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A Note on the Rational Approximations to $e^{1/k}$

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Introduction.

C. S. Davis [1] proved the following theorem: *Let k be a positive integer and* $c=1/(2k)$. *Then, for any* $\varepsilon>0$, *the inequality*

$$\left|e^{1/k}-\frac{p}{q}\right|<(c+\varepsilon)\frac{\log\log q}{q^2\log q}$$

has an infinity of solutions in integers p and q. Further, there exists a number q′, depending only on ε and k, such that

$$\left|e^{1/k}-\frac{p}{q}\right|>(c-\varepsilon)\frac{\log\log q}{q^2\log q}$$

for all integers p and q with $q\geq q'$.

In the previous paper [2], we showed that q' is an effectively computable number depending only on ε for $k=1$. The aim of this note is to give, for any $k\geq 2$, explicit lower bound for q'.

Let k and N be positive integers with $k\geq 2$ and $N\geq 5$, and let p_n/q_n be the n-th convergent of $e^{1/k}$. Let γ_N, δ_m, and γ_N^* be defined by

$$\gamma_N=2\left(k+\frac{2k+1}{2N-1}\right)\left(1+\frac{\log\log(4k(N+1)/e)}{\log(N+1)}\right),$$

$$\delta_m=\frac{(k(2m+1)+1)\log\log q_{3m}}{\log q_{3m}},$$

and

$$\gamma_N^*=\max\{\delta_m \mid 1\leq m<N\},$$

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respectively.

THEOREM. *Let $k \geq 2$ be positive integers. Then*

$$\left| e^{1/k} - \frac{p}{q} \right| > \frac{\log\log q}{\gamma q^2 \log q}$$

for all integers p and q with $q \geq 2$, where

$$\gamma \geq \max\{\gamma_N, \gamma_N^*\}$$

for any positive integer $N \geq 5$.

In [2], we showed that

$$\left| e - \frac{p}{q} \right| > \frac{\log\log q}{3q^2 \log q}$$

for all integers p and q with $q \geq 2$. As corollaries of Theorem, we have

COROLLARY 1. *For all integers p and q with $q \geq 2$,*

$$\left| e^{1/2} - \frac{p}{q} \right| > \frac{\log\log q}{6q^2 \log q}.$$

COROLLARY 2. *For all integers p and q with $q \geq 2$,*

$$\left| e^{1/3} - \frac{p}{q} \right| > \frac{\log\log q}{9q^2 \log q}.$$

§1. Lemma.

LEMMA. *Under the same assumptions as in Theorem,*

$$\left| e^{1/k} - \frac{p}{q} \right| > \frac{\log\log q}{\gamma_N q^2 \log q}$$

for all integers p and q with $q \geq q_{3N}$.

PROOF. We may assume that p/q is a convergent of $e^{1/k}$, since otherwise

$$\left| e^{1/k} - \frac{p}{q} \right| > \frac{1}{2q^2}.$$

The continued fraction of $e^{1/k}$ is

$$e^{1/k} = [a_0, a_1, a_2, \cdots] = [\overline{1, (2n-1)k-1, 1}]_{n=1}^{\infty}.$$

In other words, $a_{3m} = a_{3m+2} = 1$ and $a_{3m+1} = k(2m+1) - 1$ for $m \geq 0$.

Case 1. Let $n = 3m$ $(m \geq N)$. Since $q_{3m+1} = a_{3m+1} q_{3m} + q_{3m-1} = (k(2m+1) -$

$1)q_{3m}+q_{3m-1}<k(2m+1)q_{3m}$, we have

$$\left|e^{1/k}-\frac{p_{3m}}{q_{3m}}\right|>\frac{1}{q_{3m}(q_{3m+1}+q_{3m})}>\frac{1}{(k(2m+1)+1)q_{3m}^2}.$$

Now we must estimate q_{3m}. Since $q_{3m}\geq 2k(2m-1)q_{3m-3}\geq\cdots\geq(2k)^m(2m-1)(2m-3)\cdots5\cdot3\cdot1$, we have

$$\log q_{3m}\geq m\log(2k)+\sum_{v=1}^{m}\log(2v-1)\geq m\log(2k)+(m-1/2)\log(2m-1)-(m-1)$$
$$\geq(m-1/2)\log(2m-1).$$

Conversely, since $q_{3m}\leq 4kmq_{3m-3}\leq(4k)^m q_0\prod_{v=1}^{m}v=(4k)^m m!$, we have

$$\log q_{3m}\leq m\log(4k)+\sum_{v=1}^{m}\log v\leq m\log(4k)+(m+1)\log(m+1)-m$$
$$\leq(m+1)\log(4k(m+1)/e),$$
$$\log\log q_{3m}\leq\log(m+1)+\log\log(4k(m+1)/e).$$

Since

$$l(x)=\frac{\log\log(4k(x+1)/e)}{\log(x+1)}\qquad(x\geq5)$$

is a strictly decreasing function, we have

$$\log\log q_{3m}\leq(1+l(N))\log(m+1)\leq(1+l(N))\log(2m-1).$$

From these inequalities, we find

$$\frac{\log\log q_{3m}}{\log q_{3m}}\leq\frac{1+l(N)}{m-1/2}$$
$$\leq\left(2k+\frac{2k+1}{N-1/2}\right)\left(1+\frac{\log\log(4k(N+1)/e)}{\log(N+1)}\right)\cdot\frac{1}{k(2m+1)+1}$$
$$=\frac{\gamma_N}{k(2m+1)+1}.$$

Therefore,

$$\left|e^{1/k}-\frac{p_{3m}}{q_{3m}}\right|>\frac{\log\log q_{3m}}{\gamma_N q_{3m}^2\log q_{3m}}.$$

Case 2. Let $n=3m+1$ $(m\geq N)$. Since $q_{3m+2}=a_{3m+2}q_{3m+1}+q_{3m}=q_{3m+1}+q_{3m}$, we have

$$\left|e^{1/k}-\frac{p_{3m+1}}{q_{3m+1}}\right|>\frac{1}{q_{3m+1}(q_{3m+2}+q_{3m+1})}>\frac{1}{3q_{3m+1}^2}.$$

Therefore, the lemma is proved in this case. The same inequality also holds for $n=3m+2$ ($m\geq N$). This completes the proof.

§2. Proof of the theorem.

It suffices only to consider that p/q is a $(3m)$-th convergent of $e^{1/k}$. From the definition of γ_N^*, we have the following inequalities

$$\left|e^{1/k}-\frac{p_{3m}}{q_{3m}}\right|>\frac{1}{(k(2m+1)+1)q_{3m}^2}=\frac{\log\log q_{3m}}{\delta_m q_{3m}^2\log q_{3m}}\geq\frac{\log\log q_{3m}}{\gamma_N^* q_{3m}^2\log q_{3m}} \qquad (1\leq m<N).$$

And from Lemma, we have

$$\left|e^{1/k}-\frac{p_{3m}}{q_{3m}}\right|>\frac{\log\log q_{3m}}{\gamma_N q_{3m}^2\log q_{3m}} \qquad (m\geq N).$$

This completes the proof of the theorem.

§3. Proof of corollaries.

PROOF OF COROLLARY 1. For $N=30$, we have $\gamma_{30}=5.9993\cdots$ and $\gamma_{30}^*=\delta_{29}=4.4817\cdots$. Hence we can choose γ so that $\gamma=6$.

PROOF OF COROLLARY 2. For $N=36$, we have $\gamma_{36}=8.9919\cdots$ and $\gamma_{36}^*=\delta_{35}=6.2699\cdots$. Hence we can choose γ so that $\gamma=9$.

References

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