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Transformation $\tilde{G}$ of Analytic Functionals with Unbounded Carriers and Its Applications

Kunio YOSHINO

Sophia University

1. Introduction.

In [1] and [8], Avanissian and Supper considered transformation $\tilde{G}$ for analytic functionals. They applied transformation $\tilde{G}$ to study arithmetic entire functions of exponential type in Abel sense and derived Abel summation formula for entire functions of exponential type in C^n by using the sequence $\{D^n F(n)\}$. They also showed some relations between analytic functionals and classical special functions using transformation $\tilde{G}$. In this paper we will consider transformation $\tilde{G}$ for analytic functionals *with unbounded carrier*. As application, we derive some theorems for holomorphic (non-entire) functions of exponential type defined in direct product of half planes by using the sequence $\{D^{-n}F(-n)\}$.

2. Notations.

$U=\{\zeta=r\exp(i\theta)\in C^n: 0\leq r<(\pi-|\theta|)/|\sin(\theta)|\}$. $\Phi(\zeta)=\exp(-\zeta)/\zeta$. Φ is biholomorphic mapping between $U/\{0\}$ and $C/[-e, 0]$ ([4]). We put $\psi=\Phi^{-1}$.

For set L in C^n, $L_i=\mathrm{pr}_i(L)$ denotes i-th projection of L. $\langle z, t\rangle=z_1t_1+\cdots+z_nt_n$ for $z=(z_1, \cdots, z_n)$, $t=(t_1, \cdots, t_n)\in C^n$. $H_L(z)=\sup_{t\in L}\mathrm{Re}\langle z, t\rangle$.

$d(F)$ denotes transfinite diameter of compact set F in C. For the details of transfinite diameter, we refer the reader to [6].

$$D_x^{-n}f(x)=\frac{1}{(n-1)!}\int_0^\infty f(x-y)y^{n-1}dy\,. \qquad (n\in N)$$

Under suitable conditions, $D_x^n D_x^{-n}f(x)=f(x)$ valids for ordinary differential operator D_x. We put

$$D^{-m}=D_{x_1}^{-m_1}\cdots D_{x_n}^{-m_n}\,,\quad |m|=m_1+\cdots+m_n \qquad \text{for } m=(m_1, \cdots, m_n)\in N^n\,.$$

$\mathrm{Exp}(D:L)=\{$holomorphic functions in D satisfying condition $(*)$ in theorem 2$\}$.

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Let K be a number field of degree $d=[K:Q]=r+2s$. $K^{(i)}$ $(1\leq i\leq r)$ and $\bar{K}^{(r+j)}=K^{(r+s+j)}$ $(1\leq j\leq s)$ are its conjugate fields. $\mathcal{O}_K$ and $\mathcal{O}_K^{(j)}$ denote the rings of albebraic integers on K and $K^{(j)}$ respectively. If $a\in\mathcal{O}_K^{(i)}$, $a^{(j)}\in\mathcal{O}_K^{(j)}$ are conjugates of a. For $f(w)=\sum a_n w^n\in K[w]$, we denote $f^{(j)}(w)=\sum a_n^{(j)}w^n\in K^{(j)}[w]$. We put $\delta=d$ if $K\subset \boldsymbol{R}$, $\delta=d/2$ if $K\not\subset \boldsymbol{R}$. For algebraic integer a, $|a|=\max\{|a_1|,\cdots,|a_k|\}$ where $a_1,\cdots,a_k$ are conjugates of a over $\boldsymbol{Q}$.

3. Fourier-Borel transform of analytic functionals with unbounded carrier.

In what follows, L denotes closed convex set bounded in imaginary direction in $\boldsymbol{C}^n$. We put

$$Q(L:k')=\lim_{\varepsilon>0,\varepsilon'>0}\text{ind}\ Q_b\ (L_\varepsilon:k'+\varepsilon')$$

where $Q_b(L_\varepsilon:k'+\varepsilon')=\{f\in C(\bar{L}_\varepsilon)\cap H(L_\varepsilon):\sup_{z\in L_\varepsilon}|f(z)\exp((k'+\varepsilon')z)|<\infty\}$, L_ε is ε-neighbourhood of L and $\bar{L}_\varepsilon$ is closure of L_ε. $C(\bar{L}_\varepsilon)$ and $H(L_\varepsilon)$ denote the space of continuous functions on $\bar{L}_\varepsilon$ and holomorphic functions in L_ε respectively. $Q'(L:k')$ denotes the dual space of $Q(L:k')$. An element of $Q'(L:k')$ is called an analytic functional with carrier in L and of type k'. $T\in Q'(L:k')$ can be represented by measure. Namely, we have

PROPOSITION 1 ([7]). *Let $T\in Q'(L:k')$. For all $\varepsilon>0$, $\varepsilon'>0$, there exists a measure μ on L_ε such that $\langle T,h\rangle=\int h(t)\exp(k't+\varepsilon'|t|/2)d\mu(t)$, for $h\in Q_b\ (L_\varepsilon:k'+\varepsilon')$.*

Fourier-Borel transform $\tilde{T}(z)$ of $T\in Q'(L:k')$ is defined as follows:

$$\tilde{T}(z)=<T_t,\exp(\langle t,z\rangle)>\,.$$

Following Paley-Wiener type theorem characterizes Fourier-Borel transform of $Q'(L:k')$.

THEOREM 2 ([7]). *Suppose that $T\in Q'(L:k')$. Then $\tilde{T}(z)$ is holomorphic function in $D=\prod_{i=1}^n\{z_i\in C:\text{Re}\,z_i<-k'\}$ and satisfies following estimate: For any $\varepsilon>0$ and $\varepsilon'>0$, there exists $C_{\varepsilon,\varepsilon'}$ such that*

$$(*)\qquad |\tilde{T}(z)|\leq C_{\varepsilon,\varepsilon'}\exp(H_L(z)+\varepsilon|z|)\qquad (\text{Re}\,z_i\leq -k'-\varepsilon',\ 1\leq i\leq n)\,.$$

Conversely, if holomorphic function $F(z)$ in D satisfies above estimate $(*)$, then there exists $T\in Q'(L:k')$ such that $\tilde{T}(z)=F(z)$.

4. Transformation $\tilde{G}$.

In this section we put following assumptions (i) and (ii).

(i) $0\leq k'<1$ and

(ii) For some positive constants a_i and b_i, L_i is contained in

$$U \cap \{\zeta \in \boldsymbol{C} : \mathrm{Re}\,\zeta \geq a_i, |\mathrm{Im}\,\zeta| \leq b_i < \pi\} \qquad \text{for} \quad i=1, \cdots, n .$$

We put

$$\tilde{G}_L(T)(w) = \left\langle T_t, \prod_{i=1}^{n} (1 - w_i t_i \exp(t_i))^{-1} \right\rangle$$

for $w=(w_1, \cdots, w_n) \in \prod_{i=1}^{n} (\boldsymbol{C}/\overline{\Phi(L_i)})$. $\tilde{G}_L(T)(w)$ has following properties.

PROPOSITION 3. (1) *$\tilde{G}_L(T)(w)$ is holomorphic in $\Pi_{i=1}^{n}(\boldsymbol{C}/\overline{\Phi(L_i)})$ and vanishes at the infinity.*

(2) *Following expansion holds:*

$$\tilde{G}_L(T)(w) = (-1)^n \sum_{m \in \boldsymbol{N}^n} D^{-m} \tilde{T}(-m) w^{-m} \qquad (|w| \gg 1) .$$

(3) *For any $\varepsilon > 0$ and $\varepsilon' > 0$, there exists $C_{\varepsilon,\varepsilon'} > 0$ such that*

$$|\tilde{G}_L(T)(w)| \leq C_{\varepsilon,\varepsilon'} |w|^{-k'-\varepsilon'} \qquad (b + \varepsilon \leq |\arg w| \leq \pi) .$$

(4) (Inversion formula)

$$\tilde{T}(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \tilde{G}_L(T)(w) \exp(z \Psi(w)) dw/w$$

where $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and Γ_j is a contour surrounding $\Phi(L_j)$.

PROOF. (1) can be proved by proposition 1, Morera's theorem and Fubini's theorem.

(2)
$$(1 - wt \exp(t))^{-1} = -\sum_{m=1}^{\infty} w^{-m} t^{-m} \exp(-mt) ,$$

$$t^{-n} \exp(-nt) = \frac{1}{(n-1)!} \int_0^\infty \exp(-t(a+n)) a^{n-1} da .$$

From these equalities we obtain

$$\tilde{G}_L(T)(\mathrm{w}) = (-1)^n \sum_{m \in \boldsymbol{N}^n} D^{-m} \tilde{T}(-m) w^{-m} ,$$

for sufficiently large w.

(3) Our proof is almost similar to that of proposition 3 in [5]. By the definition of toplogy of $Q(L:k')$, we have

$$|\tilde{G}_L(T)(w)| \leq C_{\varepsilon,\varepsilon'} \sup_{t \in L_\varepsilon} \left| \exp((k'+\varepsilon')t) \prod_{i=1}^{n} (1 - w_i t_i \exp(t_i))^{-1} \right|$$

$$\cdot |\exp((k'+\varepsilon')t)(1-wte^t)^{-1}|$$

$$= |te^t|^{-1+k'+\varepsilon'} |t|^{-k'-\varepsilon'} |w - t^{-1}e^{-t}|^{-k'-\varepsilon'} |w - t^{-1}e^{-t}|^{-1+k'+\varepsilon'}$$

$$\cdot |w - t^{-1}e^{-t}| \geq |w| \sin \varepsilon ,$$

$$|te^t|^{-1-k'+\varepsilon'} |w - t^{-1}e^{-t}|^{-1+k'+\varepsilon'} \leq (\sin \varepsilon)^{-1+k'+\varepsilon'}$$

for w such that $b+\varepsilon \leq |\arg w| \leq \pi$. From these estimate, we obtain our desired estimate (cf. [6]).

(4) By virture of (3) in this proposition and proposition 1, we can change the order of integration. After change of variable $t = \Psi(w)$, we obtain inversion formula by residue theorem.

REMARK. If L is a compact convex set satisfying assumption (ii) in this section, then inversion formula (4) is equals to Supper's formula. By Lagrange-Bruman formula for inverse function ([6]), we have

$$\Psi(w) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} w^{-n} \qquad (|w| > e) .$$

Hence $\lim_{|w| \to \infty} \Psi(w) = 0$. So our inversion formula is equals to Supper's one by Cauchy's integral theorem.

5. Relations between transformations.

Suppose that $L = \prod_{i=1}^{n} L_i$ and L satisfies condition (ii) in sec. 4. For $F \in \mathrm{Exp}(D:L)$, we define $M^{-1}(F)(w)$ as follows:

$$(M^{-1}F)(z) = (2i)^{-n} \int_{\infty}^{(0+)} (\sin(\pi z)\Gamma(z))^{-1} w^{-z} \int_{0}^{\infty} F(-z-a) a^{z-1} da\, dz .$$

Integral signs denote n-fold integrals.

$\mathcal{O}_0(\prod_{i=1}^{n}(C/\overline{\Phi(L_i)}))$ denotes the space of all holomorphic functions which satisfies (1), (2) and (3) in proposition 3. For $g \in \mathcal{O}_0(\prod_{i=1}^{n}(C/\overline{\Phi(L_i)}))$, we put $M(g)(z)$ as follows:

$$M(g)(z) = (2\pi i)^{-n} \int_{\Gamma} g(w) \exp(z\Psi(w)) dw/w ,$$

where Γ is the same countour as in (4) in proposition 3.

We have following commutative diagram:

$$\begin{array}{ccc} \mathrm{Exp}(D:L) & & \\ \mathrm{FB}\uparrow & \underset{M}{\nwarrow}\;\overset{M^{-1}}{\searrow} & \\ Q'(L:k) & \xrightarrow[\tilde{G}]{\qquad} & \mathcal{O}_0\Big(\prod_{i=1}^{n}(C/\overline{\Phi(L_i)})\Big) \end{array}$$

where FB denotes the Fourier-Borel transform.

6. Applications.

In this section we will show some applications. In what follows we assume (i) and (ii) in Sec. 4.

THEOREM 4. *Suppose that* $F(z)\in \mathrm{Exp}(D:L)$. *If* $D^{-m}F(-m)=0$ $(m\in N^n)$, *then* F *vanishes identically.*

PROOF. By theorem 2, there exists $T\in Q'(L:k')$ such that $\tilde{T}(z)=F(z)$. From (2) in proposition 3 and the assumption $D^{-m}F(-m)=0$, $\tilde{G}_L(T)(w)=0$. By Inversion formula (4) in proposition 3, $F(z)=0$.

REMARK. In theorem 4, condition (ii) in section 4 is crucial. $F(z)=\exp(sz)-\exp(\bar{s}z)$ $(\mathrm{Re}\, s>0$ and $s\in\partial U)$ satisfies all conditions in theorem 4. But $F(z)$ does not vanish.

COROLLARY 5. *Suppose that* $F\in\mathrm{Exp}(D:L)$ *and satisfies following conditions*:

$$D^{-i}F(-i)\in \boldsymbol{Z}\,, \qquad (i\in N^n)\,.$$

$$D^{-i-j}F(-i-j)=D^{-i}F(-i)D^{-j}F(-j)\,, \qquad (i,j\in N^n)\,.$$

If $a>\Psi(1)=0.567\cdots$, *then* F *vanishes identically.*

REMARK. $\Psi(1)$ is crucial. $F(z)=\exp(z\Psi(1))$ satisfies above condition. But it does not vanish.

THEOREM 6. *Suppose that* $F\in\mathrm{Exp}(D:L)$ *satisfies following conditions*:
(iii) $D^{-m}F(-m)\in\mathcal{O}_k$ $(m\in N^n)$.
(iv) $\limsup_{|m|\to\infty}\frac{1}{|m|}\log|\overline{D^{-m}F(-m)}|\le c$, (*c is a positive constant*).

If $\log(d(\overline{\Phi(L_i)}))<-(\delta-1)c$ *for* $i=1,\cdots,n$, *then* $F(z)$ *is exponential polynomial.*

PROOF. Put $a_n=D^{-n}F(-n)$, $S_i=\overline{\Phi(L_i)}$ ($\overline{\ \ }$ denotes the closure),

$$S_i^{(j)}=\begin{cases} S_i & \text{if } K=K^{(j)} \\ \bar{S}_i & \text{if } \bar{K}=K^{(j)} \\ \{w\in \boldsymbol{C}:|w|\le\Phi(c)\} & \text{other case,} \end{cases}$$

$$f^{(j)}(w)=\sum_{m\in N^n} a_m^{(j)} w^{-m}.$$

Then $f^{(j)}$ and $S_i^{(j)}$ satisfies all conditions in theorem 1 in [2]. So $\tilde{G}_L(T)(w)$ is rational function as follows:

$$\tilde{G}_L(T)(w)=P(w_1, \cdots, w_n)\Big/\prod_{i=1}^{n} Q_i(w_i)$$

where $P\in\mathcal{O}_k[w_1, \cdots, w_n]$, $Q_i(w_i)\in\mathcal{O}_k[w_i]$ and $\deg_{w_i}P<\deg Q_i$. Furthermore $Q_i(w_i)$ are monic (i.e. coefficient of highest degree term is unit). By virture of inversion formula (4) in proposition 3, we have our desired result.

COROLLARY 7. *Put* $L=\prod_{i=1}^{n}[a_i, \infty)$. *Suppose that* $F(z)\in \mathrm{Exp}(D:L)$ *and* D^{-m}-$F(-m)\in \boldsymbol{Z}$ *for* $m\in \boldsymbol{N}^n$. r *is a real number such that* $r\exp(r)=4^{-1}$ $(r=0.204\cdots)$. *If* $a_i>r$ *for* $i=1, \cdots, n$, *then* $F(z)$ *is exponential polynomial as follows*:

$$F(z)=\sum P_{j_1\cdots j_n}(z_1, \cdots, z_n)(\exp(z_1\Psi(b_{1,j_1})+\cdots+z_2\Psi(b_{n,j_n}))$$

where b_{k,j_k}*'s are algebraic integers contained in* $[0, \Phi(a_i)]$ *together with their conjugates.*

PROOF. In this corollary, $\delta=1$ and $d(\Phi(L_i))=(a_i\exp(a_i))^{-1}/4$ ([6]). By assumption on a_i, $d(\overline{\Phi(L_i)})$ is less than 1. Hence we can apply theorem 6 with arbitrary positive number c.

Now we define real numbers r_k's as follows:

$$r_k\exp(r_k)=k^{-1} \qquad (k=1, 2),$$
$$r_1=0.567\cdots, \qquad r_2=0.35\cdots,$$
$$r_3\exp(r_3)=(3+\sqrt{5})/2, \qquad r_0\exp(r_0)=(3-\sqrt{5})/2,$$
$$r_3=0.9814\cdots, \qquad r_0=0.286\cdots.$$

COROLLARY 8. *Put same assumptions in corollary* 7. If $a_i>r_1$ *for* $i=1, \cdots, n$, *then* F *vanishes identically.*

COROLLARY 9. *Put same assumptions in corollary* 6 *and* $n=1$. *If* $a>r_3$, *then* $F(z)=P_0(z)\exp(r_0z)+P_1(z)\exp(r_1z)+P_2(z)\exp(r_2z)$, *where* P_i*'s are polynomials in* $\boldsymbol{Q}[z]$.

PROOF. All algebraic integers contained in $[0, (3+\sqrt{5})/2]$ with their conjugates are 0,1,2,3, $(3+\sqrt{5})/2$, $(3-\sqrt{5})/2$ ([3]). So, we obtain corollary 8 and 9.

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Present Address:
DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO, 102 JAPAN.