of $(\mathfrak{m}, \langle\,,\,\rangle)$, where $U_j\in\mathfrak{n}$ and $V_j\in\mathfrak{h}$. Then it holds that

$$0=C=f_1(t)^{-2}C_1+f_2(t)^{-2}C_2\,,$$

which is equivalent to

$$C_1:=\sum_{j=1}^{2m-2}[U_j, V_j]=0\,, \quad \text{and} \quad C_2:=[U_{2m-1}, V_{2m-1}]=0\,.$$

Indeed, $A(j)$, $j\in J$, acts on $\mathfrak{n}$ by

$$A(j)\cdot W=A(j)\circ W\circ A(j)^{-1}\,, \qquad j\in J=SU(m-1)\,, \quad W\in\mathfrak{n}\,.$$

Then $A(j)\cdot C_i=C_i$ for $j\in J$ and $i=1, 2$. Unless $C_i=0$, $A(j)\in SO(n-2)$ for all $j\in J$ which contradicts the above fact (1) that the fixed point space of V under the action of $SU(m-1)$ is one-dimensional.

It is easy to see both the inner products α_t and β_r on $\mathfrak{m}$ and $\mathfrak{n}$ satisfy the condition in Theorem 3.4 respectively.

Moreover, we get

$$\sum_{j=1}^{2m-2}\langle Y_{2n-1}, A(X_j)\rangle^2=\sum_{j=1}^{2m-2}\|A(X_j)_{\mathfrak{n}_2}\|^2=0\,,$$

$$\sum_{i=1}^{2n-2}\langle Y_i, A(X_{2m-1})\rangle^2=\|A(X_{2m-1})_{\mathfrak{n}_1}\|^2=0\,.$$

We denote

$$\lambda:=\sum_{j=1}^{2m-2}\sum_{i=1}^{2n-2}\langle Y_i, A(X_j)\rangle^2=\sum_{j=1}^{2m-2}\|A(X_j)_{\mathfrak{n}_1}\|^2\,,$$

$$\mu:=\langle Y_{2n-1}, A(X_{2m-1})\rangle^2=\|A(X_{2m-1})_{\mathfrak{n}_2}\|^2\,.$$

Note that for the homomorphism of K into G, denoted by A_k, associated to the irreducible representation $V_k:=V_{2k\lambda_1+k\lambda_2+\cdots+k\lambda_{m-1}}$,

$$\lambda=\lambda_k=4k(k+m-2)\,, \qquad \mu=0\,.$$

Case 4. $(\boldsymbol{CP}^m, g)\to(\boldsymbol{CP}^n, \bar{g})$. In this case, take the above homomorphism $A: K=SU(m)\to G=SU(n)$. The Riemannian metrics g and $\bar{g}$ correspond to the inner products

$$\alpha_t(X, Y)=f_1(t)^2\langle X', Y'\rangle+f_2(t)^2\langle X'', Y''\rangle\,,$$

$$\beta_r(Z, W) = h_1(r)^2 \langle Z', W' \rangle + h_2(r)^2 \langle Z'', W'' \rangle ,$$

for $X = X' + X''$, $Y = Y' + Y'' \in \mathfrak{m}$ and $Z = Z' + Z''$, $W = W' + W'' \in \mathfrak{n}$. Here the functions $f_i(t)$, $h_i(r)$, $i = 1, 2$, satisfy the corresponding conditions in Example 4.4. Then the ordinary differential equation of equivariant harmonic map is given by

$$\begin{aligned} &\ddot{r}(t) + \left\{ (2m-2)\frac{\dot{f}_1(t)}{f_1(t)} + \frac{\dot{f}_2(t)}{f_2(t)} \right\} \dot{r}(t) \\ &-\lambda \frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2} - \mu \frac{h_2(r(t))h_2'(r(t))}{f_2(t)^2} = 0 , \qquad 0 < t < \frac{\pi}{2} . \end{aligned} \tag{5.3}$$

In particular, for the homomorphism A_k corresponding to the representation V_k, the equation (5.3) becomes

$$\begin{aligned} &\ddot{r}(t) + \left\{ (2m-2)\frac{\dot{f}_1(t)}{f_1(t)} + \frac{\dot{f}_2(t)}{f_2(t)} \right\} \dot{r}(t) \\ &-4k(k+m-2)\frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2} = 0 , \qquad 0 < t < \frac{\pi}{2} . \end{aligned} \tag{5.4}$$

Case 5. $(\boldsymbol{C}^m, g) \to (\boldsymbol{CP}^n, \bar{g})$. In this case, we can take the same homomorphism $A : K = SU(m) \to G = SU(n)$ and α_t, β_r as in Case 4. The functions $f_i(t)$ corresponding to the K-invariant Riemannian metric g on $\boldsymbol{C}^m$ satisfy the conditions in Example 4.2. The harmonic map equation is of the same form (5.3) and (5.4) defined on $(0, \infty)$.

Case 6. $(\boldsymbol{C}^m, g) \to (\boldsymbol{C}^n, \bar{g})$. In this case, we also can take the same homomorphism $A : K = SU(m) \to G = SU(n)$ and α_t, β_r as in Case 4. Also the both functions $f(t)$ on $0 < t < \infty$ and $h(r)$ on $0 < r < \infty$ satisfy the conditions in Example 4.2. The harmonic map equation is of the same form (5.3) and (5.4) defined on $(0, \infty)$.

In these cases, the energy density functional is given by

$$e(\phi)(c(t)) = \frac{1}{2} \left\{ \dot{r}(t)^2 + \lambda \frac{h_1(r(t))^2}{f_1(t)^2} \mu \frac{h_2(r(t))^2}{f_2(t)^2} \right\}$$

from (3.7). The equation (5.3) is the Euler-Lagrange equation of

$$E(\phi) = \mathrm{Vol}(K, \langle\, , \rangle) \int_0^l e(\phi)(c(t)) D(t) dt ,$$

where $l = \pi/2$ or ∞ and $D(t)$ is given by

$$D(t) = f_1(t)^{2m-2} f_2(t) .$$

The situation of the Case 1 is some generalization of Eells and Ratto [6]. The special cases, that is, the Euclidean space and hyperbolic space in Cases 2, 3 are treated in a book of Baird [3], and Case 4 is a generalization of the setting of Urakawa [15].

But Cases 5, 6 are new. The existence of solutions $r(t)$ of the equations (5.1)–(5.4) satisfying the condition (3.1) are studied by the second author [14]. Making use of this result, we shall construct new harmonic maps for our cases in the next section.

§ 6. Construction of harmonic maps.

In this section, we firstly refer to the result due to the second author on the existence of global solution for the ordinary differential equation with singularities, and apply it to our construction problem of equivariant harmonic maps.

Let $f_i(t)$, $i=1,2$, be positive C^∞ functions on $(0,\infty)$, and $h_i(r)$, $i=1,2$, also positive C^∞ functions on $(0,\infty)$. Let us consider the following ordinary differential equation on $(0,\infty)$:

$$\text{(6.1)}\qquad \begin{aligned} &\ddot{r}(t)+\left\{p\frac{\dot{f}_1(t)}{f_1(t)}+q\frac{\dot{f}_2(t)}{f_2(t)}\right\}\dot{r}(t)\\ &-\left\{\mu^2\frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2}+\nu^2\frac{h_2(r(t))h_2'(r(t))}{f_2(t)^2}\right\}=0\,. \end{aligned}$$

We want to get a global (without blow-up) solution $r=r(t)$ defined on $(0,\infty)$ satisfying $\lim_{t\to 0} r(t)=0$ regarding (3.1) and the investigation in section 5. To do this, we have to impose the following conditions on $f_i(t)$ and $h_i(r)$:

(I-1) For $i=1,2$, $\dot{f}_i(t)\geq 0$ on $[0,\infty)$.

(I-2) For $i=1,2$, there exist positive constants $a_i>0$ satisfying that

$$f_i(t)=a_it+O(t^3)\qquad (\text{as } t\to 0)\,.$$

(I-3) For all $t_0>0$ there exists a positive constant $C=C(t_0)>0$ such that

$$0\leq pt\frac{\dot{f}_1(t)}{f_1(t)}+qt\frac{\dot{f}_2(t)}{f_2(t)}-1\leq C \qquad \text{on}\quad [0,t_0]\,.$$

(I-4) For some $t_0>0$,

$$\int_{t_0}^{\infty}\left\{\frac{1}{f_1(\tau)}+\frac{1}{f_2(\tau)}\right\}d\tau<\infty\,.$$

(II-1) For $i=1,2$, $\{h_ih_i'\}'(r)\geq 0$ on $[0,\infty)$.

(II-2) For $i=1,2$, there exist positive constants $b_i>0$ satisfying that

$$h_i(r)=b_ir+O(r^3)\qquad (\text{as } r\to 0)\,.$$

(II-3) For $i=1,2$, there exist positive constants $c_i>0$ such that

$$h_i(r)\leq \sinh(c_ir)\qquad \text{on}\quad (0,\infty)\,.$$

Then we obtain the following theorem.

THEOREM 6.2 ([14, Theorem 1]). *Let $f_i(t)$, $h_i(r)$, $i=1,2$, be positive C^∞ functions on* $(0,\infty)$ *satisfying the above conditions* (I-1)–(II-3). *Let μ and ν be two non-negative real numbers and p and q two non-negative integers satisfying $p+q\geq 1$. Then for all $t_0>0$, there exist positive constants $\alpha>0$ and $\beta>0$ such that a positive solution $r=r(t)$ of* (6.1) *exists globally on* $(0,\infty)$ *and satisfies*

$$\lim_{t\to 0} r(t)=0 \quad \textit{and} \quad \lim_{t\to\infty} r(t)\nearrow r_*\,,$$

where r_* is some positive constant.

Moreover we assume the following conditions:

(II-4) For $i=1,2$, $h_i''(r)\geq 0$ on $[0,\infty)$.

(II-5) For $i=1,2$, there exist positive constants $d_i>0$ and $l>1$ such that

$$h_1(r)\geq d_1 r^l \quad \text{or} \quad h_2(r)\geq d_2 r^l$$

for sufficiently large $r>0$.

Then we can show the following theorem.

THEOREM 6.3. ([14, Theorem 2]). *There exists a global solution $r=r(t)$ to* (6.1) *satisfying*

$$\lim_{t\to 0} r(t)=0 \quad \textit{and} \quad \lim_{t\to\infty} r(t)\nearrow \infty\,.$$

Here we examine the conditions in real 2-dimensional case, that is, we consider equivariant harmonic maps

$$(\boldsymbol{R}^2, dt^2+f(t)^2d\theta^2)\ni(t,\theta)\mapsto(r(t),\phi(\theta))\in(\boldsymbol{R}^2, dr^2+h(r)^2d\phi^2)\,,$$

where (t,θ) and (r,ϕ) are polar coordinates on $\boldsymbol{R}^2$. Then the equation (6.1) becomes

$$\ddot{r}(t)+\frac{\dot{f}(t)}{f(t)}-\mu^2\frac{h(r(t))h'(r(t))}{f(t)^2}=0\,, \tag{6.4}$$

and we can obtain the explicit solution to (6.4) (see [14, §4]).

Case 1. $(\boldsymbol{R}^2, g_{can})\to(\boldsymbol{RH}^2, \bar{g}_{can})$. In this case, $f(t)=t$, $h(r)=\sinh r$, and the condition (I-4) is broken. On the other hand, the solution to (6.4) is given by

$$\tanh\frac{r(t)}{2}=Ct^\mu\,,$$

where C is a constant determined by the initial value. This solution blows up at t_* where $Ct_*^\mu=1$.

Case 2. $(\boldsymbol{R}^2, g_{can}) \to (\boldsymbol{R}^2, \bar{g}_{can})$. In this case, $f(t)=t$, $h(r)=r$, and the condition (I-4) is broken. However the solution to (6.4) is given by

$$r(t)=C't^{\mu},$$

where C' is a constant determined by the initial value, and this solution exists globally on $[0, \infty)$.

These examples imply that if $h(r)$ increases rapidly, then the condition (II-4) is necessary for the existence of a global solution. In fact, if $f(t)=t$ and $h(r) \geq \sinh cr$, then there is no global solution to (6.4). The condition (II-3) is used to estimate the initial value $\alpha > 0$ for which the solution $r=r(t)$ is bounded. We can replace the function $\sinh c_i r$ in (II-3) by $\exp c_i r$, however it is not necessary for applications. Now we shall state our main theorem.

THEOREM 6.5. (I) *Let* (M, g) *and* $(N, \bar{g})$ *be two non-compact complete Riemannian manifolds which coincide with* $(\boldsymbol{R}^m, g)$ *and* $(\boldsymbol{R}^n, \bar{g})$ *given as in* 4.1 *respectively, and let* $A: K=SO(m) \to G=SO(n)$ *be a homomorphism induced from an n-dimensional irreducible orthogonal representation of* $SO(m)$ *which is spherical with respect to* $SO(m-1)$ *as in Case* 3 *of* 5.1. *Assume that the positive* C^∞ *functions* $f_1(t)=f_2(t)=f(t)$ *and* $h_1(r)=h_2(r)=h(r)$ *satisfy the above conditions* (I-1)–(II-3). *Then there exists a full A-equivariant harmonic map* ϕ *from* (M, g) *into* $(N, \bar{g})$.

(II) *Let* (M, g) *and* $(N, \bar{g})$ *be two non-compact complete Riemannian manifolds which coincides with* $(\boldsymbol{C}^m, g)$ *and* $(\boldsymbol{C}^n, \bar{g})$ *given as in* 4.1 *respectively, and let* $A: K=SU(m) \to G=SU(n)$ *be a homomorphism induced from an n-dimensional irreducible unitary representation of* $SU(m)$ *which is spherical with respect to* $SU(m-1)$ *as in Case* 5 *of* 5.2. *Assume that the positive* C^∞ *functions* $f_i(t)$ *and* $h_i(r)$ *on* $[0, \infty)$ *with* $i=1, 2$ *satisfy the above conditions* (I-1)–(II-3). *Then there exists a full A-equivariant harmonic map* ϕ *from* (M, g) *into* $(N, \bar{g})$.

Here we call a C^∞ map ϕ from $\boldsymbol{R}^m$ (resp. $\boldsymbol{C}^m$) into $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) to be full if there exists no hyperplane of $\boldsymbol{R}^m$ (resp. $\boldsymbol{C}^m$) which includes the image of ϕ. The fullness of ϕ in Theorem 6.5 follows from A-equivariance of ϕ and irreducibility of the corresponding representations.

COROLLARY 6.6. (I) *Let* $A: K=SO(m) \to G=SO(n)$ *be a homomorphism induced from an n-dimensional irreducible orthogonal representation of* $SO(m)$ *which is spherical with respect to* $SO(m-1)$. *Then there exist full A-equivariant harmonic maps*

(1) *from the standard Euclidean space* $(\boldsymbol{R}^m, g_{can})$ *into another one* $(\boldsymbol{R}^n, \bar{g}_{can})$,

(2) *from the hyperbolic space* $(\boldsymbol{RH}^m, g_{can})$ *into the standard Euclidean space* $(\boldsymbol{R}^n, \bar{g}_{can})$,

(3) *from the hyperbolic space* $(\boldsymbol{RH}^m, g_{can})$ *into another hyperbolic space* $(\boldsymbol{RH}^n, \bar{g}_{can})$.

(II) *Let* $A: K=SU(m) \to G=SU(n)$ *be a homomorphism induced from an n-dimensional irreducible unitary representation of* $SU(m)$ *which is spherical with respect to* $SU(m-1)$. *Then there exist full A-equivariant harmonic maps*

(1) *from the standard complex Euclidean space* $(\boldsymbol{C}^m, g_{can})$ *into another one* $(\boldsymbol{C}^n, \bar{g}_{can})$,

(2) *from the complex hyperbolic space* $(\boldsymbol{CH}^m, g_{can})$ *into the standard complex Euclidean space* $(\boldsymbol{C}^n, \bar{g}_{can})$,
(3) *from the complex hyperbolic space* $(\boldsymbol{CH}^m, g_{can})$ *into another complex hyperbolic space* $(\boldsymbol{CH}^n, \bar{g}_{can})$.

Here we note that the cases (I)-(1) and (II)-(1) are not covered by Theorem 6.5. However, in this case, we can reduce the problem to prove the existence of harmonic functions.

REMARK 6.7. (1) For Case 1: $(S^m, g) \to (S^n, \bar{g})$ in 5.1 and Case 4: $(\boldsymbol{CP}^m, g) \to (\boldsymbol{CP}^n, \bar{g})$ in 5.2, see Urakawa [15].

(2) For any given $R > 0$, there exists a global solution of (6.1) satisfying $r(t) \leq R$ on $[0, \infty)$. We do not know, however, the $\lim_{t\to\infty} r(t) = R$ holds or not. If $\lim_{t\to\infty} r(t) = R$, then ϕ may be an onto map from $(\boldsymbol{R}^m, \bar{g})$ to $B_R(0)$. Here we denote $B_R(0)$ the geodesic ball of radius R in $(\boldsymbol{R}^n, \bar{g})$ or $(\boldsymbol{C}^n, \bar{g})$.

(3) We can show that there exists an equivariant harmonic maps ϕ from $(\boldsymbol{R}^m, g)$ (resp. $(\boldsymbol{C}^m, g)$) into the geodesic ball $B_R(0)$ in $(S^n, \bar{g})$ (resp. $(\boldsymbol{CP}^n, \bar{g})$) for some radius $R > 0$.

(4) For the case of the standard Euclidean space $(\boldsymbol{R}^m, g)$ into the hyperbolic space $(\boldsymbol{RH}^n, \bar{g})$, and the standard complex Euclidean space $(\boldsymbol{C}^m, g)$ into the complex hyperbolic space $(\boldsymbol{CH}^n, \bar{g})$, to construct full A-equivariant harmonic maps is still unsolved problem for us because these cases do not satisfy the condition (I-4).

(5) If we take the homomorphism $A = id : SO(m-1) \to SO(m-1)$, then Tachikawa [11] showed that there exists no A-equivariant harmonic map ϕ from $(\boldsymbol{R}^m, g_{can})$ into $(\boldsymbol{RH}^m, g_{can})$.

References

[1] K. AKUTAGAWA, Harmonic diffeomorphisms of the hyperbolic plane, Trans. Amer. Math. Soc. (1994), (to appear).

[2] T. ASOH, Compact transformation groups on Z_2-cohomology spheres with orbit of codimension one, Hiroshima Math. J. **11** (1981), 571–616.

[3] P. BAIRD, *Harmonic Maps with Symmetry, Harmonic Morphisms and Deformations of Metrics*, Pitman (1983).

[4] P. BERARD-BERGERY, Sur de nouvelles variétés riemannienne d'Einstein, Publ. Inst. E. Cartan **6** (1982).

[5] J. EELLS and L. LEMAIRE, Another report on harmonic maps, Bull. London Math. Soc. **20** (1988), 385–524.

[6] J. EELLS and A. RATTO, Harmonic maps between spheres and ellipsoids, Internat. J. Math. **1** (1990), 1–27.

[7] J. EELLS and J. H. SAMPSON, Harmonic mappings of Riemannian manifolds, Amer. J. Math. **86** (1964), 109–160.

[8] W. Y. HSIANG and H. B. LAWSON, Minimal submanifolds of low cohomogeneity, J. Differential Geom. **5** (1971), 1–38.

[9] A. KASUE and T. WASHIO, Growth of equivariant harmonic maps and harmonic morphisms, Osaka J. Math. **27** (1990), 899–928.

[10] P. LI and L.-F. TAM, The heat equation and harmonic maps of complete manifolds, Invent. Math.

105 (1991), 1–46.

[11] A. TACHIKAWA, A non-existence result for harmonic mappings from $\boldsymbol{R}^n$ into $\boldsymbol{H}^n$, Tokyo J. Math. **11** (1985), 311–316.

[12] R. TAKAGI and T. TAKAHASHI, On the principal curvature of homogeneous hypersurfaces in a sphere, *Differential Geometry in honor of K. Yano*, Kinokuniya (1972), 469–481.

[13] F. UCHIDA, Classification of compact transformation groups on cohomology complex projective spaces, Japanese J. Math. **3** (1977), 141–189.

[14] K. UENO, A study of ordinary differential equation arising from equivariant harmonic map, Tokyo J. Math. **17** (1994), 395–416.

[15] H. URAKAWA, Equivariant harmonic maps between compact Riemannian manifolds of cohomogeneity-one, Michigan Math. J. **40** (1993) 27–51.

[16] Y. L. XIN, Equivariant harmonic maps into the sphere via isoparametric maps, (to appear).

Present Addresses:
HAJIME URAKAWA
MATHEMATICS LABORATORIES, GRADUATE SCHOOL OF INFORMATION SCIENCES, TÔHOKU UNIVERSITY,
KAWAUCHI, SENDAI, 980 JAPAN.

KEISUKE UENO
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, YAMAGATA UNIVERSITY,
YAMAGATA, 990 JAPAN.