TOKYO J. MATH.
VOL. 18, No. 1, 1995

Invariant Bilinear Forms for Heisenberg Group

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(Communicated by Y. Shimizu)

1. Introduction.

Let G be a connected, simply connected nilpotent Lie group and $\mathfrak{g}$ be its Lie algebra. Irreducible unitary representations can be described by the orbit method due to A. A. Kirillov ([Ki]). Let f be an element of the dual space $\mathfrak{g}'$ of $\mathfrak{g}$. Let $\mathfrak{h}$ be a real polarization at f. Then f defines a one-dimensional representation τ_f of a subgroup $D=\exp(\mathfrak{g}\cap\mathfrak{h})$ of G. We can get an irreducible unitary representation $U^{f,\mathfrak{h}}$ of G, which is the induced representation from the representation τ_f of D. The unitary equivalence class of $U^{f,\mathfrak{h}}$ is independent of $\mathfrak{h}$ and depends only on the coadjoint orbit containing f. And any irreducible unitary representation of G is equivalent to one of $U^{f,\mathfrak{h}}$. Since $\mathfrak{h}$ is isotropic with respect to the alternative bilinear form $\varphi_f(X, Y)=f([X, Y])$, $X, Y\in\mathfrak{g}_{\boldsymbol{C}}$, f defines τ_f.

In the present paper we study the non-unitary representations of the Heisenberg group of $(2n+1)$-dimension. Irreducible unitary representations of the Heisenberg group are essentially parametrized by unitary characters of the center. V. S. Petrosyan ([P]) studied the irreducibility of non-unitary representations of the Heisenberg group of 3-dimension induced from non-unitary characters of the center. To prove the operator irreducibility he used the method of the invariant bilinear forms which was used in [GGV].

First we fix a real standard polarization $\mathfrak{h}$ at $f\in\mathfrak{g}'$ (see §2 for the definition of standard) and take a complex linear form $\Lambda\in(\mathfrak{g}')_{\boldsymbol{C}}$ on $\mathfrak{g}$ such that $\mathfrak{h}$ is isotropic with respect to φ_Λ. We define a representation τ_Λ of D by $\tau_\Lambda(\exp X)=\exp(\sqrt{-1}\Lambda(X))$, $X\in\mathfrak{g}\cap\mathfrak{h}$. And we define a non-unitary representation $U^{\Lambda,\mathfrak{h}}$ of G induced from τ_Λ. We realize it on the space $\mathscr{D}(G/D)$ of C^∞-functions on G/D with compact support. In our case if $f\neq 0$ on the center of $\mathfrak{g}$, then $\mathfrak{h}$ is abelian and $G/D\cong\boldsymbol{R}^n$. So we denote $\mathscr{D}(\boldsymbol{R}^n)$ by $\mathscr{D}_\Lambda^{\mathfrak{h}}$ as the representation space of $U^{\Lambda,\mathfrak{h}}$. Thus our object of study is a family of non-unitary representations $\{(U^{\Lambda,\mathfrak{h}}, \mathscr{D}_\Lambda^{\mathfrak{h}})\mid\mathfrak{h}\text{ is standard}, \Lambda\in(\mathfrak{g}')_{\boldsymbol{C}}\}$.

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Received December 20, 1993
Revised December 9, 1994

form on $\mathscr{D}_{\Lambda_1}^{\mathfrak{h}} \times \mathscr{D}_{\Lambda_2}^{\mathfrak{h}}$ (Theorem 1). And we also get a necessary and sufficient condition for the existence of an intertwining operator from $\mathscr{D}_{\Lambda_1}^{\mathfrak{h}}$ to $\mathscr{D}_{\Lambda_2}^{\mathfrak{h}}$ (Theorem 2). This theorem shows that even if Λ_1 and Λ_2 are equal on the center, there does not necessarily exist an intertwining operator from $\mathscr{D}_{\Lambda_1}^{\mathfrak{h}}$ to $\mathscr{D}_{\Lambda_2}^{\mathfrak{h}}$.

We call the representation $U^{\Lambda,\mathfrak{h}}$ is operator irreducible if any intertwining operator from $\mathscr{D}_{\Lambda}^{\mathfrak{h}}$ to itself is a scalar multiple of the identity operator. We prove that $U^{\Lambda,\mathfrak{h}}$ is operator irreducible if Λ is not zero on the center of $\mathfrak{g}$ (Theorem 3). V. S. Petrosyan ([P]) proved that the non-unitary representation induced from a non-unitary and non-trivial character of the center is operator irreducible for the Heisenberg group of 3-dimension. In §5 we determine when the representation $U^{\Lambda,\mathfrak{h}}$ is unitary (Theorem 4).

In Theorem 5 we prove that two representations $U^{\Lambda_1,\mathfrak{h}}$ and $U^{\Lambda_2,\mathfrak{h}}$ are equivalent if and only if they are on the same orbit in $(\mathfrak{g}')_C$ of a subgroup B of G_C containing G. In §7 we study an invariant bilinear form for representations corresponding to different types of polarizations. Generally, the intertwining operators are not operators of $\mathscr{D}$. In §8 we get an intertwining operator for unitary representations corresponding to different types of polarizations $\mathfrak{h}_{T_1}^k$ and $\mathfrak{h}_{T_2}^k$ on the Schwartz space $\mathscr{S}$ (see §2 for the notation of polarization). This is an integral operator whose kernel is exponential of a polynomial of degree 2 (Theorem 10). By changing the basis any polarization turns out to be a canonical polarization which means that $T=O$ for our notation. G. Lion ([L1], [L2]) has given an intertwining operator between unitary representations corresponding to $\mathfrak{h}_O^{k_1}$ and $\mathfrak{h}_O^{k_2}$ in another way. In the first version of this paper ([Ku]), we considered only the case where $k=n$, $T=tI$ ($t \in \boldsymbol{R}$) and $k=0$, $T=O$. Extending our observation to any k and any $T \in M_k(\boldsymbol{R}) \oplus M_{n-k}(\boldsymbol{R})$, we can understand the subgroup B (Theorem 5).

2. Representations of Heisenberg group.

Let G be the Heisenberg group of $(2n+1)$-dimension. We realize G as a real Lie group whose underlying manifold is $\boldsymbol{R}^{2n+1}$ and multiplication is

$$(a, b, c) \cdot (a', b', c') = (a+a', b+b', c+c'+a \cdot b'),$$

where $a, a', b, b' \in \boldsymbol{R}^n$, $c, c' \in \boldsymbol{R}$ and $a \cdot b' = \sum a_j b'_j$.

Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g} = \{X=(x, y, z) \mid x, y \in \boldsymbol{R}^n, z \in \boldsymbol{R}\}$ with $[(x, y, z), (x', y', z')] = (0, 0, x \cdot y' - x' \cdot y)$. We denote by $\mathfrak{g}' = \{\Lambda = (\lambda, \mu, \nu) \mid \lambda, \mu \in \boldsymbol{R}^n, \nu \in \boldsymbol{R}\}$ the real dual space of $\mathfrak{g}$ defined by

$$\langle \Lambda, X \rangle = \lambda \cdot x + \mu \cdot y + \nu z .$$

The group G acts on $\mathfrak{g}'$ by coadjoint action:

$$\langle g \cdot \Lambda, X \rangle = \langle \Lambda, \mathrm{Ad}(g^{-1}) X \rangle .$$

Then

$$(a, b, c) \cdot (\lambda, \mu, \nu) = (\lambda + \nu b, \mu - \nu a, \nu) . \tag{2.1}$$

For any $\Lambda \in \mathfrak{g}'$ we denote by G_Λ the isotropy subgroup of Λ in G. If $\nu \neq 0$, then $G_\Lambda = \{(0, 0, c) \in G\}$, which is the center of G. The Lie algebra of G_Λ is $\mathfrak{g}_\Lambda = \{(0, 0, z) \in \mathfrak{g}\}$. The *polarization* at Λ is, by definition ([AK]), a complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}_C$ such that

(1) $\mathfrak{g}_\Lambda \subseteq \mathfrak{h}$ and $\mathfrak{h}$ is stable under $\mathrm{Ad}(G_\Lambda)$,

(2) $2 \dim_C \mathfrak{g}_C/\mathfrak{h} = \dim_R \mathfrak{g}/\mathfrak{g}_\Lambda$,

(3) $\Lambda | [\mathfrak{h}, \mathfrak{h}] = 0$,

(4) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}_C$.

A polarization $\mathfrak{h}$ is called *real* if $\mathfrak{h} = \bar{\mathfrak{h}}$.

We consider the standard polarizations for Heisenberg group at $\Lambda = (\lambda, \mu, \nu)$ for $\nu \neq 0$. We denote by $M_n(K)$ and $Sym_n(K)$ the set of all matrices and symmetric matrices on a field K, respectively. We fix an integer k $(0 \leq k \leq n)$. For $x = (x_1, \cdots, x_n) \in C^n$ we put

$$x' = (x_1, \cdots, x_k), \qquad x'' = (x_{k+1}, \cdots, x_n)$$

if $k \geq 1$ and put $x = x''$ if $k = 0$. And we write $(x, y, z) = (x', x'', y', y'', z)$ in $\mathfrak{g}_C$. In the same way, we write $(a, b, c) = (a', a'', b', b'', c) \in G$. Let $T' \in M_k(C)$ and $T'' \in M_{n-k}(C)$ and put $T = T' \oplus T'' \in M_n(C)$. We define polarizations $\mathfrak{h}_T^k$ by

$$\mathfrak{h}_T^k = \{(T'y', x'', y', T''x'', z) \mid x, y \in C^n, z \in C\}.$$

By the condition (3) of the definition of the polarization, T is symmetric. It is easy to see that $\mathfrak{h}_T^k$ is real if and only if $T \in Sym_k(R) \oplus Sym_{n-k}(R)$. We assume that $T \in Sym_k(R) \oplus Sym_{n-k}(R)$. We call $\mathfrak{h}_T^k$ a *standard* polarization of rank k and $\mathfrak{h}_O^k$ a *canonical* polarization of rank k.

Let $D_T^k = \exp(\mathfrak{g} \cap \mathfrak{h}_T^k)$. Then $D_T^k = \{p = (T'b', a'', b', T''a'', c) \in G\}$. Each $\Lambda = (\lambda, \mu, \nu) \in (\mathfrak{g}')_C$ defines a one-dimensional representation of D_T^k. We denote it by τ_Λ:

$$\begin{aligned}\tau_\Lambda(h) &= \tau_\Lambda((T'b', a'', b', T''a'', c)) \\ &= \exp\sqrt{-1}\{\lambda' \cdot T'b' + \lambda'' \cdot a'' + \mu' \cdot b' + \mu'' \cdot T''a'' \\ &\qquad + \nu(c - (T'b' \cdot b' + T''a'' \cdot a'')/2)\}\end{aligned}$$

for $h = (T'b', a'', b', T''a'', c)$. Let $A_k = \{(a', 0, 0, b'', 0) \in G\}$. Then we can get a decomposition $G = A_k D_T^k$ by

$$(a, b, c) = (a' - T'b', 0, 0, b'' - T''a'', 0)(T'b', a'', b', T''a'', c - a' \cdot b' + T'b' \cdot b').$$

Let $U^{\Lambda,k,T}$ be a continuous representation of G induced by the representation τ_Λ of D_T^k. Let $\mathscr{D}(R^n)$ be the space of infinitely differentiable functions with compact support on R^n with usual topology. We realize $U^{\Lambda,k,T}$ on $C_c^\infty(A_k) \cong \mathscr{D}(R^k \times R^{n-k}) = \mathscr{D}(R^n)$, which we denote by $\mathscr{D}_\Lambda^{k,T}$:

$$\begin{aligned}(2.2)\qquad &(U_g^{\Lambda,k,T} F)(x) \\ &= e^{\sqrt{-1}\{\lambda' \cdot T'b' + \lambda'' \cdot a'' + \mu'b' + \mu'' \cdot T''a'' + \nu(c - b' \cdot x' + a'' \cdot x'' - a'' \cdot b'' - (T'b' \cdot b' - T''a'' \cdot a'')/2)\}} \\ &\quad \times F(x' - a' + T'b', x'' - b'' + T''a'')\end{aligned}$$

for $g=(a, b, c)\in G$ and $F\in\mathscr{D}_{\Lambda}^{k.T}$.

3. Invariant bilinear forms on $\mathscr{D}_{\Lambda_1}^{k,T}\times\mathscr{D}_{\Lambda_2}^{k,T}$.

Let B be a non-zero continuous bilinear form on $\mathscr{D}_{\Lambda_1}^{k,T}\times\mathscr{D}_{\Lambda_2}^{k,T}$ ($T\in Sym_k(\boldsymbol{R})\oplus Sym_{n-k}(\boldsymbol{R})$) which is invariant under G, that is

$$B(U_g^{\Lambda_1,k,T}F_1,\ U_g^{\Lambda_2,k,T}F_2)=B(F_1, F_2)$$

for all $g\in G$, $F_1\in\mathscr{D}_{\Lambda_1}^{k,T}$ and $F_2\in\mathscr{D}_{\Lambda_2}^{k,T}$.

Let $\Lambda_j=(\lambda_j, \mu_j, \nu_j)=(\lambda'_j, \lambda''_j, \mu'_j, \mu''_j, \nu_j)$. Then

$$\begin{aligned}&B(F_1, F_2)\\&=B(e^{\sqrt{-1}\{\lambda'_1\cdot T'b'+\lambda''_1\cdot a''+\mu'_1b'+\mu''_1\cdot T''a''+\nu_1(c-b'\cdot x'+a''\cdot x''-a''\cdot b''-(T'b'\cdot b'-T''a''\cdot a'')/2)\}}\\&\qquad F_1(x'-a'+T'b', x''-b''+T''a''),\ e^{\sqrt{-1}\{\lambda'_2\cdot T'b'+\lambda''_2\cdot a''+\mu'_2b'+\mu''_2\cdot T''a''+\nu_2(c-b'\cdot x'}\\&\qquad\qquad{}^{+a''\cdot x''-a''\cdot b''-(T'b'\cdot b'-T''a''\cdot a'')/2)\}}F_2(x'-a'+T'b', x''-b''+T''a''))\,.\end{aligned}$$

If we put $a=b=0$, then

$$B(F_1, F_2)=e^{\sqrt{-1}\{(\nu_1+\nu_2)c\}}B(F_1, F_2)$$

for all $c\in\boldsymbol{R}$. Hence we have $\nu_1=-\nu_2$. We put $\nu=\nu_1$.

If we put $a''=0$, $b'=0$, $c=0$, we have

$$B(F_1, F_2)=B(F_1(x'-a', x''-b''), F_2(x'-a', x''-b''))$$

for all $a'\in\boldsymbol{R}^k$ and $b''\in\boldsymbol{R}^{n-k}$. Thus B is a translation invariant bilinear form on $\mathscr{D}(\boldsymbol{R}^n)$. Then ([GV, GGV]) there exists a distribution $B_0\in\mathscr{D}'(\boldsymbol{R}^n)$ such that

$$B(F_1, F_2)=\left\langle B_0, \int_{\boldsymbol{R}^n}F_1(y)F_2(x+y)dy\right\rangle=\langle B_0, F_1*F_2\rangle$$

for any F_1 and F_2.

If we put $a'=0$, $b''=0$,

$$\begin{aligned}\langle B_0, F_1*F_2\rangle=\langle B_0,\ &e^{\sqrt{-1}\{(T'\lambda'_1+\mu'_1+T'\lambda'_2+\mu'_2)\cdot b'+(\lambda''_1+T''\mu''_1+\lambda''_2+T''\mu''_2)\cdot a''\}}\\&e^{\sqrt{-1}\,\nu(b'\cdot x'-a''\cdot x'')}F_1*F_2\rangle\,.\end{aligned}$$

Since the functions of the form F_1*F_2 ($F_1, F_2\in\mathscr{D}$) make a dense subset in $\mathscr{D}$,

$$\begin{aligned}\langle B_0, F\rangle=\langle B_0,\ &e^{\sqrt{-1}\{(T'\lambda'_1+\mu'_1+T'\lambda'_2+\mu'_2)\cdot b'+(\lambda''_1+T''\mu''_1+\lambda''_2+T''\mu''_2)\cdot a''\}}\\&e^{\sqrt{-1}\,\nu(b'\cdot x'-a''\cdot x'')}F\rangle\end{aligned}\tag{3.1}$$

for all $F\in\mathscr{D}(\boldsymbol{R}^n)$.

First we assume that $\nu=0$. Then

$$T'\lambda'_1+\mu'_1+T'\lambda'_2+\mu'_2=0\,,\qquad \lambda''_1+T''\mu''_1+\lambda''_2+T''\mu''_2=0\,.\tag{3.2}$$

Conversely, if Λ_1 and Λ_2 satisfy the relation (3.2), for any $B_0 \in \mathscr{D}'$ the bilinear form

$$B(F_1, F_2) = \langle B_0, F_1 * F_2 \rangle$$

is continuous and invariant.

Next we assume that $\nu \neq 0$. We set $|q'| = q_1 + \cdots + q_k$, $|q''| = q_{k+1} + \cdots + q_n$, $|q| = |q'| + |q''|$ and

$$D_{b'}^{q'} = \frac{\partial^{|q'|}}{\partial b_1'^{q_1} \cdots \partial b_k'^{q_k}}, \qquad D_{a''}^{q''} = \frac{\partial^{|q''|}}{\partial a_{k+1}''^{q_{k+1}} \cdots \partial a_n''^{q_n}}$$

for $q = (q_1, \cdots, q_n) \in \mathbf{Z}_+^n$. We let operate $D_{b'}^{q'} D_{a''}^{q''}$ ($|q| \neq 0$) to the both side of (3.1) and we put $b' = 0$ and $a'' = 0$. Then

$$(3.3) \qquad 0 = \sqrt{-1}^{|q|} \Big\langle B_0, \prod_{j=1}^{k} ((T'\lambda_1')_j + (\mu_1')_j + (T'\lambda_2')_j + (\mu_2')_j + \nu x_j')^{q_j} \prod_{j=k+1}^{n} ((\lambda_1'')_j + (T''\mu_1'')_j + (\lambda_2'')_j + (T''\mu_2'')_j - \nu x_j'')^{q_j} F \Big\rangle .$$

We put

$$\alpha_j = \begin{cases} -\dfrac{(T'\lambda_1')_j + (\mu_1')_j + (T'\lambda_2')_j + (\mu_2')_j}{\nu} & (1 \le j \le k) \\[2ex] \dfrac{(\lambda_1'')_j + (T''\mu_1'')_j + (\lambda_2'')_j + (T''\mu_2'')_j}{\nu} & (k < j \le n) . \end{cases}$$

Then, in particular, we have

$$(3.4) \qquad \langle B_0, (x_j - \alpha_j) F \rangle = 0$$

for any $F \in \mathscr{D}$. If $\alpha_j \notin \mathbf{R}$, for any $F \in \mathscr{D}$,

$$\langle B_0, F \rangle = \Big\langle B_0, (x_j - \alpha_j) \frac{F}{x_j - \alpha_j} \Big\rangle = 0 .$$

This contradicts the non-triviality of B. Hence $\alpha_j \in \mathbf{R}$ for all $j = 1, \cdots, n$.

From (3.3) we have

$$\Big\langle B_0, \sum_{j=1}^{n} (x_j - \alpha_j)^2 F \Big\rangle = 0 .$$

Let $U(\alpha)$ be any open neighborhood of $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $\mathbf{R}^n$ and $\chi_{U(\alpha)}$ be its characteristic function. If $F \in \mathscr{D}$ is zero on $U(\alpha)$, then

$$\langle B_0, F \rangle = \langle B_0, \chi_{U(\alpha)} F \rangle + \Big\langle B_0, \Big(\sum_{j=1}^{n} (x_j - \alpha_j)^2 \Big) \frac{(1 - \chi_{U(\alpha)}) F}{\sum_{j=1}^{n} (x_j - \alpha_j)^2} \Big\rangle = 0 .$$

Hence the support of B_0 is a single point α. So there exist $p \in \mathbf{Z}_+$ and $a_q \in \mathbf{C}$ for $q \in \mathbf{Z}_+^n$

such that

$$B_0 = \sum_{|q| \le p} a_q D_x^q \delta(x-\alpha)\,,$$

where δ is the Dirac's delta function. Then we have

$$\sum_{1 \le |q| \le p} (-1)^{|q|} a_q D^q((x_j-\alpha_j)F)(x)\big|_{x=\alpha} = 0 \tag{3.5}$$

for $j=1, \cdots, n$ by (3.4). If $|q| \ge 1$, then $q_j \ge 1$ for some j. We choose $F \in \mathscr{D}$ so that

$$F(x) = (x_1-\alpha_1)^{q_1} \cdots (x_j-\alpha_j)^{q_j-1} \cdots (x_n-\alpha_n)^{q_n}$$

on a neighbourhood of $x=\alpha$. Then by (3.5), $(-1)^{|q|} a_q q_1! \cdots q_n! = 0$. Hence

$$B(F_1, F_2) = a_0 \int_{\boldsymbol{R}_n} F_1(x) F_2(x+\alpha) dx \qquad (a_0 \neq 0)\,.$$

Thus we have the following theorem.

THEOREM 1. *Let* $\Lambda_1 = (\lambda_1, \mu_1, \nu_1)$, $\Lambda_2 = (\lambda_2, \mu_2, \nu_2) \in \boldsymbol{C}^{2n+1}$ *and* $T = T' \oplus T'' \in Sym_k(\boldsymbol{R}) \oplus Sym_{n-k}(\boldsymbol{R})$. *There exists a continuous non-trivial invariant bilinear form* $B = B_{\Lambda_1, \Lambda_2}$ *on* $\mathscr{D}_{\Lambda_1}^{k,T} \times \mathscr{D}_{\Lambda_2}^{k,T}$ *if and only if*

$$\nu_1 = -\nu_2 \neq 0\,, \qquad \frac{T'(\lambda_1' + \lambda_2') + \mu_1' + \mu_2'}{\nu_1} \in \boldsymbol{R}^k \tag{1}$$

and

$$\frac{\lambda_1'' + \lambda_2'' + T''(\mu_1'' + \mu_2'')}{\nu_1} \in \boldsymbol{R}^{n-k}\,,$$

or

$$\nu_1 = \nu_2 = 0\,, \quad T'(\lambda_1' + \lambda_2') + \mu_1' + \mu_2' = 0 \quad \textit{and} \quad \lambda_1'' + \lambda_2'' + T''(\mu_1'' + \mu_2'') = 0\,. \tag{2}$$

When (1) *holds,*

$$B(F_1, F_2) = C \int_{\boldsymbol{R}^k \times \boldsymbol{R}^{n-k}} F_1(x', x'') F_2\left(x' - \frac{T'(\lambda_1' + \lambda_2') + \mu_1' + \mu_2'}{\nu_1},\right.$$
$$\left. x'' + \frac{\lambda_1'' + \lambda_2'' + T''(\mu_1'' + \mu_2'')}{\nu_1}\right) dx' dx''$$

for non zero $C \in \boldsymbol{C}$. *In the case of* (2)

$$B(F_1, F_2) = \left\langle B_0, \int_{\boldsymbol{R}^n} F_1(y) F_2(x+y) dy \right\rangle$$

for any non zero distribution B_0 *on* $\boldsymbol{R}^n$.

COROLLARY. *There always exists a continuous invariant bilinear form*

$$B_{\Lambda,-\Lambda}(F_1, F_2)=C\int_{\mathbf{R}^n} F_1(x)F_2(x)dx \qquad (C\neq 0)$$

on $\mathscr{D}^k_\Lambda \times \mathscr{D}^k_{-\Lambda}$.

4. Intertwining operators between $\mathscr{D}^k_{\Lambda_1}$ and $\mathscr{D}^k_{\Lambda_2}$.

Let $A=A(\Lambda_1, \Lambda_2)$ be a non-trivial intertwining operator between $\mathscr{D}^{k,T}_{\Lambda_1}$ and $\mathscr{D}^{k,T}_{\Lambda_2}$, *i.e.* it is a non-trivial continuous linear mapping of $\mathscr{D}^{k,T}_{\Lambda_1}$ to $\mathscr{D}^{k,T}_{\Lambda_2}$ such that

$$AU^{\Lambda_1,k,T}_g=U^{\Lambda_2,k,T}_g A$$

for all $g\in G$.

We assume that $\nu_2\neq 0$. Then there exists a non-trivial invariant bilinear form on $\mathscr{D}^{k,T}_{\Lambda_2}\times\mathscr{D}^{k,T}_{-\Lambda_2}$:

$$B_{\Lambda_2,-\Lambda_2}(F_2, F_3)=C'\int_{\mathbf{R}^n} F_2(x)F_3(x)dx\,, \qquad C'\neq 0\,.$$

We put

$$B(F_1, F_3)=B_{\Lambda_2,-\Lambda_2}(A(\Lambda_1, \Lambda_2)F_1, F_3)\,.$$

This is a non-trivial invariant bilinear form on $\mathscr{D}^k_{\Lambda_1}\times\mathscr{D}^k_{-\Lambda_2}$. From Theorem 1 we have

$$\nu_1=\nu_2\,, \quad \frac{T'(\lambda'_1-\lambda'_2)+\mu'_1-\mu'_2}{\nu_1}\in\mathbf{R}^k\,, \quad \frac{\lambda''_1-\lambda''_2+T''(\mu''_1-\mu''_2)}{\nu_1}\in\mathbf{R}^{n-k}\,,$$

and

$$B(F_1, F_3)=C''\int_{\mathbf{R}^n} F_1(x)F_3\left(x'-\frac{T'(\lambda'_1-\lambda'_2)+\mu'_1-\mu'_2}{\nu_1},\right.$$
$$\left. x''+\frac{\lambda''_1-\lambda''_2+T''(\mu''_1-\mu''_2)}{\nu_1}\right)dx\,,$$

($C''\neq 0$). We have, therefore,

$$AF(x)=CF\left(x'+\frac{T'(\lambda'_1-\lambda'_2)+\mu'_1-\mu'_2}{\nu_1}, x''-\frac{\lambda''_1-\lambda''_2+T''(\mu''_1-\mu''_2)}{\nu_1}\right), \qquad C\neq 0\,.$$

Next we assume that $\nu_2=0$. Then we put $g=(0, 0, c)\in G$ in

$$(AU^{\Lambda_1,k,T}_g F)(x)=(U^{\Lambda_2,k,T}_g AF)(x)\,. \tag{4.1}$$

Then

$$e^{\sqrt{-1}\nu_1 c}(AF)(x)=(AF)(x)$$

for all $c\in \boldsymbol{R}$. Thus we have that $\nu_1=0$. Now we put $g=(a', b'', 0)\in G$ in (4.1). Since the operator $U_g^{\Lambda_j,k,T}$ is a translation on $\mathscr{D}$, the operator A is a continuous operator which commutes with translations. Finally, we put $g=(a, b, 0)\in G$. Then

$$e^{\sqrt{-1}\{(T'\lambda_1'+\mu_1')\cdot b'+(\lambda_1''+T''\mu_1'')\cdot a''\}}(AF)(x'-a'+T'b', x''-b''+T''a'')$$
$$=e^{\sqrt{-1}\{(T'\lambda_2'+\mu_2')\cdot b'+(\lambda_2''+T''\mu_2'')\cdot a''\}}(AF)(x'-a'+T'b', x''-b''+T''a'') .$$

We put $a'=T'b'$ and $b''=T''a''$. Then

$$e^{\sqrt{-1}\{\{(T'\lambda_1'+\mu_1')-(T'\lambda_2'+\mu_2')\}\cdot b'+\{(\lambda_1''+T''\mu_1'')-(\lambda_2''+T''\mu_2'')\}\cdot a''\}}(AF)(x)=(AF)(x)$$

for all $b'\in \boldsymbol{R}^k$ and $a''\in \boldsymbol{R}^{n-k}$. We have, therefore,

$$T'\lambda_1'+\mu_1'=T'\lambda_2'+\mu_2' \quad \text{and} \quad \lambda_1''+T''\mu_1''=\lambda_2''+T''\mu_2'' .$$

Thus we have the following theorem.

THEOREM 2. *There exists a non-trivial intertwining operator $A(\Lambda_1, \Lambda_2)$ between $\mathscr{D}_{\Lambda_1}^{k,T}$ and $\mathscr{D}_{\Lambda_2}^{k,T}$ if and only if*

$$(1)\qquad \nu_1=\nu_2\neq 0 , \quad \frac{(T'\lambda_1'+\mu_1')-(T'\lambda_2'+\mu_2')}{\nu_1}\in \boldsymbol{R}^k$$

and

$$\frac{(\lambda_1''+T''\mu_1'')-(\lambda_2''+T''\mu_2'')}{\nu_1}\in \boldsymbol{R}^{n-k} ,$$

or

$$(2)\qquad \nu_1=\nu_2=0 , \quad T'\lambda_1'+\mu_1'=T'\lambda_2'+\mu_2' \quad \textit{and} \quad \lambda_1''+T''\mu_1''=\lambda_2''+T''\mu_2'' .$$

In the case of (1)

$$(A(\Lambda_1, \Lambda_2)F)(x)=CF\left(x'+\frac{(T'\lambda_1'+\mu_1')-(T'\lambda_2'+\mu_2')}{\nu_1}, x''-\frac{(\lambda_1''+T''\mu_1'')-(\lambda_2''+T''\mu_2'')}{\nu_1}\right),$$

$C\neq 0$. And in the case of (2) *$A(\Lambda_1, \Lambda_2)$ is a continuous operator on $\mathscr{D}$ which commutes with translations.*

In the case (1) of Theorem 2 we put $\Lambda_1=\Lambda_2$, then we have

$$A(\Lambda_1, \Lambda_2)=CI .$$

Thus we have the following theorem.

THEOREM 3. *The representation $U^{\Lambda,k,T}$ ($\Lambda=(\lambda, \mu, \nu)$) on $\mathscr{D}_{\Lambda}^{k,T}$ ($T=T'\oplus T''\in Sym_k(\boldsymbol{R})\oplus Sym_{n-k}(\boldsymbol{R})$) is operator irreducible if $\nu\neq 0$.*

5. Invariant hermitian form on $\mathscr{D}_{\Lambda}^{k,T}$.

A hermitian form $H(F_1, F_2)$ on $\mathscr{D}_{\Lambda}^{k,T}$ is said invariant if

$$H(U_g^{\Lambda,k,T}F_1,\ U_g^{\Lambda,k,T}F_2)=H(F_1, F_2)$$

for all $F_1, F_2 \in \mathscr{D}_{\Lambda}^{k,T}$. Let $H(F_1, F_2)$ be a non-trivial continuous hermitian form on $\mathscr{D}_{\Lambda}^{k,T}$. We put

$$B(F_1, F_2)=H(F_1, \bar{F}_2)\,,$$

where the upper bar denotes the complex conjugate. For $\Lambda=(\lambda, \mu, \nu)=(\lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n, \nu)$ we put $\bar{\Lambda}=(\bar{\lambda}, \bar{\mu}, \bar{\nu})=(\bar{\lambda}_1, \cdots, \bar{\lambda}_n, \bar{\mu}_1, \cdots, \bar{\mu}_n, \bar{\nu})$. Then we have

$$\overline{(U_g^{\Lambda,k,T}F)(x)}=(U_g^{-\bar{\Lambda},k,T}\bar{F})(x)\,.$$

Hence

$$B(U_g^{\Lambda,k,T}F_1,\ U_g^{-\bar{\Lambda},k,T}F_2)=H(U_g^{\Lambda,k,T}F_1,\ U_g^{\Lambda,k,T}\bar{F}_2)=H(F_1, \bar{F}_2)=B(F_1, F_2)\,.$$

Thus B is a non-trivial continuous invariant bilinear form on $\mathscr{D}_{\Lambda}^{k,T}\times\mathscr{D}_{-\bar{\Lambda}}^{k,T}$. By Theorem 1 we have $\nu=\bar{\nu}$ and so $\nu\in\boldsymbol{R}$. If $\nu\neq 0$, then $(T'\lambda'+\mu'-\overline{(T\lambda'+\mu')})/\nu\in\boldsymbol{R}^k$ and $(\lambda''+T''\mu''-\overline{(\lambda''+T''\mu'')})/\nu\in\boldsymbol{R}^{n-k}$. Hence $T'\lambda'+\mu'=\overline{(T\lambda'+\mu')}$ and $\lambda''+T''\mu''=\overline{(\lambda''+T''\mu'')}$. If $\nu=0$, then the same result holds. Thus we proved the following theorem.

THEOREM 4. *There exists a non-trivial continuous invariant hermitian form H on $\mathscr{D}_{\Lambda}^{k,T}$ if and only if Λ is of the form $(\lambda'+\sqrt{-1}\,\xi', \lambda''-\sqrt{-1}\,T''\xi'', \mu'-\sqrt{-1}\,T'\xi', \mu''+\sqrt{-1}\,\xi'', \nu)$, where λ, μ and ξ are in $\boldsymbol{R}^n$ and ν is in $\boldsymbol{R}$. If $\nu\neq 0$, then*

$$H(F_1, F_2)=C\int_{\boldsymbol{R}^n} F_1(x)\bar{F}_2(x)dx\,, \qquad C\in\boldsymbol{R}-\{0\}\,.$$

6. G- and B_T^k-orbits in $(\mathfrak{g}')_{\boldsymbol{C}}$.

Let $\Lambda_1, \Lambda_2\in(\mathfrak{g}')_{\boldsymbol{C}}$. Two elements $\Lambda_1=(\lambda_1, \mu_1, \nu_1)$ and $\Lambda_2=(\lambda_2, \mu_2, \nu_2)$ are on the same G-orbit if and only if $\Lambda_2=g\cdot\Lambda_1$ for some $g=(a, b, c)\in G$. Then by (2.1)

$$\lambda_2=\lambda_1+\nu_1 b\,, \quad \mu_2=\mu_1-\nu_1 a \quad \text{and} \quad \nu_1=\nu_2\,.$$

Hence if $\nu_1=0$, then

$$\lambda_1=\lambda_2\,, \quad \mu_1=\mu_2 \quad \text{and} \quad \nu_2=0\,.$$

If $\nu_1\neq 0$, then

$$\nu_1=\nu_2\,, \qquad \frac{(T'\lambda_2'+\mu_2')-(T'\lambda_1'+\mu_1')}{\nu_1}=T'b'-a'\in\boldsymbol{R}^k\,,$$

$$\frac{(\lambda_2''+T''\mu_2'')-(\lambda_1''+T''\mu_1'')}{\nu_1}=b''-T''a''\in \boldsymbol{R}^{n-k}.$$

Then $U^{\Lambda_1,k,T}$ is equivalent to $U^{\Lambda_2,k,T}$ from Theorems 2, 3. However, even if two representations $U^{\Lambda_1,k,T}$ and $U^{\Lambda_2,k,T}$ are equivalent, Λ_1 and Λ_2 are not necessarily on the same G-orbit.

Let $G_{\boldsymbol{C}}$ be the complexification of G, *i.e.* the group of elements $g=(\alpha, \beta, \gamma)\in \boldsymbol{C}^{2n+1}$ with the same multiplication as that of G. The Lie algebra of $G_{\boldsymbol{C}}$ is $\mathfrak{g}_{\boldsymbol{C}}$. Then $G_{\boldsymbol{C}}$ acts naturally on $(\mathfrak{g}')_{\boldsymbol{C}}$:

$$(\alpha, \beta, \gamma)\cdot(\lambda, \mu, \nu)=(\lambda+\nu\beta, \mu-\nu\alpha, \nu).$$

Let B_T^k be a subgroup of $G_{\boldsymbol{C}}$ consisting of elements

$$g=(a'+\sqrt{-1}\,T'u', a''+\sqrt{-1}\,u'', b'+\sqrt{-1}\,u', b''+\sqrt{-1}\,T''u'', \gamma), \tag{6.1}$$

where $a, b, u\in \boldsymbol{R}^n$, $\gamma\in \boldsymbol{C}$. Then $B_T^k=\exp(\mathfrak{g}+\sqrt{-1}(\mathfrak{h}_T^k\cap\mathfrak{g}))$, where $T=T'\oplus T''$. We assume that Λ_1 and Λ_2 satisfy the condition (1) of Theorem 2. We put

$$\frac{\lambda_1'-\lambda_2'}{\nu_1}=-b'-\sqrt{-1}\,u', \qquad \frac{\mu_1'-\mu_2'}{\nu_1}=a'+\sqrt{-1}\,T'u',$$

$$\frac{\lambda_1''-\lambda_2''}{\nu_1}=b''+\sqrt{-1}\,T''u'', \qquad \frac{\mu_1''-\mu_2''}{\nu_1}=-a''-\sqrt{-1}\,u'',$$

where $a, b, u\in \boldsymbol{R}^n$. We put

$$g=(a'+\sqrt{-1}\,T'u', -a''-\sqrt{-1}\,u'', b'+\sqrt{-1}\,u', -b''-\sqrt{-1}\,T''u'', 0).$$

Then $g\cdot\Lambda_1=\Lambda_2$.

Conversely, if $g\cdot\Lambda_1=\Lambda_2$ for $g\in B_T^k$ of the form (6.1), then it is easy to see that Λ_1 and Λ_2 satisfy the condition (1) of Theorem 2.

Thus we have the following theorem.

THEOREM 5. *Let* $T=T'\oplus T''\in Sym_k(\boldsymbol{R})\oplus Sym_{n-k}(\boldsymbol{R})$. *We assume that* $\Lambda_1=(\lambda_1, \mu_1, \nu_1)$, $\Lambda_2=(\lambda_2, \mu_2, \nu_2)\in(\mathfrak{g}')_{\boldsymbol{C}}$ *and* $\nu_1\neq 0$, $\nu_2\neq 0$. *Let* $B_T^k=\exp(\mathfrak{g}+\sqrt{-1}(\mathfrak{h}_T^k\cap\mathfrak{g}))$. *Then two representations* $U^{\Lambda_1,k,T}$ *and* $U^{\Lambda_2,k,T}$ *are equivalent if and only if* Λ_1 *and* Λ_2 *are on the same* B_T^k*-orbit in* $(\mathfrak{g}')_{\boldsymbol{C}}$. *Especially, if* Λ_1 *and* Λ_2 *are on the same G-oibit, then* $U^{\Lambda_1,k,T}$ *and* $U^{\Lambda_2,k,T}$ *are equivalent.*

7. Invariant bilinear forms on $\mathcal{D}_{\Lambda_1}^{k,T_1}\times\mathcal{D}_{\Lambda_2}^{k,T_2}$.

In this section we consider an invariant bilinear form on $\mathcal{D}_{\Lambda_1}^{k,T_1}\times\mathcal{D}_{\Lambda_2}^{k,T_2}$ for the two cases: (1) T_1-T_2 is regular or (2) T_1-T_2 is diagonal.

Let B be a non-trivial continuous bilinear form on $\mathcal{D}_{\Lambda_1}^{k,T_1}\times\mathcal{D}_{\Lambda_2}^{k,T_2}$, that is,

$$
\begin{aligned}
&B(F_1, F_2) \\
&= B\big(e^{\sqrt{-1}\{(T'_1\lambda'_1+\mu'_1)\cdot b'+(\lambda''_1+T''_1\lambda''_1)\cdot a''+\nu_1(c-b'\cdot x'+a''\cdot x''-a''\cdot b''-(T'_1b'\cdot b'-T''_1a''\cdot a'')/2)\}} \\
&\qquad F_1(x'-a'+T'_1b', x''-b''+T''_1a''), e^{\sqrt{-1}\{(T'_2\lambda'_2+\mu'_2)\cdot b'+(\lambda''_2+T''_2\mu''_2)\cdot a''} \\
&\qquad\quad {}^{+\nu_2(c-b'\cdot x'+a''\cdot x''-a''\cdot b''-(T'_2b'\cdot b'-T''_2a''\cdot a'')/2)\}}F_2(x'-a'+T'_2b', x''-b''+T''_2a'')\big)
\end{aligned}
$$

for all $a, b\in \boldsymbol{R}^n$, $c\in \boldsymbol{R}$, $F_1\in \mathscr{D}^{k,T_1}_{\Lambda_1}$ and $F_2\in\mathscr{D}^{k,T_2}_{\Lambda_2}$. By the same arguments as in §2, we have $\nu_2=-\nu_1$ and there exists a distribution $B_0\in\mathscr{D}'(\boldsymbol{R}^n)$ such that

$$B(F_1, F_2)=\left\langle B_0, \int_{\boldsymbol{R}^n} F_1(y)F_2(x+y)dy\right\rangle .$$

Then we have

$$
\begin{aligned}
&\langle B_0, F\rangle \\
&=\langle B_0, e^{\sqrt{-1}\{(T'_1\lambda'_1+\mu'_1+T'_2\lambda'_2+\mu'_2+\nu_1x')\cdot b'+(\lambda''_1+T''_1\mu''_1+\lambda''_2+T''_2\mu''_2-\nu_1x'')\cdot a''} \\
&\qquad {}^{-\nu_1(T'_1-T'_2)b'\cdot b'/2+\nu_1(T''_1-T''_2)a''\cdot a''/2\}}F(x'-(T'_1-T'_2)b', x''-(T''_1-T''_2)a'')\rangle .
\end{aligned}
$$

We differentiate the both sides by b_j and a_j at $b'=0$, $a''=0$. Then, for $1\le j\le k$,

$$\left\langle B_0, \sqrt{-1}\{(T'_1\lambda'_1)_j+(\mu_1)_j+(T'_2\lambda'_2)_j+(\mu_2)_j+\nu_1x_j\}F - \sum_{i=1}^{k}(T'_1-T'_2)_{ji}\frac{\partial}{\partial x_i}F\right\rangle=0$$

and, for $k+1\le j\le n$,

$$\left\langle B_0, \sqrt{-1}\{(\lambda_1)_j+(T''_1\mu''_1)_j+(\lambda_2)_j+(T''_2\mu''_2)_j-\nu_1x_j\}F - \sum_{i=k+1}^{n}(T''_1-T''_2)_{ji}\frac{\partial}{\partial x_i}F\right\rangle=0$$

for any $F\in\mathscr{D}(\boldsymbol{R}^n)$. Hence the distribution B_0 satisfies the following differential equations

$$(7.1)\qquad (T'_1-T'_2)\begin{pmatrix}\partial/\partial x_1\\ \vdots \\ \partial/\partial x_k\end{pmatrix}B_0=-\sqrt{-1}\,{}^t\{T'_1\lambda'_1+\mu'_1+T'_2\lambda'_2+\mu'_2+\nu_1x'\}B_0 ,$$

$$(7.2)\qquad (T''_1-T''_2)\begin{pmatrix}\partial/\partial x_{k+1}\\ \vdots \\ \partial/\partial x_n\end{pmatrix}B_0=-\sqrt{-1}\,{}^t\{\lambda''_1+T''_1\mu''_1+\lambda''_2+T''_2\mu''_2-\nu_1x''\}B_0 .$$

(1) We assume that T_1-T_2 is regular. Then

$$\begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_k \end{pmatrix} B_0 = -\sqrt{-1}(T'_1 - T'_2)^{-1\,t}\{T'_1\lambda'_1 + \mu'_1 + T'_2\lambda'_2 + \mu'_2 + \nu_1 x'\}B_0\,,$$

$$\begin{pmatrix} \partial/\partial x_{k+1} \\ \vdots \\ \partial/\partial x_n \end{pmatrix} B_0 = -\sqrt{-1}(T''_1 - T''_2)^{-1\,t}\{\lambda''_1 + T''_1\mu''_1 + \lambda''_2 + T''_2\mu''_2 - \nu_1 x''\}B_0\,.$$

Therefore, B_0 is a function on $\boldsymbol{R}^n$ such that

$$B_0(x) = Ce^{-\sqrt{-1}\{\nu_1((T'_1 - T'_2)^{-1}x'\cdot x' - (T''_1 - T''_2)^{-1}x''\cdot x'')/2 + (T'_1 - T'_2)^{-1}(T'_1\lambda'_1 + \mu'_1 + T'_2\lambda'_2 + \mu'_2)\cdot x' + (T''_1 - T''_2)^{-1}(\lambda''_1 + T''_1\mu''_1 + \lambda''_2 + T''_2\mu''_2)\cdot x''\}}\,,$$

where C is a non-zero constant. Then we have the following theorem.

THEOREM 6. *Let $T_1, T_2 \in Sym_k(\boldsymbol{R}) \oplus Sym_{n-k}(\boldsymbol{R})$. We assume that $T_1 - T_2$ is regular. If $\nu_2 = -\nu_1$, then there exists a non-trivial continuous invariant bilinear form on $\mathscr{D}_{\Lambda_1}^{T_1,k} \times \mathscr{D}_{\Lambda_2}^{T_2,k}$. It is of the form*

$$B(F_1, F_2) = C\iint_{\boldsymbol{R}^{2n}} e^{-\sqrt{-1}\varphi(y-x)}F_1(x)F_2(y)dxdy\,,$$

where

$$\varphi(x) = \nu_1((T'_1 - T'_2)^{-1}x'\cdot x' - (T''_1 - T''_2)^{-1}x''\cdot x'')/2 + (T'_1 - T'_2)^{-1}(T'_1\lambda'_1 + \mu'_1 + T'_2\lambda'_2 + \mu'_2)\cdot x' + (T''_1 - T''_2)^{-1}(\lambda''_1 + T''_1\mu''_1 + \lambda''_2 + T''_2\mu''_2)\cdot x''$$

and C is a non zero constant.

(2) Next we assume that $T_1 - T_2$ is a diagonal matrix and $\nu_2 = -\nu_1 \neq 0$. For simplicity we consider for diagonal matrix $T_1 - T_2 = (T'_1 - T'_2) \oplus (T''_1 - T''_2)$,

$$T'_1 - T'_2 = \mathrm{diag}(t_1, \cdots, t_r, 0, \cdots, 0)\,, \qquad T''_1 - T''_2 = \mathrm{diag}(t_{k+1}, \cdots, t_s, 0, \cdots, 0)\,,$$

where $t_1 \cdots t_r t_{k+1} \cdots t_s \neq 0$. Then, by (7.1) and (7.2), we have

$$\frac{\partial}{\partial x_j}B_0 = -\sqrt{-1}\,t_j^{-1}\{(T'_1\lambda'_1)_j + (\mu'_1)_j + (T'_2\lambda'_2)_j + (\mu'_2)_j + \nu_1 x_j\}B_0 \qquad (1 \leq j \leq r)\,,$$

$$\frac{\partial}{\partial x_j}B_0 = -\sqrt{-1}\,t_j^{-1}\{(\lambda''_1)_j + (T''_1\mu''_1)_j + (\lambda''_2)_j + (T''_2\mu''_2)_j - \nu_1 x_j\}B_0 \qquad (k+1 \leq j \leq s)\,.$$

For $r < j \leq k$ and $s < j \leq n$ we have

$$0=\left\langle B_0, \prod_{j=r+1}^{k}((T'\lambda'_1)_j+(\mu'_1)_j+(T'\lambda'_2)_j+(\mu'_2)_j+\nu x_j)^{q_j}\right.$$
$$\left.\prod_{j=s+1}^{n}((\lambda''_1)_j+(T''\mu''_1)_j+(\lambda''_2)_j+(T''\mu''_2)_j-\nu x_j)^{q_j}F\right\rangle$$

as (3.3). Hence, if

$$\alpha_j\equiv-\frac{(T'(\lambda'_1+\lambda'_2)+\mu'_1+\mu'_2)_j}{\nu_1}\in\boldsymbol{R}\qquad(r<j\leq k)$$
$$\alpha_j\equiv\frac{(\lambda''_1+\lambda''_2+T''(\mu''_1+\mu''_2))_j}{\nu_1}\in\boldsymbol{R}\qquad(s<j\leq n)\,,$$

the distribution B_0 is a direct product of two distributions B_1 with respect to variables $x_1, \cdots, x_r, x_{k+1}, \cdots, x_s$ and B_2 with respect to variables $x_{r+1}, \cdots, x_k, x_{s+1}, \cdots, x_n$:

$$\langle B_1, f\rangle=C_1\int_{\boldsymbol{R}^{r+s-k}}e^{-\sqrt{-1}\,\varphi(x)}f(x)dx\qquad(f\in\mathscr{D}(\boldsymbol{R}^{r+s}))\,,$$
$$B_2=C_2\delta(x_{r+1}-\alpha_{r+1}, \cdots, x_k-\alpha_k, x_{s+1}-\alpha_{s+1}, \cdots, x_n-\alpha_n)\,,$$

where

$$\varphi(x)=\sum_{j=1}^{r}t_j^{-1}(\nu_1x_j/2+(T'_1\lambda'_1)_j+(\mu'_1)_j+(T'_2\lambda'_2)_j+(\mu'_2)_j)x_j$$
$$+\sum_{j=k+1}^{s}t_j^{-1}(-\nu_1x_j/2+(\lambda''_1)_j+(T''_1\mu''_1)_j+(\lambda''_2)_j+(T''_2\mu''_2)_j)x_j\,.$$

Thus we have the following theorem.

THEOREM 7. *Let $T_1, T_2\in Sym_k(\boldsymbol{R})\oplus Sym_{n-k}(\boldsymbol{R})$. We assume that $T'_1-T'_2=\mathrm{diag}(t_1, \cdots, t_r, 0, \cdots, 0)$, $T''_1-T''_2=\mathrm{diag}(t_{k+1}, \cdots, t_s, 0, \cdots, 0)$, $(t_1\cdots t_rt_{k+1}\cdots t_s\neq0)$. If $\nu_2=-\nu_1\neq0$ and*

$$\alpha_j\equiv-\frac{(T'(\lambda'_1+\lambda'_2)+\mu'_1+\mu'_2)_j}{\nu_1}\in\boldsymbol{R}\qquad(r<j\leq k)$$
$$\alpha_j\equiv\frac{(\lambda''_1+\lambda''_2+T''(\mu''_1+\mu''_2))_j}{\nu_1}\in\boldsymbol{R}\qquad(s<j\leq n)\,,$$

then there exists a non-trivial continuous invariant bilinear form on $\mathscr{D}_{\Lambda_1}^{k,T_1}\times\mathscr{D}_{\Lambda_2}^{k,T_2}$. It is of the form

$$B(F_1, F_2)=C\int_{R^{n+r+s-k}} e^{-\sqrt{-1}\varphi(y-x)}F_1(x)F_2(y_1, \cdots, y_r, x_{r+1}+\alpha_{r+1}, \cdots, x_k+\alpha_k, y_{k+1}, \cdots, y_s, x_{s+1}+\alpha_{s+1}, \cdots, x_n+\alpha_n)dx_1\cdots dx_n dy_1\cdots dy_r dy_{k+1}\cdots dy_s,$$

where

$$\varphi(x)=\sum_{j=1}^{r} t_j^{-1}(\nu_1 x_j/2+(T'_1\lambda'_1)_j+(\mu'_1)_j+(T'_2\lambda'_2)_j+(\mu'_2)_j)x_j + \sum_{j=k+1}^{s} t_j^{-1}(-\nu_1 x_j/2+(\lambda''_1)_j+(T''_1\mu''_1)_j+(\lambda''_2)_j+(T''_2\mu''_2)_j)x_j$$

and C is a non-zero constant.

8. Application to intertwining operators of the unitary representations $U^{\Lambda,k}$.

We assume that $\Lambda=(\lambda, \mu, \nu)$, $T'\lambda'+\mu'\in \boldsymbol{R}^k$, $\lambda''+T''\mu''\in \boldsymbol{R}^{n-k}$ and $\nu\in \boldsymbol{R}$. Then the representation $U^{\Lambda,k,T}$ can be realized on the Schwartz space $\mathcal{S}(\boldsymbol{R}^n)$ of rapidly decreasing functions by (2.2). We denote $\mathcal{S}(\boldsymbol{R}^n)$ by $\mathcal{S}_\Lambda^{k,T}$ as the representation space of $U^{\Lambda,k,T}$. Then we can prove the following theorems in the same way as the proof of Theorem 1 and Theorem 6.

THEOREM 8. *Let* $\Lambda_1=(\lambda_1, \mu_1, \nu_1)$, $\Lambda_2=(\lambda_2, \mu_2, \nu_2)\in \boldsymbol{C}^{2n+1}$ *and* $T=T'\oplus T''\in Sym_k(\boldsymbol{R})\oplus Sym_{n-k}(\boldsymbol{R})$. *We assume that* $\nu_i\in \boldsymbol{R}$, $T'_i\lambda'_i+\mu'_i\in \boldsymbol{R}^k$ *and* $\lambda''_i+T''_i\mu''_i\in \boldsymbol{R}^{n-k}$ $(i=1, 2)$. *There exists a continuous non-trivial invariant bilinear form* $B=B_{\Lambda_1,\Lambda_2}$ *on* $\mathcal{S}_{\Lambda_1}^{k,T}\times\mathcal{S}_{\Lambda_2}^{k,T}$ *if and only if*

(1) $$\nu_1=-\nu_2\neq 0$$

or

(2) $\nu_1=\nu_2=0$, $T'(\lambda'_1+\lambda'_2)+\mu'_1+\mu'_2=0$ *and* $\lambda''_1+\lambda''_2+T''(\mu''_1+\mu''_2)=0$.

When (1) *holds,*

$$B(F_1, F_2)=C\int_{R^k\times R^{n-k}} F_1(x', x'')F_2\left(x'-\frac{T'(\lambda'_1+\lambda'_2)+\mu'_1+\mu'_2}{\nu_1}, x''+\frac{\lambda''_1+\lambda''_2+T''(\mu''_1+\mu''_2)}{\nu_1}\right)dx'dx''$$

for non zero $C\in \boldsymbol{C}$. *In the case of* (2)

$$B(F_1, F_2)=\left\langle B_0, \int_{R^n} F_1(y)F_2(x+y)dy\right\rangle$$

for any non zero tempered distribution B_0 *on* $\boldsymbol{R}^n$.

THEOREM 9. *Let $T_1, T_2 \in Sym_k(\boldsymbol{R}) \oplus Sym_{n-k}(\boldsymbol{R})$. We assume that $T_1 - T_2$ is regular and that $\nu_i \in \boldsymbol{R}$, $T'_i\lambda'_i + \mu'_i \in \boldsymbol{R}^k$ and $\lambda''_i + T''_i\mu''_i \in \boldsymbol{R}^{n-k}$ $(i=1,2)$. If $\nu_2 = -\nu_1$, then there exists a non-trivial continuous invariant bilinear form on $\mathscr{S}^{k,T_1}_{\Lambda_1} \times \mathscr{S}^{k,T_2}_{\Lambda_2}$. It is of the form*

$$B(F_1, F_2) = C\iint_{\boldsymbol{R}^{2n}} e^{-\sqrt{-1}\varphi(y-x)} F_1(x)F_2(y)dxdy\,,$$

where

$$\varphi(x) = \nu_1((T'_1 - T'_2)^{-1}x'\cdot x' - (T''_1 - T''_2)^{-1}x''\cdot x'')/2 + (T'_1 - T'_2)^{-1}(T'_1\lambda'_1 + \mu'_1 + T'_2\lambda'_2 + \mu'_2)\cdot x' + (T''_1 - T''_2)^{-1}(\lambda''_1 + T''_1\mu''_1 + \lambda''_2 + T''_2\mu''_2)\cdot x''$$

and C is a non-zero constant.

Let T_1, T_2, Λ_1 and Λ_2 be as in Theorem 9. We assume that $\nu_2 \neq 0$. Let $A(\Lambda_1, \Lambda_2)$ be a non-trivial continuous intertwining operator of $\mathscr{S}^{k,T_1}_{\Lambda_1}$ to $\mathscr{S}^{k,T_2}_{\Lambda_2}$. By Theorem 8 there exists an essentially unique invariant bilinear form $B_{-\Lambda_2,\Lambda_2}$ on $\mathscr{S}^{k,T_2}_{-\Lambda_2} \times \mathscr{S}^{k,T_2}_{\Lambda_2}$:

$$B_{-\Lambda_2,\Lambda_2}(F_3, F_2) = C_1\int_{\boldsymbol{R}^n} F_3(x)F_2(x)dx\,,$$

for $F_2 \in \mathscr{S}^{k,T_2}_{\Lambda_2}$, $F_3 \in \mathscr{S}^{k,T_2}_{-\Lambda_2}$. Then the bilinear form B on $\mathscr{S}^{k,T_2}_{-\Lambda_2} \times \mathscr{S}^{k,T_1}_{\Lambda_1}$ defined by

$$B(F_3, F_1) = B_{-\Lambda_2,\Lambda_2}(F_3, A(\Lambda_1, \Lambda_2)F_1) = C_1\int_{\boldsymbol{R}^n} F_3(x)(A(\Lambda_1, \Lambda_2)F_1)(x)dx$$

is invariant. Hence by Theorems 8, 9 we have $\nu_1 = \nu_2$ and

$$B(F_3, F_1) = C_2\iint_{\boldsymbol{R}^{2n}} e^{-\sqrt{-1}\varphi(y-x)} F_3(x)F_1(y)dxdy\,,$$

where

$$\varphi(x) = \nu_1((T'_2 - T'_1)^{-1}x'\cdot x' - (T''_2 - T''_1)^{-1}x''\cdot x'')/2 + (T'_2 - T'_1)^{-1}(T'_1\lambda'_1 + \mu'_1 - T'_2\lambda'_2 - \mu'_2)\cdot x' + (T''_2 - T''_1)^{-1}(\lambda''_1 + T''_1\mu''_1 - \lambda''_2 - T''_2\mu''_2)\cdot x''$$

and C_2 is a non-zero constant. Since F_3 is arbitrary, we have the following theorem.

THEOREM 10. *Let $T_1, T_2 \in Sym_k(\boldsymbol{R}) \oplus Sym_{n-k}(\boldsymbol{R})$. We assume that $T_1 - T_2$ is regular and that $\nu_i \in \boldsymbol{R}$, $T'_i\lambda'_i + \mu'_i \in \boldsymbol{R}^k$ and $\lambda''_i + T''_i\mu''_i \in \boldsymbol{R}^{n-k}$ $(i=1,2)$. If $\nu_1 = \nu_2 \neq 0$, then any non-trivial intertwining operator $\mathscr{S}^{k,T_1}_{\Lambda_1}$ to $\mathscr{S}^{k,T_2}_{\Lambda_2}$ is given by*

$$(A(\Lambda_1, \Lambda_2)F)(x) = C\int_{\boldsymbol{R}^n} e^{-\sqrt{-1}\varphi(y-x)} F(y)dy\,,$$

where

$$\varphi(x) = \nu_1((T'_2 - T'_1)^{-1}x'\cdot x' - (T''_2 - T''_1)^{-1}x''\cdot x'')/2 + (T'_2 - T'_1)^{-1}(T'_1\lambda'_1 + \mu'_1 - T'_2\lambda'_2 - \mu'_2)\cdot x' + (T''_2 - T''_1)^{-1}(\lambda''_1 + T''_1\mu''_1 - \lambda''_2 - T''_2\mu''_2)\cdot x''$$

and C is a non-zero constant.

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