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A Study of Ordinary Differential Equations Arising from Equivariant Harmonic Maps

Keisuke UENO

Tôhoku University

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1. Introduction.

In the theory of harmonic maps between Riemannian manifolds, the problems of existence and construction are basic and important. One of the methods of constructing harmonic maps is the one by making use of ordinary differential equations. For example, Ding [2], Eells and Ratto [3], Ratto [5] and Smith [6] use the join of two maps and derive ordinary differential equations. They construct harmonic maps between spheres or between spheres and ellipsoids. Xin [10] studies equivariant harmonic maps with respect to Riemannian submersion. He applies this method to the existence of harmonic representatives of $\pi_{2m+1}(S^{2m+1})$([11]). Also Urakawa [8], Urakawa and author [9] investigate the theory of equivariant harmonic maps between Riemannian manifolds admitting large isometry group actions.

For constructing equivariant harmonic maps, it is important to study ordinary differential equations with singularities. In particular, in relation with the regularity of solutions and the problem asking how the image of equivariant harmonic maps expands, we want to know the asymptotic behavior of a solution of the ordinary differential equation nearby its singularities. On the other hand, from the viewpoint of ordinary differential equation theory it is interesting to investigate a solution on the blow-up phenomena or its regularity of a solution at singularities.

In this paper, we study the existence of a positive solution satisfying the following equations (1.1) and (1.2):

$$
\begin{aligned}
&\ddot{r}(t)+\left\{p\frac{\dot{f}_1(t)}{f_1(t)}+q\frac{\dot{f}_2(t)}{f_2(t)}\right\}\dot{r}(t) \\
(1.1)\qquad &-\left\{\mu^2\frac{h_1(r(t))h_1'(r(t))}{f_1(t)^2}+\nu^2\frac{h_2(r(t))h_2'(r(t))}{f_2(t)^2}\right\}=0 \qquad \text{on} \quad [0,\infty)\,;
\end{aligned}
$$

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$$\lim_{t\to 0} r(t)=0\,, \tag{1.2}$$

where $\dot r$ (resp. h') means dr/dt (resp. dh/dr), and μ and ν are non-negative numbers, p and q are non-negative integers which satisfy $p+q\geq 2$. The equation (1.1) appears in the study of equivariant harmonic maps between two complete non-compact Riemannian manifolds, for instance, the real (resp. complex) Euclidean space $(\boldsymbol{R}^m, g_{can})$ (resp. $(\boldsymbol{C}^m, g_{can})$), the real hyperbolic space $(\boldsymbol{RH}^m, g_{can})$ of constant negative curvature -1 and the complex hyperbolic space $(\boldsymbol{CH}^m, g_{can})$.

Throughout this paper, we assume that C^∞-functions $f_i=f_i(t)$ $(i=1,2)$ on $[0,\infty)$ satisfy the following conditions:

$$\left\{\begin{array}{l}\cdot\ f_i(t)>0 \text{ on } (0,\infty), \text{ and } \dot f_i(t)\geq 0 \text{ on } [0,\infty)\,;\\ \cdot\ \text{there exist positive constants } a_i>0 \text{ such that}\\ \qquad f_i(t)=a_i t+O(t^3) \qquad (\text{as } t\to 0)\,;\\ \cdot\ \text{for any } t_0>0 \text{ there exists a positive constant } C=C(t_0)>0 \text{ such that}\\ \qquad 0\leq pt\dfrac{\dot f_1(t)}{f_1(t)}+qt\dfrac{\dot f_2(t)}{f_2(t)}-1\leq C \qquad \text{on} \quad [0,t_0]\,;\end{array}\right. \tag{1.3}$$

and

$$\int_{t_0}^{\infty}\left\{\frac{1}{f_1(\tau)}+\frac{1}{f_2(\tau)}\right\}d\tau<\infty\,. \tag{1.4}$$

Moreover we assume the following conditions for C^∞-functions $h_i(r)$ $(i=1,2)$ on $[0,\infty)$:

$$\left\{\begin{array}{l}\cdot\ 0<h_i(r)\leq \sinh br \text{ on } (0,\infty) \text{ for some constant } b>0 \text{ and}\\ \quad \{h_i h_i'\}'(r)\geq 0 \text{ on } [0,\infty);\\ \cdot\ \text{there exist positive constants } b_i>0 \text{ such that}\\ \qquad h_i(r)=b_i r+O(r^3) \qquad (\text{as } r\to 0);\end{array}\right. \tag{1.5}$$

and

$$\left\{\begin{array}{l}\cdot\ h_i''(r)\geq 0 \text{ on } [0,\infty)\,;\\ \cdot\ \text{there exist positive constants } d_i>0 \text{ and } l>1 \text{ such that}\\ \qquad h_1(r)\geq d_1 r^l \quad \text{or} \quad h_2(r)\geq d_2 r^l\\ \quad \text{for sufficiently large } r>0\,.\end{array}\right. \tag{1.6}$$

For example, if we consider equivariant harmonic maps from $(\boldsymbol{RH}^m, g_{can})$ into $(\boldsymbol{RH}^n, h_{can})$, then (1.1) becomes

$$\ddot r(t)+(m-1)\frac{\cosh t}{\sinh t}\dot r(t)-\mu^2\frac{\sinh r(t)\cosh r(t)}{\sinh^2 t}=0 \qquad \text{on } [0,\infty)\,, \tag{1.1$'$}$$

moreover conditions (1.3)–(1.6) are fulfilled. In this case, for $\mu>0$, we can find a solution which yields equivariant harmonic map from $\boldsymbol{RH}^m$ into $\boldsymbol{RH}^n$.

We shall consider the initial value problem (1.1) with (1.7) under the conditions (1.3)–(1.6):

$$r(t_0)=\alpha\,, \qquad \dot{r}(t_0)=\beta\,, \tag{1.7}$$

where $\alpha>0$ and $\beta>0$. Then the main theorem of this paper is

THEOREM 1. *Under the assumptions* (1.3)–(1.5), *for any fixed* $t_0>0$, *there exist positive numbers* α *and* β *such that the initial value problem* (1.1) *with* (1.7) *has a positive solution* $r=r_\alpha(t)$ *which satisfies*

(i) $\lim_{t\to 0} r_\alpha(t)=0$,

(ii) $T_\alpha=\infty$, $r_\alpha(t)\geq 0$ *and* $r_\alpha(t)$ *is bounded on* $[0, \infty)$,

where T_α *is the life span of the solution* $r_\alpha(t)$.

The proof of this theorem is divided into two parts. In §2, we shall show that for any $\alpha>0$ there exists $\beta>0$ uniquely determined by α such that the solution $r=r(t)$ of (1.1) with (1.7) tends to 0 as t tends to 0 (see Theorem 2.1). We denote this solution by $r_\alpha(t)$. The proof of this part is basically due to Baird ([1]), Kasue and Washio ([4]). In §3 we shall investigate the asymptotic behavior of $r_\alpha(t)$ as t tends to ∞ and prove the existence of a solution satisfying the condition in Theorem 1. Moreover, under the assumptions (1.3)–(1.6), we can show the following theorem.

THEOREM 2. *There exists a global solution* r *to* (1.1)–(1.2) *satisfying*

$$\lim_{t\to\infty} r(t)\nearrow\infty\,.$$

Note that the condition (1.4) is essential for the existence of $\alpha>0$ with $T_\alpha=\infty$. For example, this condition is not satisfied if $f_1(t)=f_2(t)=t$. (In this case, the domain manifold is the real Euclidean space.) Tachikawa [7] proves that there exists no rotationally symmetric harmonic map from $\boldsymbol{R}^m$ into $\boldsymbol{RH}^m$ in some case. Thus we cannot omit the condition (1.4).

Also in the last section, we study the case $p+q=1$. In this case we can solve the equation (1.1) explicitly, and determine α so that $r=r_\alpha(t)$ satisfies Theorems 1 and 2.

As an application of Theorem 1 we can construct equivariant harmonic maps between some non-compact Riemannian manifolds, for example, from $(\boldsymbol{RH}^m, g_{can})$ into $(\boldsymbol{RH}^n, h_{can})$, from $(\boldsymbol{CH}^m, g_{can})$ into $(\boldsymbol{CH}^n, h_{can})$ for some $m\geq 2$ and n determined by m (see [9] for details).

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2. Asymptotic behavior as $t \to 0$.

In this section, we consider the equation (1.1), and discuss the behavior of a solution when t tends to 0. From now on we fix $t_0 > 0$ to the end of this paper. First, let us reparametrize the equation (1.1) with parameter $u = \log t$ to remove singularity at $t = 0$. Then (1.1), (1.7) and (1.3) become respectively

$$\text{(2.1)} \quad \begin{aligned} &\frac{d^2 r}{du^2}(u) + \left\{ \frac{p}{f_1(u)} \frac{df_1}{du}(u) + \frac{q}{f_2(u)} \frac{df_2}{du}(u) - 1 \right\} \frac{dr}{du}(u) \\ &\qquad - e^{2u} \left\{ \mu^2 \frac{h_1(r(u))h'_1(r(u))}{f_1(u)^2} + \nu^2 \frac{h_2(r(u))h'_2(r(u))}{f_2(u)^2} \right\} = 0 \, ; \end{aligned}$$

$$\text{(2.2)} \quad r(u_0) = \alpha \, , \qquad r'(u_0) = \alpha' \, ;$$

and

$$\text{(2.3)} \quad \begin{cases} \cdot \ f_i \in C^\infty(-\infty, \infty), \ f_i(u) > 0, \ (df_i/du)(u) \geq 0 \ (u \in (-\infty, \infty)) \, ; \\ \cdot \ f_i(u) = a_i e^u + O(e^{3u}) \ (\text{as } u \to -\infty) \, ; \\ \cdot \ \text{for any } u_0 > -\infty \text{ there exists constant } C = C(u_0) \text{ such that} \\ \qquad 0 \leq \dfrac{p}{f_1(u)} \dfrac{df_1}{du}(u) + \dfrac{q}{f_2(u)} \dfrac{df_2}{du}(u) - 1 \leq C \qquad \text{on} \quad (-\infty, u_0] \, . \end{cases}$$

For any fixed $\alpha > 0$, define a set $\mathscr{A}(\alpha)$ by

$$\mathscr{A}(\alpha) := \{ \alpha' \in \boldsymbol{R} \mid \text{a solution of (2.1) with (2.2) decreases monotonically to zero within finite time as } u \text{ decreases from } u_0 \} \, .$$

Then, for any fixed $\alpha > 0$, $\mathscr{A}(\alpha)$ is an open, non-empty set and $\inf \mathscr{A}(\alpha) > 0$ (cf. [1], Chapter 6 pp. 99–100). We put $\alpha' := \inf \mathscr{A}(\alpha)$. Since α' is uniquely determined by α, a solution $r = r(u)$ of the initial value problem (2.1) with (2.2) depends only on α. We denote this solution by $r_\alpha(u)$. Then we obtain

THEOREM 2.1. *Under the conditions* (1.5) *and* (2.3), *for any* $\alpha > 0$, *put* $\alpha' := \inf \mathscr{A}(\alpha)$. *Then the solution* $r = r_\alpha(u)$ *of* (2.1) *with* (2.2) *exists on* $(-\infty, u_0]$ *and satisfies the following*:

(i) $r_\alpha(u) > 0$, $\dfrac{dr_\alpha}{du}(u) > 0$ *on* $(-\infty, u_0)$, *and*

(ii) $\lim_{u \to -\infty} r_\alpha(u) = 0$, $\lim_{u \to -\infty} \dfrac{dr_\alpha}{du}(u) = 0$ *and* $\lim_{u \to -\infty} \dfrac{d^2 r_\alpha}{du^2}(u) = 0$.

We first show the following lemma concerning the life span of r_α.

LEMMA 2.2. *For the positive solution* $r = r_\alpha(u)$, $(dr_\alpha/du)(u)$ *remains bounded if* $r_\alpha(u)$ *is bounded. Moreover, if* $r_\alpha(u)$ *blows up at some* $\bar{u} < u_0$, *then* $\lim_{u \to \bar{u}} r_\alpha(u) = +\infty$.

PROOF. Equation (2.1) is equivalent to

$$\frac{d}{du}\left\{f_1(u)^p f_2(u)^q e^{-u}\frac{dr}{du}(u)\right\}=\mu^2 f_1(u)^{p-2}f_2(u)^q e^u h_1(r(u))h_1'(r(u))$$
$$+\nu^2 f_1(u)^p f_2(u)^{q-2}e^u h_2(r(u))h_2'(r(u))\,.$$

Integrating both sides from u ($<u_0$) to u_0, we obtain

$$-f_1(u)^p f_2(u)^q e^{-u}\frac{dr}{du}(u)+f_1(u_0)^p f_2(u_0)^q e^{-u_0}\frac{dr}{du}(u_0)$$
$$=\int_u^{u_0}\{\mu^2 f_1(v)^{p-2}f_2(v)^q e^v h_1(r(v))h_1'(r(v))+\nu^2 f_1(v)^p f_2(v)^{q-2}e^v h_2(r(v))h_2'(r(v))\}dv\,.$$

Hence

$$f_1(u)^p f_2(u)^q e^{-u}\left|\frac{dr}{du}(u)\right|\le C_0(u_0)+C_1(u_0)\max_{v\in[u,u_0]}|h_1(r(v))h_1'(r(v))|$$
$$+C_2(u_0)\max_{v\in[u,u_0]}|h_2(r(v))h_2'(r(v))|\,,$$

where $C_0(u_0)$, $C_1(u_0)$ and $C_2(u_0)$ are constants given by

$$\begin{cases} C_0(u_0)=f_1(u_0)^p f_2(u_0)^q e^{-u_0}\dfrac{dr}{du}(u_0)\,,\\ C_1(u_0)=\mu^2\displaystyle\int_{-\infty}^{u_0} f_1(v)^{p-2}f_2(v)^q e^v dv\,,\\ C_2(u_0)=\nu^2\displaystyle\int_{-\infty}^{u_0} f_1(v)^p f_2(v)^{q-2}e^v dv\,.\end{cases}$$

The assertion follows from this inequality. □

PROOF OF THEOREM 2.1. First, for simplicity, set

$$Q(u,s):=e^{2u}\left\{\mu^2\frac{h_1(s)h_1'(s)}{f_1(u)^2}+\nu^2\frac{h_2(s)h_2'(s)}{f_2(u)^2}\right\}.$$

Then $Q(u,s)$ satisfies the following properties

$$\begin{cases}\cdot\ Q(u,0)=0,\ Q(u,s)>0 \text{ for } 0<s, \text{ and}\\ \cdot\ Q(u,s_1)<Q(u,s_2) \text{ for } 0<s_1<s_2\,.\end{cases}$$

(i) Suppose $(u_\alpha, u^\alpha]$ is a maximal existence interval of r_α. Then the assertions

$$r_\alpha(u)>0 \quad\text{and}\quad \frac{dr_\alpha}{du}(u)>0 \qquad\text{on}\quad (u_\alpha, u_0]$$

follow from the same arguments as in [1]. It remains to prove that $u_\alpha = -\infty$. If $u_\alpha > -\infty$, then $r_\alpha(u)$ blows up at u_α. By virtue of Lemma 2.2, we have

$$r_\alpha(u) \to +\infty \qquad (\text{as } u \to u_\alpha).$$

In this case, however, there exists $u_* \in (u_\alpha, u_0)$ such that $(dr_\alpha/du)(u_*) < 0$. This contradicts $(dr_\alpha/du)(u) > 0$ on $(u_\alpha, u_0]$. Thus $u_\alpha = -\infty$.

(ii) It holds that $\lim_{u\to-\infty}(dr_\alpha/du)(u) = 0$. This is proved in the same way as in [4]. We shall show that $\lim_{u\to-\infty} r_\alpha(u) = 0$. In view of (i), there exists $\eta \geq 0$ such that

$$r_\alpha(u) \searrow \eta \qquad (\text{as } u \to -\infty).$$

On the other hand, it follows from (2.1), (2.3) and (i) that for sufficiently small $\bar{u} < u_0$, there exists a positive constant $C(\bar{u})$ such that

$$\frac{d^2 r_\alpha}{du^2}(u) \geq -C(\bar{u})\frac{dr_\alpha}{du}(u) + Q(u, r_\alpha(u)) \qquad \text{on} \quad (-\infty, \bar{u}). \tag{2.4}$$

Suppose that $\eta > 0$. Then

$$Q(u, r_\alpha(u)) > Q(u, \eta) > 0 \qquad \text{on} \quad (-\infty, \bar{u}).$$

Since $\lim_{u\to-\infty}(dr_\alpha/du)(u) = 0$, we obtain from (2.4)

$$\frac{d^2 r_\alpha}{du^2}(u) \geq \tilde{C}(\bar{u}) > 0 \qquad \text{on} \quad (-\infty, \bar{u})$$

for sufficiently small $\bar{u}$. This inequality implies $(dr_\alpha/du)(u) < 0$ for sufficiently small u. This is a contradiction. Thus

$$\lim_{u\to-\infty} r_\alpha(u) = 0.$$

Finally the equation (2.1) implies

$$\frac{d^2 r_\alpha}{du^2}(u) = -\left\{\frac{p}{f_1(u)}\frac{df_1}{du}(u) + \frac{q}{f_2(u)}\frac{df_2}{du}(u) - 1\right\}\frac{dr_\alpha}{du}(u) + Q(u, r_\alpha(u)).$$

We see from the conditions (2.3) that

$$\begin{cases} \displaystyle\lim_{u\to-\infty}\left\{\frac{p}{f_1(u)}\frac{df_1}{du}(u) + \frac{q}{f_2(u)}\frac{df_2}{du}(u) - 1\right\} = p + q - 1 \quad \text{and} \\ \displaystyle\lim_{u\to-\infty} Q(u, r_\alpha(u)) = 0 \end{cases}$$

which implies

$$\lim_{u\to-\infty}\frac{d^2 r_\alpha}{du^2}(u) = 0.$$

Thus the proof is completed. □

As a consequence of Theorem 2.1, we have

COROLLARY 2.3. *Under the conditions* (1.3) *and* (1.5), *for any fixed* $t_0>0$ *and* $\alpha>0$ *there exists* $\beta>0$ *such that the solution* $r=r_\alpha(t)$ *of the initial value problem* (1.1) *with* (1.7) *exists on* $[0, t_0]$ *and satisfies the following*:

(i) $r_\alpha(t)>0$, $\dot r_\alpha(t)>0$ *on* $(0, t_0]$.

(ii) $\lim_{t\to 0} r_\alpha(t)=0$ *and* $\dot r_\alpha(t)=o(1/t)$ (*as* $t\to 0$).

Now we can prove the first assertion of Theorem 1 as follows. For any fixed $t_0>0$ and α, if we set $\beta:=t_0^{-1}\inf\mathscr{A}(\alpha)$ $(=\beta(\alpha)$ for simplicity), then the solution $r=r_\alpha(t)$ with the initial values $r_\alpha(t_0)=\alpha$, $\dot r_\alpha(t_0)=\beta$ satisfies the first assertion of Theorem 3.1.

Note that we can take $\alpha>0$ arbitrary. In the next section, we shall show the existence of $\alpha>0$ which satisfies the second assertion of Theorem 1.

Next we want to show the following proposition, which concerns the regularity of the solution $r=r_\alpha(t)$ at $t=0$.

PROPOSITION 2.4. *Let* $r=r_\alpha(t)$ *be a solution of* (1.1) *with* (1.2). *Then the following assertions hold.*

(i) *There exist* $a>0$ *and* $t_1\in(0, t_0)$ *such that*

$$0<r_\alpha(t)\le r_\alpha(t_1)\frac{t^a}{t_1^a} \qquad on \quad (0, t_1)\,.$$

(ii) *Set* $\lambda_1=p+q-1$, $\lambda_2=\mu^2b_1^2/a_1^2+\nu^2b_2^2/a_2^2$ (a_i *and* b_i *are defined in* (1.3) *and* (1.5)). *If* $\lambda_2>k\lambda_1+k^2$, *then*

$$\lim_{t\to 0} r^{(l)}(t)=0 \qquad (0\le l\le k)\,,$$

where $r^{(l)}(t)$ *stands for the* l-*th derivative of* $r(t)$.

Hence if $\lambda_2>\lambda_1+1$, $r_\alpha(t)$ is of class C^1 on $[0, T_\alpha)$. We first show the following lemma.

LEMMA 2.5. *Let* $A_i(t)$, $B_i(t)\in C^0((0, t_0])$ *such that* $0<B_i(t)\le A_i(t)$ *on* $(0, t_0]$. *Suppose* $r(t)$ *and* $\rho(t)$ *satisfy*

$$\frac{d}{dt}[f_1(t)^pf_2(t)^q\dot r(t)]=\{f_1(t)^{p-2}f_2(t)^qA_1(t)+f_1(t)^pf_2(t)^{q-2}A_2(t)\}r(t)\,, \tag{2.5}$$

$$\frac{d}{dt}[f_1(t)^pf_2(t)^q\dot\rho(t)]\le\{f_1(t)^{p-2}f_2(t)^qB_1(t)+f_1(t)^pf_2(t)^{q-2}B_2(t)\}\rho(t) \tag{2.6}$$

on $(0, t_0)$ *and*

$$\begin{cases} r(t_0)=\rho(t_0)\,, \quad r(t)>0\,, \quad \rho(t)>0 \qquad \textit{on} \quad (0, t_0] \quad \textit{and} \\ \lim_{t\to 0} r(t)=0\,, \quad \lim_{t\to 0} \rho(t)=\delta \qquad \textit{for some} \ \ \delta\ge 0\,. \end{cases}$$

Then we have

$$r(t)\le\rho(t) \qquad \textit{on} \quad [0, t_0]\,.$$

PROOF. Set $w(t):=r(t)-\rho(t)$. Then from (2.5) and (2.6)

$$\begin{aligned} \frac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{w}(t)\} &\ge f_1(t)^{p-2} f_2(t)^q \{A_1(t) r(t) - B_1(t)\rho(t)\} \\ &\quad + f_1(t)^p f_2(t)^{q-2}\{A_2(t) r(t) - B_2(t)\rho(t)\} \\ &\ge f_1(t)^{p-2} f_2(t)^{q-2}\{f_2(t)^2 B_1(t) + f_1(t)^2 B_2(t)\} w(t)\,. \end{aligned}$$

Hence $w=w(t)$ satisfies

$$\text{(2.7)} \qquad \begin{cases} \dfrac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{w}(t)\} \ge f_1(t)^{p-2} f_2(t)^{q-2}\{f_2(t)^2 B_1(t) + f_1(t)^2 B_2(t)\} w(t)\,, \\ w(t_0)=0 \quad \text{and} \quad \lim_{t\to 0} w(t) = -\delta\,(\le 0)\,, \end{cases}$$

which implies that $w(t)\le 0$. In fact, we assume that there exists $\bar{t}\in(0, t_0)$ such that $w(\bar{t})>0$. Then we can find $t_1\in(0, t_0)$ such that $w(t_1)>0$, $\dot{w}(t_1)=0$ and $\ddot{w}(t_1)\le 0$. On the other hand, the inequality in (2.7) implies

$$\begin{aligned} &f_1(t_1)^p f_2(t_1)^q \ddot{w}(t_1) \\ &\qquad \ge f_1(t_1)^{p-2} f_2(t_1)^{q-2}\{f_2(t_1)^2 B_1(t_1) + f_1(t_1)^2 B_2(t_1) w(t_1)\} > 0\,. \end{aligned}$$

Since $f_1(t_1)^p f_2(t_1)^q>0$, this contradicts $\ddot{w}(t_1)\le 0$. □

PROOF OF PROPOSITION 2.4. (i) The equation (1.1) is equivalent to

$$\text{(2.8)} \qquad \begin{aligned} \frac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{r}_\alpha(t)\} &= f_1(t)^{p-2} f_2(t)^{q-2}\{\mu^2 f_2(t)^2 h_1(r_\alpha(t)) h_1'(r_\alpha(t)) \\ &\qquad + \nu^2 f_1(t)^2 h_2(r_\alpha(t)) h_2'(r_\alpha(t))\}\,. \end{aligned}$$

We want to apply Lemma 2.5 with

$$A_1(t)=\mu^2 \frac{h_1(r_\alpha(t)) h_1'(r_\alpha(t))}{r_\alpha(t)}\,, \qquad A_2(t)=\nu^2 \frac{h_2(r_\alpha(t)) h_2'(r_\alpha(t))}{r_\alpha(t)}\,.$$

Since

$$\lim_{t\to 0} A_1(t)=\mu^2 b_1^2 \quad \text{and} \quad \lim_{t\to 0} A_2(t)=\nu^2 b_2^2\,,$$

there exist constants B_1 and B_2 which satisfy $0<B_i\leq A_i(t)$ on $[0, t_0]$. Put $\rho(t):=C_0t^a$, where constants $a>0$ and C_0 will be determined later. Then

$$\frac{d}{dt}\{f_1(t)^pf_2(t)^q\dot{\rho}(t)\}-f_1(t)^{p-2}f_2(t)^{q-2}\{f_2(t)^2B_1+f_1(t)^2B_2\}\rho(t)$$

$$=C_0t^{a-2}f_1(t)^pf_2(t)^q\left\{a(a-1)+apt\frac{\dot{f}_1(t)}{f_1(t)}+aqt\frac{\dot{f}_2(t)}{f_2(t)}-B_1\frac{t^2}{f_1(t)^2}-B_2\frac{t^2}{f_2(t)^2}\right\}.$$

On the other hand, since

$$\lim_{t\to 0}t\frac{\dot{f}_i(t)}{f_i(t)}=1 \quad\text{and}\quad \lim_{t\to 0}\frac{a_i^2t^2}{f_i(t)^2}=1$$

for any sufficiently small $\varepsilon>0$ satisfying

$$4\mu^2b_1^2-a_1^2\varepsilon>0 \quad\text{and}\quad 4\nu^2b_2^2-a_2^2\varepsilon>0\,,$$

there exists $t_1\in(0, t_0)$ such that the following hold on $[0, t_1]$:

$$p\left|t\frac{\dot{f}_1(t)}{f_1(t)}-1\right|<\frac{\varepsilon}{2}\,,\qquad q\left|t\frac{\dot{f}_2(t)}{f_2(t)}-1\right|<\frac{\varepsilon}{2}\,,$$

$$\frac{\mu^2b_1^2}{a_1^2}\left|\frac{a_1^2t^2}{f_1(t)^2}-1\right|<\frac{\varepsilon}{4}\,,\qquad \frac{\nu^2b_2^2}{a_2^2}\left|\frac{a_2^2t^2}{f_2(t)^2}-1\right|<\frac{\varepsilon}{4}\,,$$

$$\mu^2b_1^2-\frac{a_1^2}{4}\varepsilon\leq A_1(t) \quad\text{and}\quad \nu^2b_2^2-\frac{a_2^2}{4}\varepsilon\leq A_2(t)\,.$$

Hence if we set $B_1=\mu^2b_1^2-(a_1^2/4)\varepsilon$ and $B_2=\nu^2b_2^2-(a_2^2/4)\varepsilon$, then for all $t\in(0, t_1)$

$$\frac{d}{dt}\{f_1(t)^pf_2(t)^q\dot{\rho}(t)\}-f_1(t)^{p-2}f_2(t)^{q-2}\{f_2(t)^2B_1+f_1(t)^2B_2\}\rho(t)$$

$$\leq C_0t^{a-2}f_1(t)^pf_2(t)^q\{a^2+(\lambda_1+\varepsilon)a-(\lambda_2-\varepsilon)\}\,,$$

where we put $\lambda_1:=p+q-1$ and $\lambda_2:=\mu^2b_1^2/a_1^2+\nu^2b_2^2/a_2^2$. Choose

$$a=\tfrac{1}{2}\left\{-(\lambda_1+\varepsilon)+\sqrt{(\lambda_1+\varepsilon)^2+4(\lambda_2-\varepsilon)}\right\}>0\,.$$

Then

$$\frac{d}{dt}\{f_1(t)^pf_2(t)^q\dot{\rho}(t)\}\leq f_1(t)^{p-2}f_2(t)^{q-2}\{f_2(t)^2B_1+f_1(t)^2B_2\}\rho(t)\,.$$

Thus if we take C_0 so that $\rho(t_1)=r(t_1)$, Lemma 2.5 yields the following

$$0<r(t)\leq\rho(t)=r(t_1)t_1^{-a}t^a \qquad\text{on}\quad (0, t_1)\,.$$

(ii) In the case $\lambda_2>k\lambda_1+k^2$, take ε so as $0<\varepsilon<(1/(k+1))\{\lambda_2-k\lambda_1-k^2\}$. Then

$$a=\tfrac{1}{2}\{-(\lambda_1+\varepsilon)+\sqrt{(\lambda_1+\varepsilon)^2+4(\lambda_2-\varepsilon)}\}$$
$$>\tfrac{1}{2}\{-(\lambda_1+\varepsilon)+(\lambda_1+\varepsilon+2k)\}=k\,.$$

Combining this with the estimate in (i), we have the conclusion. □

In the rest of this section, we shall show that $\beta(\alpha)$ is a continuous function in α. This fact plays an important role in the next section. For simplicity we set

$$G(t, s):=\mu^2\frac{h_1(s)h_1'(s)}{f_1(t)^2}+\nu^2\frac{h_2(s)h_2'(s)}{f_2(t)^2}\,.$$

Then it holds for any $t>0$ that

$$G(t, s_1)\leq G(t, s_2) \qquad \text{for} \quad s_1\leq s_2\,.$$

LEMMA 2.6. *Let $r_i=r_i(t)$ $(i=1, 2)$ be solutions of (1.1) with*

$$r_1(t_0)=r_2(t_0) \quad \textit{and} \quad \lim_{t\to 0} r_1(t)=\lim_{t\to 0} r_2(t)=0\,.$$

Then

$$r_1(t)=r_2(t) \qquad \textit{on} \quad [0, t_0]\,.$$

PROOF. Since $r_1(t)$ and $r_2(t)$ are solutions of (1.1), we have

$$\frac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{r}_i(t)\}=f_1(t)^p f_2(t)^q G(t, r_i(t)) \qquad (i=1, 2)\,. \tag{2.9}$$

Then

$$\frac{d}{dt}\{f_1(t)^p f_2(t)^q(\dot{r}_1(t)-\dot{r}_2(t))\}=f_1(t)^p f_2(t)^q\{G(t, r_1(t))-G(t, r_2(t))\}\,. \tag{2.10}$$

Multiplying the both sides by $r_1(t)-r_2(t)$ and integrating from 0 to t_0, we obtain

$$\begin{aligned}&[f_1(t)^p f_2(t)^q(\dot{r}_1(t)-\dot{r}_2(t))(r_1(t)-r_2(t))]_{t=0}^{t=t_0}\\ &-\int_0^{t_0} f_1(t)^p f_2(t)^q(\dot{r}_1(t)-\dot{r}_2(t))^2 dt\\ &=\int_0^{t_0} f_1(t)^p f_2(t)^q\{G(t, r_1(t))-G(t, r_2(t))\}\{r_1(t)-r_2(t)\}dt\,.\end{aligned} \tag{2.11}$$

Now, it turns out from the assumption on $r_i(t)$ $(i=1, 2)$ and the monotonicity of $G(t, \cdot)$ that

$$\int_0^{t_0} f_1(t)^p f_2(t)^q(\dot{r}_1(t)-\dot{r}_2(t))^2 dt\leq 0\,.$$

Since $f_1(t)^p f_2(t)^q > 0$ on $(0, t_0)$, we obtain

$$\dot{r}_1(t) = \dot{r}_2(t) \qquad \text{on} \quad (0, t_0)\,.$$

Thus

$$r_1(t) = r_2(t) \qquad \text{on} \quad [0, t_0]\,. \qquad \square$$

As a corollary of Lemma 2.6, we have

COROLLARY 2.7. *Let $r = r(t)$ and $\rho = \rho(t)$ be solutions of* (1.1)–(1.2). *If $r(t_0) = \rho(t_0)$ holds for some $t_0 \in (0, T)$, then we have*

$$r(t) = \rho(t) \qquad \textit{on} \quad [0, T)\,,$$

where $[0, T)$ is the common life span of $r(t)$ and $\rho(t)$.

LEMMA 2.8. *Let $r_i = r_i(t)$ be solutions of* (1.1) *with initial values*

$$r_i(t_0) = \alpha_i\,, \quad \dot{r}_i(t_0) = \beta_i = \beta(\alpha) \qquad (i = 1, 2)\,.$$

Then

$$|r_1(t) - r_2(t)| \le |r_1(t_0) - r_2(t_0)| \qquad \textit{on} \quad [0, t_0]\,. \tag{2.12}$$

PROOF. Note that if $\alpha_1 = \alpha_2$ then the assertion holds from Lemma 2.6. We first claim the following:

CLAIM 1. If $\alpha_1 < \alpha_2$ then $\beta_1 < \beta_2$.
In fact, from (2.11) and $\lim_{t \to 0} r_1(t) = \lim_{t \to 0} r_2(t) = 0$

$$f_1(t_0)^p f_2(t_0)^q \{\dot{r}_1(t_0) - \dot{r}_2(t_0)\}\{r_1(t_0) - r_2(t_0)\} > 0\,.$$

This is equivalent to

$$(\beta_1 - \beta_2)(\alpha_1 - \alpha_2) > 0\,.$$

Since $\alpha_1 < \alpha_2$ we have $\beta_1 < \beta_2$.

Now we set $\psi(t) := r_2(t) - r_1(t)$ for $t \in (0, t_0)$, then from the equation (1.1)

$$\ddot{\psi}(t) = -\left\{p\,\frac{\dot{f}_1(t)}{f_1(t)} + q\,\frac{\dot{f}_2(t)}{f_2(t)}\right\}\dot{\psi}(t) + G(t, r_2(t)) - G(t, r_1(t))\,. \tag{2.13}$$

CLAIM 2. If $\psi(t_0) > 0$ then $\dot{\psi}(t) \ge 0$ on $[0, t_0]$.
Since $\psi(t_0) > 0$, $\dot{\psi}(t_0) > 0$ by Claim 1. We assume that there exists $t_1 \in (0, t_0)$ such that

$$\dot{\psi}(t_1) = 0 \quad \text{and} \quad \dot{\psi}(t) > 0 \qquad \text{on} \quad (t_1, t_0)\,.$$

Then either of the following two cases occurs:
(1) The case $\psi(t_1) > 0$. In this case, because of $r_2(t_1) > r_1(t_1)$ and from (2.13)

$$\ddot{\psi}(t_1) = G(t_1, r_2(t_1)) - G(t_1, r_1(t_1)) > 0\,.$$

Hence $\dot{\psi}(t)<0$ on $(t_1-\varepsilon, t_1)$ for some $\varepsilon>0$. On the other hand, since $\lim_{t\to 0}\psi(t)=0$, there exists $t_2\in(0, t_1)$ such that

$$\psi(t_2)>0\,,\quad \dot{\psi}(t_2)=0 \quad\text{and}\quad \ddot{\psi}(t_2)\leq 0\,.$$

However, we can show that $\ddot{\psi}(t_2)>0$ by using the equation (2.13) again. This yields a contradiction. Thus, if $\dot{\psi}(t_1)=0$, then $\psi(t_1)\leq 0$.

(2) The case $\psi(t_1)\leq 0$. In this case, we can find $t_3\in[t_1, t_0)$ such that $\psi(t_3)=0$. Hence $r_1(t_3)=r_2(t_3)$ and $\lim_{t\to 0}r_1(t)=\lim_{t\to 0}r_2(t)=0$. It follows from Lemma 2.6 that

$$r_1(t)=r_2(t) \qquad \text{on} \quad (0, t_3)\,.$$

By virtue of the uniqueness theorem, we can conclude

$$r_1(t)=r_2(t) \qquad \text{on} \quad (0, t_0)\,,$$

which implies $\psi(t)=0$ on $(0, t_0)$. This contradicts $\psi(t_0)>0$. Thus Claim 2 follows.

Let us complete the proof of Lemma 2.8. From the second argument, if $r_2(t_0)>r_1(t_0)$, we obtain $r_2(t)>r_1(t)$ on $(0, t_0)$. Thus, if $\psi(t_0)>0$ then $0<\psi(t)<\psi(t_0)$ on $(0, t_0)$. That is, if $r_2(t_0)>r_1(t_0)$, then

$$0\leq r_2(t)-r_1(t)\leq r_2(t_0)-r_1(t_0) \qquad \text{on} \quad [0, t_0]\,.$$

In a similar way, it follows that

$$0\leq r_1(t)-r_2(t)\leq r_1(t_0)-r_2(t_0) \qquad \text{on} \quad [0, t_0]\,,$$

if $r_1(t_0)>r_2(t_0)$. Both these inequalities imply

$$|r_1(t)-r_2(t)|\leq|r_1(t_0)-r_2(t_0)| \qquad \text{on} \quad [0, t_0]\,.$$ □

Making use of these lemmas we show the following proposition.

PROPOSITION 2.9. *$\beta=\beta(\alpha)$ is a continuous function in α. That is, for $\alpha_i>0$ $(i=1, 2)$, there exists a constant $C_0=C_0(t_0)$ such that*

$$|\beta_1-\beta_2|\leq C_0|\alpha_1-\alpha_2|\,.$$

PROOF. Integrating the both sides of (2.10) from 0 to t_0, we have

$$f_1(t_0)^p f_2(t_0)^q(\dot{r}_1(t_0)-\dot{r}_2(t_0))=\int_0^{t_0} f_1(t)^p f_2(t)^q\{G(t, r_1(t))-G(t, r_2(t))\}dt\,.$$

Applying mean value theorem to a function $(h_i h_i')'(r)$, we obtain

$$G(t, r_1(t))-G(t, r_2(t))\leq\{\mu^2 c_1 f_1(t)^{-2}+\nu^2 c_2 f_2(t)^{-2}\}\{r_1(t)-r_2(t)\}\,,$$

where c_i $(i=1, 2)$ are constants given by

$$c_i=\max_{0\leq s\leq\max\{\alpha_1, \alpha_2\}}(h_i h_i')'(s)\,.$$

Therefore

$$\begin{aligned} & f_1(t_0)^p f_2(t_0)^q |\dot{r}_1(t_0) - \dot{r}_2(t_0)| \\ & \leq \int_0^{t_0} f_1(t)^p f_2(t)^q \{\mu^2 c_1 f_1(t)^{-2} + \nu^2 c_2 f_2(t)^{-2}\} |r_1(t) - r_2(t)| dt \\ & \leq |r_1(t_0) - r_2(t_0)| \int_0^{t_0} f_1(t)^p f_2(t)^q \{\mu^2 c_1 f_1(t)^{-2} + \nu^2 c_2 f_2(t)^{-2}\} dt \quad \text{(by (2.12))} . \end{aligned}$$

Since $p+q \geq 2$, the integral $\int_0^{t_0} f_1(t)^p f_2(t)^q \{\mu^2 c_1 f_1(t)^{-2} + \nu^2 c_2 f_2(t)^{-2}\} dt$ converges. Let us denote this value by C'. Then we have

$$f_1(t_0)^p f_2(t_0)^q |\dot{r}_1(t_0) - \dot{r}_2(t_0)| \leq C' |r_1(t_0) - r_2(t_0)| .$$

Putting $C_0 := C' f_1(t_0)^{-p} f_2(t_0)^{-q}$, the assertion is obtained. □

3. Asymptotic behavior as $t \to \infty$.

In the former half of this section, we investigate the properties of a solution of the equation (1.1), and show the existence of a global solution.

For any $\alpha > 0$, β is given by $\beta(\alpha)$ as in §2. Therefore r_α exists on $[0, T_\alpha)$ and satisfies (1.2). We often omit the subscript α of r_α.

LEMMA 3.1. *For the solution $r = r(t)$ of* (1.1) *with* (1.7), *the following hold:*
(i) *$\dot{r}(t) > 0$ for $t \in (0, T_\alpha)$, and*
(ii) *If $r(t)$ is bounded, $\dot{r}(t)$ is also bounded on* $(0, T_\alpha)$.

PROOF. (i) Since $\dot{r}(t) > 0$ on $(0, t_0]$ we shall show that $\dot{r}(t) > 0$ on (t_0, T_α). Assume that there exists $t_1 \in (t_0, T_\alpha)$ such that $\dot{r}(t_1) = 0$. Since $\dot{r}(t_0) > 0$, we can choose t_1 such that

$$r(t_1) > 0 , \quad \dot{r}(t_1) = 0 \quad \text{and} \quad \ddot{r}(t_1) \leq 0 .$$

On the other hand, the equation (1.1) implies

$$\ddot{r}(t_1) = \mu^2 \frac{h_1(r(t_1)) h_1'(r(t_1))}{f_1(t)^2} + \nu^2 \frac{h_2(r(t_1)) h_2'(r(t_1))}{f_2(t)^2} > 0 .$$

This yields a contradiction. Thus $\dot{r}(t) > 0$ on $[t_0, T_\alpha)$.

(ii) Integrating the both sides of the equation (2.8) from t_0 to t, we get in a similar manner as Lemma 2.2

$$\begin{aligned} f_1(t)^p f_2(t)^q \dot{r}(t) \leq C_0(t_0) + C_1(t_0) \max_{\tau \in [t_0, t]} |h_1(r(\tau)) h_1'(r(\tau))| \\ + C_2(t_0) \max_{\tau \in [t_0, t]} |h_2(r(\tau)) h_2'(r(\tau))| , \end{aligned}$$

where C_0, C_1 and C_2 are constants depending on t_0 and t. The assertion follows from this inequality. □

LEMMA 3.2. *For a solution $r=r(t)$ of* (1.1), *it holds that*

$$\begin{aligned}\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 &= \{f_1(s)^p f_2(s)^q \dot{r}(s)\}^2 \\ &\quad + \mu^2\{f_1(t)^{2p-2} f_2(t)^{2q} h_1(r(t))^2 - f_1(s)^{2p-2} f_2(s)^{2q} h_1(r(s))^2\} \\ &\quad + \nu^2\{f_1(t)^{2p} f_2(t)^{2q-2} h_2(r(t))^2 - f_1(s)^{2p} f_2(s)^{2q-2} h_2(r(s))^2\} \\ &\quad - \int_s^t \mu^2 h_1(r(\tau))^2 \frac{d}{d\tau}\{f_1(\tau)^{2p-2} f_2(\tau)^{2q}\} d\tau \\ &\quad - \int_s^t \nu^2 h_2(r(\tau))^2 \frac{d}{d\tau}\{f_1(\tau)^{2p} f_2(\tau)^{2q-2}\} d\tau \end{aligned} \tag{3.1}$$

for any $s, t \in (0, T_\alpha)$.

PROOF. Multiplying the both sides of (2.8) by $f_1(t)^p f_2(t)^q \dot{r}(t)$, we obtain

$$\frac{d}{dt}\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 = \mu^2 f_1(t)^{2p-2} f_2(t)^{2q} \frac{d}{dt} h_1(r(t))^2 + \nu^2 f_1(t)^{2p} f_2(t)^{2q-2} \frac{d}{dt} h_2(r(t))^2 .$$

Integrating the both sides of this equation from s to t, we have the conclusion. □

LEMMA 3.3. *Let $r=r_\alpha(t)$ be a solution of* (1.1) *with* (1.7). *Then the following holds*:

$$\dot{r}_\alpha(t)^2 \le \frac{\mu^2}{f_1(t)^2} h_1(r_\alpha(t))^2 + \frac{\nu^2}{f_2(t)^2} h_2(r_\alpha(t))^2 \quad \text{on} \quad [t_0, T_\alpha) . \tag{3.2}$$

PROOF. Let s tend to 0 in (3.1), we have

$$\{f_1(t)^p f_2(t)^q \dot{r}(t)\}^2 \le \mu^2 f_1(t)^{2p-2} f_2(t)^{2q} h_1(r(t))^2 + \nu^2 f_1(t)^{2p} f_2(t)^{2q-2} h_2(r(t))^2 .$$

Dividing the both sides by $f_1(t)^{2p} f_2(t)^{2q}$, we reach the conclusion. □

Define subsets $\mathscr{B}_1$ and $\mathscr{B}_2$ of $\boldsymbol{R}$ by

$$\mathscr{B}_1 := \{\alpha > 0 \mid T_\alpha = \infty \text{ and } r_\alpha(t) \text{ is bounded on } [0, \infty)\} ,$$
$$\mathscr{B}_2 := \{\alpha > 0 \mid T_\alpha < \infty\} .$$

PROPOSITION 3.4. *Under the assumption* (1.4)–(1.5) *the set $\mathscr{B}_1$ is open and non-empty.*

PROOF. We first prove that $\mathscr{B}_1$ is non-empty. From the condition (1.5) and the equation (3.2), we have

$$\dot{r}(t)^2 \leq \sinh^2\{br(t)\}\left\{\frac{\mu^2}{f_1(t)^2}+\frac{\nu^2}{f_2(t)^2}\right\}.$$

Hence

$$\dot{r}(t) \leq \sinh\{br(t)\}\sqrt{\frac{\mu^2}{f_1(t)^2}+\frac{\nu^2}{f_2(t)^2}}$$
$$\leq \gamma \sinh\{br(t)\}\left\{\frac{1}{f_1(t)}+\frac{1}{f_2(t)}\right\},$$

where $\gamma=\max\{\mu, \nu\}$. This is equivalent to

$$\frac{\dot{r}(t)}{\sinh\{br(t)\}} \leq \gamma\left\{\frac{1}{f_1(t)}+\frac{1}{f_2(t)}\right\}.$$

Integrating the both sides and making use of the formula $\int(\sinh x)^{-1}dx=\log\tanh x/2$, we obtain

$$\tanh\frac{br(t)}{2} \leq \tanh\frac{b\alpha}{2}\exp\{b\gamma B\}, \tag{3.3}$$

where $B:=\int_{t_0}^{\infty}\{f_1(\tau)^{-1}d\tau+f_2(\tau)^{-1}d\tau\}$, which converges by virtue of the condition (1.5). If we take an α such that

$$\tanh\frac{b\alpha}{2}\exp\{b\gamma B\}<1, \tag{3.4}$$

then (3.3) implies that r_α is bounded and $T_\alpha=\infty$ from Lemma 3.1. Therefore,

$$\{\alpha>0 \mid \tanh\frac{b\alpha}{2}\exp(b\gamma B)<1\} \subset \mathscr{B}_1,$$

which implies that $\mathscr{B}_1$ is non-empty.

Next we shall show $\mathscr{B}_1$ is an open set. For any fixed $\alpha_* \in \mathscr{B}_1$, we set $K_*:=\lim_{t\to\infty} r_{\alpha_*}(t)$. Since the assumption (1.5) holds,

$$F(t):=\exp\left[b\gamma\int_t^{\infty}\left\{\frac{1}{f_1(\tau)}+\frac{1}{f_2(\tau)}\right\}d\tau\right]$$

is a well-defined and monotonically decreasing function. We take $T>0$ in such a way that

$$\tanh\frac{bK_*}{2}\exp\left[b\gamma\int_T^{\infty}\left\{\frac{1}{f_1(\tau)}+\frac{1}{f_2(\tau)}\right\}d\tau\right]<1 \tag{3.5}$$

and fix the above $T>0$ to the end of this proof. Because solutions of (1.1) depend on its initial values continuously, we can assert that for any $\varepsilon>0$ there exists $\delta>0$ such that if $|\alpha-\alpha_*|<\delta$ then $|r_\alpha(T)-r_{\alpha_*}(T)|<\varepsilon$. Here we note that $T_\alpha>T$. Hence if α is sufficiently close to α_*, (3.5) yields

$$\tanh\frac{br_\alpha(T)}{2}\exp\left[b\gamma\int_T^\infty\left\{\frac{1}{f_1(\tau)}+\frac{1}{f_2(\tau)}\right\}d\tau\right]<1,$$

which coincides with the inequality (3.4), if we set $t_0=T$ and $\alpha=r_\alpha(T)$. On the other hand, from Lemma 3.3,

$$(0<)\left[\frac{dr_\alpha}{dt}(T)\right]^2\leq\frac{\mu^2}{f_1(T)^2}h_1(r_\alpha(T))^2+\frac{\nu^2}{f_2(T)^2}h_2(r_\alpha(T))^2.$$

Therefore we can show that $T_\alpha=\infty$ and $r_\alpha=r_\alpha(t)$ is bounded by the same argument as in the proof for non-emptiness of $\mathscr{B}_1$. This means $\alpha\in\mathscr{B}_1$ if α is sufficiently close to α_*. Thus $\mathscr{B}_1$ is open. □

REMARK 1. In particular, from (3.2) we have

$$r_\alpha(t)\leq\frac{2}{b}\tanh^{-1}\left(\tanh\frac{b\alpha}{2}\exp(b\gamma B)\right).$$

PROOF OF THEOREM 1. Proposition 3.4 and Lemma 3.1 show that there exists $\alpha>0$ such that r_α satisfies the conditions in Theorem 1. □

In the rest of this section, we assume the condition (1.6). We set

$$A:=f_1(t_0)^pf_2(t_0)^q\int_{t_0}^\infty\frac{dt}{f_1(t)^pf_2(t)^q},$$

$$\phi(\alpha):=\int_\alpha^\infty\frac{dr}{\sqrt{\gamma_1^2\{h_1(r)^2-h_1(\alpha)^2\}+\gamma_2^2\{h_2(r)^2-h_2(\alpha)^2\}+\beta^2}},$$

where $\gamma_1=\mu/f_1(t_0)$ and $\gamma_2=\nu/f_2(t_0)$. The assumption (1.4) and $\dot{f}_i(t)\geq0$ imply that the integral of A converges and Lemma 3.6 below guarantees that $\phi(\alpha)$ is well-defined.

LEMMA 3.5.

$$\phi(\alpha)\geq f_1(t_0)^pf_2(t_0)^q\int_{t_0}^{T_\alpha}\frac{dt}{f_1(t)^pf_2(t)^q}.$$

PROOF. Since $\dot{f}_i(t)\geq0$ and $h_i(s)^2\leq h_i(t)^2$ for all $s\leq t$, we have

$$\int_{t_0}^t h_1(r(\tau))^2\frac{d}{d\tau}\{f_1(\tau)^{2p-2}f_2(\tau)^{2q}\}d\tau\leq h_1(r(t))^2\{f_1(t)^{2p-2}f_2(t)^{2q}-f_1(t_0)^{2p-2}f_2(t_0)^{2q}\},$$

$$\int_{t_0}^t h_2(r(\tau))^2\frac{d}{d\tau}\{f_1(\tau)^{2p}f_2(\tau)^{2q-2}\}d\tau\leq h_2(r(t))^2\{f_1(t)^{2p}f_2(t)^{2q-2}-f_1(t_0)^{2p}f_2(t_0)^{2q-2}\}.$$

Making use of the equation (3.1) and these inequalities, we obtain

$$\begin{aligned}\{f_1(t)^p f_2(t)^q \dot r(t)\}^2 \geq & \{f_1(t_0)^p f_2(t_0)^q \dot r(t_0)\}^2 \\ & + \mu^2 f_1(t_0)^{2p-2} f_2(t_0)^{2q}\{h_1(r(t))^2 - h_1(r(t_0))^2\} \\ & + \nu^2 f_1(t_0)^{2p} f_2(t_0)^{2q-2}\{h_2(r(t))^2 - h_2(r(t_0))^2\} .\end{aligned}$$

This is equivalent to

$$\frac{\dot r(t)}{\sqrt{\gamma_1^2\{h_1(r)^2 - h_1(\alpha)^2\} + \gamma_2^2\{h_2(r)^2 - h_2(\alpha)^2\} + \beta^2}} \geq f_1(t_0)^p f_2(t_0)^q \frac{1}{f_1(t)^p f_2(t)^q} .$$

Integrating the both sides from t_0 to T_α, we obtain

$$\begin{aligned}\int_\alpha^{r(T_\alpha)} & \frac{dr}{\sqrt{\gamma_1^2\{h_1(r)^2 - h_1(\alpha)^2\} + \gamma_2^2\{h_2(r)^2 - h_2(\alpha)^2\} + \beta^2}} \\ & \geq f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{T_\alpha} \frac{dt}{f_1(t)^p f_2(t)^q} .\end{aligned}$$

This implies

$$\phi(\alpha) \geq f_1(t_0)^p f_2(t_0)^q \int_{t_0}^{T_\alpha} \frac{dt}{f_1(t)^p f_2(t)^q} .$$ □

By a similar argument as in the proof of Lemma 3.5, for any $t \in (0, T_\alpha)$, we can prove

$$\phi(r(t)) \geq f_1(t)^p f_2(t)^q \int_t^{T_\alpha} \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q} . \tag{3.6}$$

We shall use this inequality (3.6) to prove Proposition 3.7 below.

LEMMA 3.6. *Under the hypothesis of* (1.3)–(1.6), *there exists* $\delta > 0$ *such that*

$$\phi(\alpha) = O(\alpha^{-\delta}) \qquad (as\ \alpha \to \infty) .$$

PROOF. We assume that $h_1(r) \geq d_1 r^l$ for sufficiently large r. Because of convexity of $h_i(r)^2$, we get for $r \geq \alpha$

$$h_i(r)^2 - h_i(\alpha)^2 \geq 2b_i h_i(\alpha)(r - \alpha) .$$

Thus, for sufficiently large α, it holds that

$$\begin{aligned}\int_\alpha^{2\alpha} & \frac{dr}{\sqrt{\gamma_1^2\{h_1(r)^2 - h_1(\alpha)^2\} + \gamma_2^2\{h_2(r)^2 - h_2(\alpha)^2\} + \beta^2}} \\ & \leq \frac{1}{\sqrt{2b_1\gamma_1^2 h_1(\alpha) + 2b_2\gamma_2^2 h_2(\alpha)}} \int_\alpha^{2\alpha} \frac{dr}{\sqrt{r - \alpha}}\end{aligned}$$

$$\leq \frac{\alpha^{-l/2}}{\sqrt{2b_1\gamma_1^2 d_1}}[2\sqrt{r-\alpha}]_{\alpha}^{2\alpha}$$
$$= O(\alpha^{(1-l)/2}) \qquad (\text{as } \alpha \to \infty).$$

On the other hand, because of $h_i''(r) \geq 0$, we have

$$\frac{h_i(\alpha)}{h_i(2\alpha)} \leq \frac{1}{2}$$

and hence, when $r \geq 2\alpha$ and α is sufficiently large,

$$h_i(r)^2 - h_i(\alpha)^2 \geq h_i(r)^2\left(1 - \frac{h_i(\alpha)^2}{h_i(2\alpha)^2}\right) \geq \frac{3}{4}\, h_i(r)^2 .$$

Therefore

$$\int_{2\alpha}^{\infty} \frac{dr}{\sqrt{\gamma_1^2\{h_1(r)^2 - h_1(\alpha)^2\} + \gamma_2^2\{h_2(r)^2 - h_2(\alpha)^2\} + \beta^2}}$$
$$\leq 2\int_{2\alpha}^{\infty} \frac{dr}{\sqrt{\gamma_1^2 h_1(r)^2 + \gamma_2^2 h_2(r)^2}}$$
$$\leq \frac{2}{\gamma_1 d_1}\int_{2\alpha}^{\infty} \frac{dr}{r^l}$$
$$= O(\alpha^{1-l}) \qquad (\text{as } \alpha \to \infty).$$

As a consequence, we obtain

$$\phi(\alpha) = O(\alpha^{-\delta}) \qquad (\text{as } \alpha \to \infty). \qquad \square$$

PROPOSITION 3.7. *Under the conditions* (1.3)–(1.6), $\mathscr{B}_2$ *is open and non-empty.*

PROOF. We prove $\mathscr{B}_2$ is non-empty. Suppose that $T_\alpha = \infty$ for all $\alpha > 0$, then by Lemma 3.5

$$\phi(\alpha) \geq A \ (>0)$$

holds for all $\alpha > 0$. On the other hand, from Lemma 3.6,

$$\phi(\alpha) \to 0 \qquad (\text{as } \alpha \to \infty).$$

This is a contradiction. Thus $T_\alpha < \infty$ for sufficiently large α. This means $\mathscr{B}_2$ is non-empty.

Next we show $\mathscr{B}_2$ is an open set. Suppose $\mathscr{B}_2$ is not open. Then there exist α_* and a sequence $\{\alpha_j\}_{j=1}^{\infty} \subset \boldsymbol{R}_+$ which satisfy the following properties:

$$\alpha_j \to \alpha_* \ (j \to \infty), \qquad T_{\alpha_*} < \infty \quad \text{and} \quad T_{\alpha_j} = \infty .$$

Let $r_j = r_j(t)$ be the solution of (1.1) with the initial values $r_j(t_0) = \alpha_j$, $\dot{r}_j(t_0) = \beta_j\ (= \beta(\alpha_j))$.

Applying (3.6) to r_j, we have

$$\phi(r_j(t)) \geq f_1(t)^p f_2(t)^q \int_t^\infty \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}$$

for any $t \geq t_0$. Since t is arbitrary, we can set $t = T_{\alpha_*} - \varepsilon$ for sufficiently small ε $(0 < \varepsilon < 1)$ in such a way that

$$\phi(r_j(T_{\alpha_*} - \varepsilon)) \geq f_1(T_{\alpha_*} - \varepsilon)^p f_2(T_{\alpha_*} - \varepsilon)^q \int_{T_{\alpha_*} - \varepsilon}^\infty \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}. \tag{3.7}$$

The assumptions $f_i(t) \geq 0$ and $\dot{f}_i(t) \geq 0$ imply the right hand side of (3.7) is bigger than

$$f_1(T_{\alpha_*} - 1)^p f_2(T_{\alpha_*} - 1)^q \int_{T_{\alpha_*}}^\infty \frac{d\tau}{f_1(\tau)^p f_2(\tau)^q}$$

which is a positive constant, denoted by δ_1. Hence

$$\phi(r_j(T_{\alpha_*} - \varepsilon)) \geq \delta_1 > 0. \tag{3.8}$$

On the other hand, by virtue of continuous dependence of the solution r_j on its initial values, we have the following:

> for any $R > 0$ there exist sufficiently small $\varepsilon > 0$ and sufficiently large $j \in \boldsymbol{N}$ such that $r_j(T_{\alpha_*} - \varepsilon) > R$.

Lemma 3.6 implies that the left hand side of (3.8) can be made arbitrarily small by taking ε sufficiently small and j sufficiently large. This contradicts (3.8). Thus $\mathscr{B}_2$ is an open set. □

REMARK 2. It follows from Lemmas 3.5, 3.6, inequality in (3.6) and continuous dependence of the solutions on their initial values that

$$\{T_\alpha \mid \alpha \in \mathscr{B}_2\} = (t_0, \infty).$$

PROOF OF THEOREM 2. Propositions 3.4 and 3.7 assert that $\mathscr{B}_2$ and $\mathscr{B}_1$ are non-empty open sets and $\mathscr{B}_2 \cap \mathscr{B}_1 = \varnothing$. Therefore, because $\boldsymbol{R}_+$ is connected

$$\boldsymbol{R}_+ \backslash (\mathscr{B}_2 \cup \mathscr{B}_1) \neq \varnothing.$$

Hence if we take $\alpha \in \boldsymbol{R}_+ \backslash (\mathscr{B}_2 \cup \mathscr{B}_1)$, then the solution r_α to (1.1)–(1.2) satisfies the property in Theorem 2. □

4. The case $p + q = 1$.

In the previous sections, we assume that $p + q \geq 2$. However, we can solve the equation (1.1) explicitly in the special case that $p + q = 1$.

We can assume that $p = 1$, $q = 0$ and $\nu = 0$. Then (1.1) with (1.2) becomes

$$(4.1)\qquad \begin{cases} \ddot{r}(t)+\dfrac{\dot{f}(t)}{f(t)}\dot{r}(t)-\mu^2\dfrac{h(r(t))h'(r(t))}{f_1(t)^2}=0\,, \\ \lim\limits_{t\to 0} r(t)=0\,. \end{cases}$$

In this case, we have

PROPOSITION 4.1. *The problem* (4.1) *has a solution.*

PROOF. Multiplying the both sides of the equation in (4.1) by $f(t)^2\dot{r}(t)$ and integrating from t_0 to t, we obtain

$$(4.2)\qquad \{f(t)\dot{r}(t)\}^2=\mu^2h(r(t))^2+C_0\,,$$

where C_0 is given by $C_0=\{f(t_0)\dot{r}(t_0)\}^2-\mu^2h(r(t_0))^2$. Because of $\lim_{t\to 0}r(t)=0$, $C_0\geq 0$. The equation (4.2) yields

$$(4.3)\qquad \int_{r(t_0)}^{r(t)}\frac{dr}{\sqrt{\mu^2h(r)^2+C_0}}=\int_{t_0}^{t}\frac{d\tau}{f(\tau)}\,.$$

The right hand side of (4.3) goes to infinity as t_0 tends to 0, which implies that C_0 should vanish. As a consequence, we have

$$(4.4)\qquad \int_{r(t_0)}^{r(t)}\frac{dr}{h(r)}=\mu\int_{t_0}^{t}\frac{d\tau}{f(\tau)}\,.$$

Denoting the primitive functions of $1/h(r)$ (resp. $1/f(t)$) by H (resp. F), we get by (4.4)

$$(4.5)\qquad H(r(t))=\mu F(t)+C(t_0)\,.$$

Since H is a monotonically increasing function, there exists a solution $r=r(t)$ of (4.1). This completes the proof. □

Now we apply this method to construct an equivariant harmonic map from the hyperbolic space $\boldsymbol{RH}^2$ into itself. The space $\boldsymbol{RH}^2$ can be considered as the space $\boldsymbol{R}^2$ equipped with the metric $dt^2+\sinh^2 t d\theta^2$ in the polar coordinate (t,θ). In this representation, any equivariant map can be expressed as

$$\psi:\boldsymbol{RH}^2\ni(t,\theta)\mapsto(r(t),\phi(\theta))\in\boldsymbol{RH}^2\,.$$

The sufficient and necessary conditions for ψ to be a harmonic map of $\boldsymbol{RH}^2$ into itself are

$$(4.6)\qquad \begin{cases} \ddot{r}(t)+\dfrac{\cosh t}{\sinh t}\dot{r}(t)-2e(\phi)\dfrac{\sinh r(t)\cosh r(t)}{\sinh^2 t}=0\,; \\ \phi: S^1\to S^1 \text{ is harmonic and } e(\phi) \text{ is constant}\,, \end{cases}$$

where $e(\phi)$ denotes the energy density of ϕ. On the other hand, any harmonic map

$\phi: S^1 \to S^1$ with constant energy density (i.e., eigenmap) is given by

$$\phi_l(e^{i\theta}) = e^{il\theta}, \qquad l \in \boldsymbol{Z},$$

and $2e(\phi_l) = l^2$. Thus the equivariant map ψ_l defined by $(t, \theta) \mapsto (r(t), \phi_l(\theta))$ is harmonic on $\boldsymbol{RH}^2$ if and only if

$$\begin{cases} \ddot{r}(t) + \dfrac{\cosh t}{\sinh t}\,\dot{r}(t) - l^2\,\dfrac{\sinh r(t)\cosh r(t)}{\sinh^2 t} = 0, & \text{and} \\ \lim\limits_{t \to 0} r(t) = 0. \end{cases}$$

Making use of the equation (4.4), we have

$$\int_{r(t_0)}^{r(t)} \frac{dr}{\sinh r} = |l| \int_{t_0}^{t} \frac{d\tau}{\sinh \tau}.$$

From this equation, all the solution $r = r_{\alpha_0}$ are classified into the following three cases:

$$\begin{cases} \text{If } \alpha_0 < \alpha_*, \text{ then } T_{\alpha_0} = \infty \text{ and } r_{\alpha_0} \text{ is bounded.} \\ \text{If } \alpha_0 = \alpha_*, \text{ then } T_{\alpha_0} = \infty \text{ and } r_{\alpha_0} \nearrow \infty. \\ \text{If } \alpha_0 > \alpha_*, \text{ then } T_{\alpha_0} < \infty. \end{cases}$$

Here $\alpha_0 = r(t_0)$ and α_* satisfies the following equation

$$\frac{\cosh \alpha_* - 1}{\cosh \alpha_* + 1} = \left\{ \frac{\cosh t_0 + 1}{\cosh t_0 - 1} \right\}^{|l|}.$$

In particular, if $\alpha_0 = \alpha_*$, then

$$r(t) = \log\left\{ \cosh^{|l|} \frac{t}{2} + \sinh^{|l|} \frac{t}{2} \right\} - \log\left\{ \cosh^{|l|} \frac{t}{2} - \sinh^{|l|} \frac{t}{2} \right\}.$$

Now, $\boldsymbol{RH}^2$ can be identified with the Poincaré disc $\boldsymbol{D}^2$ by the stereographic projection. Using this identification, in the case $\alpha_0 = \alpha_*$, $\psi_l: \boldsymbol{D}^2 \to \boldsymbol{D}^2$ coincides with $\boldsymbol{D}^2 \ni z \mapsto z^l \in \boldsymbol{D}^2$. Therefore, ψ_l is holomorphic.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, YAMAGATA UNIVERSITY,
YAMAGATA, 990 JAPAN.