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Differential Forms on Ringed Spaces of Valuation Rings

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In memory of Professor Yukiyosi Kawada

Introduction.

Given a field K and a subring A of K, we consider the set of valuation rings of K which contain A. This set has a structure of a local ringed space, denoted by $\mathrm{Zar}(K|A)$ (see [6] or [7]).

In this paper, we shall show that normal integral schemes $(X, \mathcal{O}_X)$ proper over $\mathrm{Spec}\,A$ with rational function field K are quotient spaces of $\mathrm{Zar}(K|A)$ and $\mathcal{O}_X=\Phi_{X*}\mathcal{O}_Z$. Here $\mathcal{O}_Z$ is the structure sheaf of $Z=\mathrm{Zar}(K|A)$ and $\Phi_X: Z\to X$ is the quotient mapping. In order to show this, we introduce a category $\mathscr{C}_0(K|A)$ of local ringed spaces, which contains both $\mathrm{Zar}(K|A)$ and all integral schemes proper over $\mathrm{Spec}\,A$ with rational function field K (see Theorems 1 and 1′).

For objects X of $\mathscr{C}_0(K|A)$, we introduce sheaves Ω_X^m of differential forms as in the case of schemes over $\mathrm{Spec}\,A$. In particular if A is a perfect field and X is a regular scheme, then Ω_X^m coincides with the ordinary sheaf of regular differential forms and $\Omega_X^m=\Phi_{X*}\Omega_Z^m$ for any multi-index m (see Theorem 2). From this, the birational invariance of regular differential forms of regular varieties follows immediately.

To define structure sheaves on quotient spaces of $\mathrm{Zar}(K|A)$ and sheaves Ω_X^m on objects X of $\mathscr{C}_0(K|A)$ in a unified way, we shall introduce the notion of intersection sheaf in §0.

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§0. Let $\mathscr{A}$ be the category of A-modules or the category of A-rings, where A is a commutative ring with unity. For an object N of $\mathscr{A}$, we denote by $\mathrm{Sub}_{\mathscr{A}}(N)$ the totality of subobjects of N. For a subset E of N, we put

$$\mathrm{Sub}_{\mathscr{A}}(N|E)=\{M\in\mathrm{Sub}_{\mathscr{A}}(N) \mid E\subset M\}\,.$$

Let $(E_i)_{i\in I}$ be a family of subsets of N. Then

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(1) $$\mathrm{Sub}_{\mathscr{A}}\left(N \middle| \bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} \mathrm{Sub}_{\mathscr{A}}(N|E_i)\,,$$

(2) $$\mathrm{Sub}_{\mathscr{A}}\left(N \middle| \bigcap_{i\in I} E_i\right) \supset \bigcup_{i\in I} \mathrm{Sub}_{\mathscr{A}}(N|E_i)\,.$$

There exists a unique topology on $\mathrm{Sub}_{\mathscr{A}}(N)$ with open base

(3) $$\Sigma = \{\mathrm{Sub}_{\mathscr{A}}(N|E) \mid E \text{ is a finite subset of } N\}\,.$$

In what follows, given a subset X of $\mathrm{Sub}_{\mathscr{A}}(N)$, we consider the relative topology, which is called the Zariski topology on X. Then for $M\in X$, we obtain

(4) $$\overline{\{M\}} = \{M'\in X \mid M'\subset M\}\,.$$

Hence X is a T_0-space. Let Y be an irreducible closed subset of X and put $\xi_Y = \bigcup_{M\in Y} M$. Then we obtain

(5) $$Y = \{M\in X \mid M\subset \xi_Y\}\,.$$

Thus Y has a generic point if and only if $\xi_Y\in X$. In this case ξ_Y is the unique generic point of Y.

Let X be a topological space and let $s\colon X\to\mathrm{Sub}_{\mathscr{A}}(N)$ be a mapping (not necessary continuous). For any open subset V of X, we put

(6) $$\mathscr{F}(V) = \begin{cases} \bigcap_{x\in V} s(x)\,, & \text{where } V\neq\varnothing\,, \\ 0 & \text{where } V=\varnothing\,. \end{cases}$$

Let U and V be open subsets of X such that $U\subset V$. If $U\neq\varnothing$, then $\mathscr{F}(V)\subset\mathscr{F}(U)\subset N$, and we denote by $\rho_{V,U}$ the inclusion mapping. If $U=\varnothing$, we define $\rho_{V,U}$ to be the 0-mapping. Then it is clear that $\mathscr{F}$ is an $\mathscr{A}$-valued presheaf on X. For a family $(V_i)_{i\in I}$ of non empty open subsets of X, we have

(7) $$\mathscr{F}\left(\bigcup_{i\in I} V_i\right) = \bigcap_{i\in I} \mathscr{F}(V_i)\,.$$

LEMMA 1. *Let X, $\mathscr{A}$, N and s be as above. Then*

(i) *the preshearf $\mathscr{F}$ defined by* (6) *satisfies the local uniqueness conditions.*

(ii) *For any non empty subset E of X, we have*

$$\text{ind.lim}\,\mathscr{F}(V) \simeq \bigcup \mathscr{F}(V) \subset \bigcap_{x\in E} s(x)\,,$$

where V runs over all open subsets of X containing E. Especially if we put $E=\{x\}$, then

$$\mathscr{F}_x \simeq \bigcup_V \mathscr{F}(V) \subset s(x)\,.$$

(iii) *If X is either irreducible or empty, then $\mathscr{F}$ satisfies the local existence conditions. Especially if* $\operatorname{card}\mathscr{F}(X)\geq 2$, *then the converse holds.*

(iv) *The following three conditions are equivalent*:

(a) *The mapping s is continuous.*

(b) $\bigcup_{\substack{V:\text{open}\\ V\ni x}} \mathscr{F}(V)=s(x)$ *for any* $x\in X$.

(c) $\bigcup_{\substack{V:\text{open}\\ V\supset E}} \mathscr{F}(V)=\bigcap_{x\in E} s(x)$ *for any non empty subset E of X.*

The proof is similar to that of Lemma 1 in [6].

COROLLARY. *Suppose that X is irreducible and that s is continuous. Then $\mathscr{F}$ is an $\mathscr{A}$-valued sheaf on X and $\mathscr{F}_x=s(x)$ for any $x\in X$.*

Thus we also have

$$\mathscr{F}(V)=\bigcap_{x\in V}\mathscr{F}_x, \tag{8}$$

for any non empty open subset V of X.

The sheaf $\mathscr{F}$ is said to be the intersection sheaf of X with respect to the mapping s.

LEMMA 2. *Let $\mathscr{A}$ and N be as above. Let Y and X be irreducible topological spaces and let*

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & X \\ & {\scriptstyle s_Y}\searrow & \downarrow{\scriptstyle s_X} \\ & & \operatorname{Sub}_{\mathscr{A}}(N) \end{array} \tag{9}$$

be a diagram in topological spaces (not necessarily commutative). Assume that $\mathscr{F}_Y$ (resp. $\mathscr{F}_X$) is the intersection sheaf of Y (resp. X) with respect to s_Y (resp. s_X).

(i) *If $s_X(f(y))\subset s_Y(y)$ holds for any $y\in Y$, then there exists a morphism $f^{\#}: \mathscr{F}_X\to f_*\mathscr{F}_Y$ of sheaves on X such that $f^{\#}(V)$: $\mathscr{F}_X(V)\to\mathscr{F}_Y(f^{-1}(V))$ is the inclusion mapping for any non empty open subset V of X. Then $f^{\#}_y$: $\mathscr{F}_{X,f(y)}\to\mathscr{F}_{Y,y}$ is also the inclusion mapping for any $y\in Y$.*

(ii) *If the triangle (9) commutes, then we have $\mathscr{F}_Y=f^{-1}\mathscr{F}_X$.*

The proof is easy.

§1. Let K be a field and A a subring of K. We denote by $\operatorname{Loc}(K|A)$ the set of local subrings R of K which contain A. We do not assume $QR=K$, where QR is the quotient field of R. By Lemma 1, we have the local ringed space $L=\operatorname{Loc}(K|A)$ consisting of the Zariski topology and the intersection sheaf $\mathcal{O}_L$ with respect to the inclusion mapping:

$$\mathrm{Loc}(K|A) \hookrightarrow \mathrm{Sub}_{(A\text{-rings})}(K)\,.$$

Next we consider the mapping $\pi_{K|A}\colon \mathrm{Loc}(K|A) \to \mathrm{Spec}\,A$ defined by

$$\pi_{K|A}(R) = A \cap m(R)\,, \qquad \text{for} \quad R \in \mathrm{Loc}(K|A)\,, \tag{10}$$

where $m(R)$ is the unique maximal ideal of R. Then $\pi_{K|A}$ is surjective and continuous. Moreover, we have

$$\tilde{A} = (\pi_{K|A})_* \mathcal{O}_L\,. \tag{11}$$

Letting $\pi^\#_{K|A}$ be the natural isomorphism in (11), we obtain a morphism $(\pi_{K|A}, \pi^\#_{K|A})$ of local ringed spaces. By restriction, we also obtain a local ringed space $\mathrm{Zar}(K|A)$ and a morphism

$$\Phi_{K|A}\colon \quad \mathrm{Zar}(K|A) \to \mathrm{Spec}\,A \tag{10'}$$

of local ringed spaces. Then $\Phi_{K|A}$ is a surjective and closed mapping (see Theorem 2.5 in [1] or Lemma 4 (p. 117) in [8]).

For a local ringed space $(X, \mathcal{O}_X)$, we define a mapping $\pi_X\colon X \to \mathrm{Spec}\,\mathcal{O}_X(X)$ by $x \mapsto \rho^{-1}_{X,x}(m(\mathcal{O}_{X,x}))$, where $\rho_{X,x}\colon \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ are the canonical mappings. Then we consider the following condition for a local ringed space $(X, \mathcal{O}_X)$:

(12) X has an open base consisting of open sets V such that π_V is dominant.

This is the condition (8) in [7] (see also [7], sec. 2).

Both the local ringed spaces $\mathrm{Loc}(K|A)$ and $\mathrm{Zar}(K|A)$ satisfy the condition (12). We remark that $\pi_L = \pi_{K|A}$ for $L = \mathrm{Loc}(K|A)$.

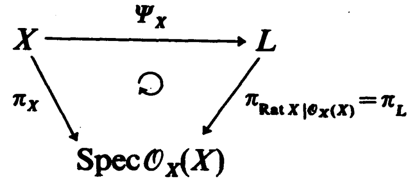
LEMMA 3. *Let $(X, \mathcal{O}_X)$ be an integral local ringed space satisfying the condition* (12) *and let $L = \mathrm{Loc}(\mathrm{Rat}\,X|\mathcal{O}_X(X))$. Then*

(i) *the mapping $\Psi_X\colon X \to L$ defined by*

$$\Psi_X(x) = \mathcal{O}_{X,x}\,, \qquad \textit{for} \quad x \in X \tag{13}$$

is continuous, dominant and $\mathcal{O}_X = \Psi_X^{-1}\mathcal{O}_L$ is the intersection sheaf of X with respect to Ψ_X.

(ii) *There exists a morphism $\Psi^\#_X\colon \mathcal{O}_L \to \Psi_{X*}\mathcal{O}_X$ of sheaves on L such that $(\Psi_X, \Psi^\#_X)\colon (X, \mathcal{O}_X) \to (L, \mathcal{O}_L)$ is a morphism of local ringed spaces over* $\mathrm{Spec}\,\mathcal{O}_X(X)$:



The proof is easy from Lemma 7 in [7], Lemma 1, (iv) and Lemma 2.

COROLLARY. *For a topological space X, the following two conditions are equivalent*:

(a) *There exists a sheaf $\mathcal{O}_X$ of rings over X such that $(X, \mathcal{O}_X)$ is an integral local ringed space satisfying the condition* (12).

(b) *X is irreducible, and there exist a field K and a dominant continuous mapping $s: X \to \mathrm{Loc}\,K$.*

For an integral domain A, we also write

$$\Psi_A = \Psi_{\mathrm{Spec}\,A}: \quad \mathrm{Spec}\,A \to \mathrm{Loc}(QA|A)\,. \tag{13'}$$

We consider the following condition for a local ringed space $(X, \mathcal{O}_X)$:

(14) For any $x \in X$, there exists a morphism of local ringed spaces $(j_x, j_x^{\#})$: $\mathrm{Spec}\,\mathcal{O}_{X,x} \to X$ such that $\mathrm{Im}(j_x) = \{y \in X \mid x \in \overline{\{y\}}\}$ and $(j_x^{\#})_P: \mathcal{O}_{X,j_x(P)} = (\mathcal{O}_{X,x})_P$ for any $P \in \mathrm{Spec}\,\mathcal{O}_{X,x}$.

This is the condition (21) in [7] (see also Lemmas 8 and 9 in [7]).

In general, the local ringed space $\mathrm{Loc}(K|A)$ does not satisfy the condition (14), but $\mathrm{Zar}(K|A)$ satisfies (14).

Let $(X, \mathcal{O}_X)$ be an integral local ringed space satisfying the conditions (12) and (14). Then the triangle:

$$\begin{array}{ccc} X & \xrightarrow{\quad \Psi_X \quad} & \mathrm{Loc}(\mathrm{Rat}\,X|\mathcal{O}_X(X)) \\ & {\scriptstyle j_x}\nwarrow \qquad \nearrow {\scriptstyle \Psi_{\mathcal{O}_{X,x}}} & \\ & \mathrm{Spec}\,\mathcal{O}_{X,x} & \end{array}$$

commutes for any $x \in X$. Therefore j_x is an into-homeomorphism for any $x \in X$, because $\Psi_{\mathcal{O}_{X,x}}$ is an into-homeomorphism.

§2. Let K be a field and A a subring of K. Then the category $\mathscr{C}(K|A)$ was defined in [7] as follows.

The objects $(X, \mathcal{O}_X)$ satisfy the following conditions:

(15) X is a topological space with a generic point and satisfies the separable condition T_0.

(16) $(X, \mathcal{O}_X)$ is an integral local ringed space satisfying the conditions (12) and (14).

(17) $K = \mathrm{Rat}(X, \mathcal{O}_X)$.

(18) $A \subset \mathcal{O}_X(X)$ and the morphism $X \to \mathrm{Spec}\,A$ is valuative-proper.

The morphisms (f, θ) satisfy the condition:

(19) (f, θ) is a dominant morphism of local ringed spaces over $\operatorname{Spec} A$ and $\operatorname{Rat}(f, \theta)$ is the identity mapping of K.

Let $(X, \mathcal{O}_X)$ be an object of $\mathscr{C}(K|A)$. Then the mapping $\Phi_X: \operatorname{Zar}(K|A) \to X$ is defined by

(20) $$\Phi_X(R) = x \quad \text{if and only if} \quad R \text{ dominates } \mathcal{O}_{X,x}$$

for any $R \in \operatorname{Zar}(K|A)$ and $x \in X$ (see also Lemma 16 in [7]).

Since the triangle

(21) $$\begin{array}{ccc} \operatorname{Zar}(K|\mathcal{O}_{X,x}) & \xrightarrow{\Phi_X} & X \\ & \searrow^{\Phi_{K|\mathcal{O}_{X,x}}} & \nearrow_{j_x} \\ & \operatorname{Spec} \mathcal{O}_{X,x} & \end{array}$$

commutes for any $x \in X$, we obtain:

(22) $$\Phi_X^{-1}(x) = \Phi_{K|\mathcal{O}_{X,x}}^{-1}(m(\mathcal{O}_{X,x})), \qquad \text{for any } x \in X.$$

(23) $$\Phi_X(\operatorname{Zar}(K|\mathcal{O}_{X,x})) = \operatorname{Im}(j_x), \qquad \text{for any } x \in X.$$

(24) $$x \in \overline{\{y\}} \Leftrightarrow \mathcal{O}_{X,x} \subset \mathcal{O}_{X,y} \Leftrightarrow \mathcal{O}_{X,y} = (\mathcal{O}_{X,x})_P \text{ for some } P \in \operatorname{Spec} \mathcal{O}_{X,x}, \quad \text{for any } x, y \in X.$$

(25) If $\mathcal{O}_{X,y}$ dominates $\mathcal{O}_{X,x}$, then $x = y$, for any $x, y \in X$.

(26) $$X \text{ is normal} \Leftrightarrow \mathcal{O}_X = \Phi_{X*}\mathcal{O}_Z \Leftrightarrow \mathcal{O}_{X,x} = \bigcap_{R \in \Phi_X^{-1}(x)} R \quad \text{for any } x \in X.$$

DEFINITION. Let K be a field and A a subring of K. We denote by $\mathscr{C}_0(K|A)$ the full subcategory of $\mathscr{C}(K|A)$ consisting of all objects $(X, \mathcal{O}_X)$ such that

(27) $$\Phi_X: \operatorname{Zar}(K|A) \to X \quad \text{is a quotient mapping}.$$

It is clear that $\operatorname{Zar}(K|A)$ is the initial object of $\mathscr{C}_0(K|A)$ and all the morphisms of $\mathscr{C}_0(K|A)$ are quotient mappings.

LEMMA 4. *Let K, A and $\mathscr{C}_0(K|A)$ be as above.*

(i) *Let X be a normal object of $\mathscr{C}_0(K|A)$. Then the equivalence relation $\sim$ of $\operatorname{Zar}(K|A)$ associated to the mapping Φ_X satisfies the following four conditions*:

(28) *For any $R \in \operatorname{Zar}(K|A)$, the ring $\bigcap_{R' \sim R} R'$ is a local ring with quotient field K and*

$$m\Big(\bigcap_{R' \sim R} R'\Big) = \bigcap_{R' \sim R} R' \cap m(R).$$

(29) *For any* $\alpha \in K$, *the set* $\{R \in \mathrm{Zar}(K|A) \mid R' \sim R \Rightarrow \alpha \in R'\}$ *is open in* $\mathrm{Zar}(K|A)$.

(30) $R_1 \sim R_2$ *if and only if* $\bigcap_{R\sim R_1} R = \bigcap_{R\sim R_2} R$, *for any* $R_1, R_2 \in \mathrm{Zar}(K|A)$.

(31)
$$\bigcap_{R\sim R_1} R \subset R_2 \Leftrightarrow \bigcap_{R\sim R_1} R \subset \bigcap_{R\sim R_2} R$$
$$\Leftrightarrow \bigcap_{R\sim R_2} R = \Big(\bigcap_{R\sim R_1} R\Big)_P \quad \textit{for some } P \in \mathrm{Spec}\Big(\bigcap_{R\sim R_1} R\Big)$$
$$\Leftrightarrow p(R_1) \in \overline{\{p(R_2)\}} \textit{ in } \mathrm{Zar}(K|A)_{/\sim},$$
for any $R_1, R_2 \in \mathrm{Zar}(K|A)$.

Here we denote by $p\colon \mathrm{Zar}(K|A) \to \mathrm{Zar}(K|A)_{/\sim}$ *the natural projection, and introduce the quotient topology for* $\mathrm{Zar}(K|A)_{/\sim}$.

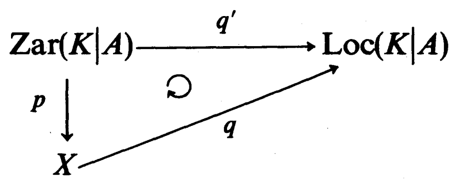
(ii) *Conversely, assume that an equivalence relation* $\sim$ *of* $\mathrm{Zar}(K|A)$ *satisfies the conditions* (28), (29), (30) *and* (31), *and let* $X = \mathrm{Zar}(K|A)_{/\sim}$. *Then there exists a sheaf* $\mathcal{O}_X$ *of rings on* X *with respect to the quotient topology such that* $(X, \mathcal{O}_X)$ *is a normal object in* $\mathscr{C}_0(K|A)$.

PROOF. (i) Let $R \in \mathrm{Zar}(K|A)$ and put $x = \Phi_X(R)$. Then by (26), we obtain

(26′)
$$\mathcal{O}_{X,x} = \bigcap_{R'\sim R} R' \subset R.$$

(28) is induced from Lemma 9 in [7], (20) and (26′). (29) is verified from (26′) and the continuity of Φ_X and Ψ_X. (30) follows immediately from (25) and (26′). (31) is proved by (23), (24) and (26′).

(ii) By (28) and (29), we can define a continuous mapping $q'\colon \mathrm{Zar}(K|A) \to \mathrm{Loc}(K|A)$ by $R \mapsto \bigcap_{R'\sim R} R'$. Then by (30), q' induces a continuous injection q such that



Let $\mathcal{O}_X$ be the intersection sheaf of X with respect to q. Then $(X, \mathcal{O}_X)$ becomes a normal object of $\mathscr{C}_0(K|A)$. Q.E.D.

We denote by $\mathscr{M}_0(K|A)$ the totality of equivalence relations $\sim$ on $Z = \mathrm{Zar}(K|A)$ satisfying the conditions (28), (29), (30) and (31). The set $\mathscr{M}_0(K|A)$ is abbreviated as $\mathscr{M}_0$. For relations $\sim_1, \sim_2 \in \mathscr{M}_0$, we define $\sim_1 \leq \sim_2$ if the following two conditions hold:

(32) If $R\sim_2 R'$, then $R\sim_1 R'$, for any $R, R'\in Z$,

(33) $$m\Big(\bigcap_{R'\sim_1 R} R'\Big)=\bigcap_{R'\sim_1 R} R'\cap m\Big(\bigcap_{R''\sim_2 R} R''\Big),\qquad \text{for any } R\in Z.$$

Then it is clear that $(\mathcal{M}_0, \leq)$ is an ordered set.

THEOREM 1. *Let K be a field and A a subring of K.*

(i) *Take a normal object X of* $\mathscr{C}_0(K|A)$. *If* $\sim\in\mathcal{M}_0$ *is the equivalence relation associated to* Φ_X, *then we obtain an isomorphism* $\mathrm{Zar}(K|A)_{/\sim}\simeq X$ of $\mathscr{C}_0(K|A)$.

(i′) *Take a relation* $\sim\in\mathcal{M}_0$. *Letting* $X=\mathrm{Zar}(K|A)_{/\sim}$, *we see that the equivalence relation associated to* Φ_X *coincides with the relation* $\sim$.

Therefore, there exists a bijection between the totality of isomorphic classes of normal objects in $\mathscr{C}_0(K|A)$ *and the set* $\mathcal{M}_0$.

(ii) *Suppose that* X_i *and* $\sim_i$ $(i=1, 2)$ *correspond to each other by the above mapping. Then there exists a morphism* $X_2\to X_1$ *of* $\mathscr{C}_0(K|A)$, *if and only if* $\sim_1\leq\sim_2$.

PROOF. (i) It is clear that the mapping $\mathrm{Zar}(K|A)_{/\sim}\to X$ defined by $p(R)\mapsto\Phi_X(R)$ is a homeomorphism. By (26′) and the definition of the intersection sheaves, this is an isomorphism of $\mathscr{C}_0(K|A)$. (i′) is induced from $p=\Phi_X$. (ii) is easy to verify.

COROLLARY. *Let* $\mathscr{C}_0^N(K|A)$ *be the full subcategory of* $\mathscr{C}_0(K|A)$ *consisting of all normal objects. Then we obtain an order-isomorphism:* $\mathscr{C}_0^N(K|A)_{/\simeq}\simeq\mathcal{M}_0$.

Next we consider the schemes. Assume that

(34) A is noetherian and K is a finitely generated extension over QA.

Then we denote by $\mathscr{C}_1(K|A)$ the category of integral schemes proper over $\mathrm{Spec}\,A$ with rational function field K (the morphisms are the same as in $\mathscr{C}_0(K|A)$). Replacing "proper" by "projective", we obtain the category $\mathscr{C}_2(K|A)$.

Since $\Phi_{K|B}$ defined by (10′) is a closed mapping for any subring B of K, Φ_X: $\mathrm{Zar}(K|A)\to X$ is also a closed mapping for any object X of $\mathscr{C}_1(K|A)$. Therefore $\mathscr{C}_1(K|A)$ and $\mathscr{C}_2(K|A)$ are the full subcategories of $\mathscr{C}_0(K|A)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}_0$ consisting of all relations $\sim\in\mathcal{M}_0$ satisfying the following condition:

(35) For any $R_0\in\mathrm{Zar}(K|A)$, there exists a ring B such that
- (i) $A\subset B\subset R_0$, B is of finite type over A.
- (ii) there exists $P\in\mathrm{Spec}\,B$ such that $B_P=\bigcap_{R'\sim R}R'$, if and only if $B\subset R$, for any $R\in\mathrm{Zar}(K|A)$.
- (iii) $R_1\sim R_2$ if and only if $B\cap m(R_1)=B\cap m(R_2)$, for any R_1, $R_2\in\mathrm{Zar}(K|B)$.

THEOREM 1′. *Supposing that K and A satisfy* (34), *we assume that a normal object*

$(X, \mathcal{O}_X)$ in $\mathscr{C}_0(K|A)$ and a relation $\sim\in\mathscr{M}_0$ correspond to each other by the mapping in Theorem 1. *Then X is an object in $\mathscr{C}_1(K|A)$ if and only if $\sim\in\mathscr{M}_1$.*

PROOF. For any $x_0\in X$, there exists $R_0\in \mathrm{Zar}(K|A)$ such that $x_0=\Phi_X(R_0)$. Let B be as in (35), and put $V=\Psi_X^{-1}(\mathrm{Im}\,\Psi_B)$. Then V is an affine open neighborhood of x_0 in X. Conversely, for any $R_0\in \mathrm{Zar}(K|A)$, we put $x_0=\Phi_X(R_0)$. Let V be an affine open neighborhood of x_0 in X, and put $B=\mathcal{O}_X(V)$. Then B satisfies (35). Q.E.D.

COROLLARY. *There exists an order-isomorphism*:

$$\mathscr{C}_1^N(K|A)_{/\simeq} \;\simeq\; \mathscr{M}_1 .$$

§3. Let K be a field, A a subring of K and $n=\dim \mathrm{Zar}(K|A)\geq 1$. Then Y is said to be a prime divisor of $\mathrm{Zar}(K|A)$, if Y is an irreducible closed subset of $\mathrm{Zar}(K|A)$ and $\dim Y=n-1$. It is also called a prime divisor of $K|A$.

If Y is a prime divisor of $K|A$, then by (4) and (5), there exists a unique $R\in \mathrm{Zar}(K|A)$ such that $Y=\overline{\{R\}}$. This R is also called a prime divisor of $K|A$. If necessary, we call R a Zariski prime divisor (see also [8], sec. 14).

Let X be an object of $\mathscr{C}_0(K|A)$. Then R is said to be a (Zariski) prime divisor of X, if R is a prime divisor of $K|A$ and there exists an element x of X such that $R=\mathcal{O}_{X,x}$.

We denote by $N(X)$ the totality of prime divisors R of X and K. The free abelian group $\mathrm{Div}\,X$ generated by the prime divisors of X is said to be the divisor group of X. This is an ordered abelian group. If $X=\mathrm{Zar}(K|A)$, then we also write $N(X)=N(K|A)$ and $\mathrm{Div}\,X=\mathrm{Div}(K|A)$.

Let $f: Y\to X$ be a morphism of $\mathscr{C}_0(K|A)$. Then we have

(36) $$N(X)\subset N(Y)\subset N(K|A)\,,$$

(37) $$\mathrm{Div}\,X\subset \mathrm{Div}\,Y\subset \mathrm{Div}(K|A)\,.$$

Therefore, we obtain a contravariant functor

$$\mathrm{Div}:\quad \mathscr{C}_0(K|A)\to(\mathrm{Mod})\,.$$

Next we assume that

(34′) $$\begin{array}{l}A \text{ is of finite type over a subfield } k \text{ and}\\ K \text{ is finitely generated over } QA\,.\end{array}$$

Then we have

(38) $$N(K|A)=N(K|k)\cap \mathrm{Zar}(K|A)\,.$$

LEMMA 5. *Suppose that K and A satisfy* (34′). *Then for any $R_1,\cdots,R_q\in N(K|A)$, there exists an integrally closed integral domain B such that $K=QB$, $A\subset B\subset\bigcap_{i=1}^q R_i$, B is of finite type over A and $R_i=B_{P_i}$ for some $P_i\in \mathrm{Spec}\,B$ $(1\leq i\leq q)$.*

The proof is similar to that of Theorem 31 in [8].

COROLLARY. *Let $\mathscr{C}$ be a subcategory of $\mathscr{C}_1(K|A)$ satisfying the condition:*

(39) *If a ring B is an intermediate ring between A and K such that $QB=K$ and B is of finite type over A, then there exists an object X of $\mathscr{C}$ and an affine open subset V of X such that $B=\mathcal{O}_X(V)$.*

Then

(40) $$N(K|A)=\bigcup N(X),$$

(41) $$\mathrm{Div}(K|A)=\mathrm{ind.lim}\,\mathrm{Div}\,X,$$

where X runs over all the objects in $\mathscr{C}$.

Note that $\mathscr{C}_1(K|A)$ and $\mathscr{C}_2(K|A)$ satisfy the condition (39). See also Theorem 2 in [7].

Let K be a field, A a subring of K and $N=N(K|A)$. Then N is irreducible, because N has the generic point K. Therefore N becomes a local ringed space with the Zariski topology and the intersection sheaf $\mathcal{O}_N$ with respect to the inclusion mapping:

$$N \hookrightarrow \mathrm{Sub}_{(A\text{-rings})}(K).$$

LEMMA 6. *Suppose that K and A satisfy* (34′). *Let $N=N(K|A)$ and take an object X from $\mathscr{C}_1(K|A)$. Then*

(i) *the restriction mapping $\Phi_X|_N\colon N\to X$ is surjective.*

(ii) *If we put $N_x=N\cap\Phi_X^{-1}(x)$ for any $x\in X$, then the integral closure of $\mathcal{O}_{X,x}$ in K is $\bigcap_{R\in N_x}R$.*

(iii) *If we put $Z=\mathrm{Zar}(K|A)$ and let $i\colon N\to Z$ be the inclusion mapping, then we obtain*

(42) $$\mathcal{O}_N=i^{-1}\mathcal{O}_Z=\mathcal{O}_Z|_N,$$

(43) $$\mathcal{O}_Z=i_*\mathcal{O}_N.$$

PROOF. (i) and (ii) are proved in a way similar to the proof of Theorem 35 in [8]. (iii) is verified from Lemma 2 and (ii).

COROLLARY. (i) *If $\dim\mathcal{O}_{X,x}=1$, then $1\le\mathrm{card}\,N_x\ne\infty$. Especially if x is a normal point of X, then $N_x=\{\mathcal{O}_{X,x}\}$ that is the one point set.*

(ii) *If $\dim\mathcal{O}_{X,x}\ge 2$, then there exists $\alpha\in K$ such that the set $\{R\in N_x\mid\alpha\notin R\}$ is infinite. Thus N_x is also infinite.*

PROOF. (i) is verified by using the similar method to the proof of Theorems 32 and 33 in [8]. (ii) is induced from Theorem 12.3 in [5].

§4. For a ring homomorphism $A\to B$, we denote by $\Omega_{B|A}$ the B-module of regular differential forms of B over A. For a positive integer r, we denote by $\Omega^r_{B|A}$ the r-th exterior power of $\Omega_{B|A}$ as B-modules. Moreover, for any multi-index $m=(m_1, \cdots, m_n)$ of non negative integers, we define

(44) $$\Omega^m_A(B)=(\Omega^1_{B|A})^{\otimes m_1}\otimes\cdots\otimes(\Omega^n_{B|A})^{\otimes m_n}\,.$$

Putting $|m|=m_1+2m_2+\cdots+nm_n$, we have an A-multilinear mapping $d^m_{B|A}: B^{|m|}\to \Omega^m_A(B)$ and

(45) $$\Omega^m_A(B)=\sum_{x_1,\cdots,x_{|m|}\in B} B\, d^m_{B|A}(x_1, \cdots, x_{|m|})\,.$$

For any homomorphism $\varphi: B_1\to B_2$ of A-rings, there exists a homomorphism $\Omega^m_A(\varphi)$ of B_1-modules such that

$$\begin{array}{ccc} B_1^{|m|} & \xrightarrow{\varphi^{|m|}} & B_2^{|m|} \\ {\scriptstyle d^m_{B_1|A}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle d^m_{B_2|A}} \\ \Omega^m_A(B_1) & \xrightarrow[\Omega^m_A(\varphi)]{} & \Omega^m_A(B_2) \end{array}\quad .$$

Thus a functor Ω^m_A: (A-rings)$\to$(Mod) is obtained.

For any multiplicative subset S of B, we have a commutative diagram of B-modules and an isomorphism of $S^{-1}B$-modules:

(46) $$\begin{array}{ccc} \Omega^m_A(B) & & \\ {\scriptstyle \text{cano}}\downarrow & \searrow{\scriptstyle \Omega^m_A(i)} & \\ S^{-1}\Omega^m_A(B) & \simeq & \Omega^m_A(S^{-1}B) \end{array}$$

where $i: B\to S^{-1}B$ is the canonical mapping. Especially if B is an integral domain, $K=QB$ and $i_{B|K}: B\to K$ is the inclusion mapping, then by (46),

(47) $\Omega^m_A(i_{B|K})$ is injective if and only if $\Omega^m_A(B)$ is a torsion free B-module.

LEMMA 7. *Let B be a ring, A a subring of B and m a multi-index. Then the mapping* $s^m_{B|A}: \mathrm{Sub}_{(\mathrm{Rings})}(B|A)\to\mathrm{Sub}_{(\mathrm{Mod})}(\Omega^m_A(B))$ *defined by*

$$s^m_{B|A}(R)=\mathrm{Im}\,\Omega^m_A(i_{R|B})\,, \qquad \textit{for} \quad R\in\mathrm{Sub}_{(\mathrm{Rings})}(B|A)$$

is continuous. Here let $i_{R|B}: R\to B$ denote the inclusion mapping.

PROOF. For any $\omega\in\Omega^m_A(B)$, we put $V=\mathrm{Sub}_{(\mathrm{Mod})}(\Omega^m_A(B)\mid\{\omega\})$. Then it suffices to prove that $s^m_{B|A}{}^{-1}(V)$ is open in $\mathrm{Sub}_{(\mathrm{Rings})}(B|A)$. For any $R\in s^m_{B|A}{}^{-1}(V)$, there exists $\omega_0\in\Omega^m_A(R)$ such that $\omega=\Omega^m_A(i_{R|B})(\omega_0)$. By (45), we can write

$$\omega_0=\sum_{i=1}^r y_i d^m_{R|A}(x_{i1}, \cdots, x_{i|m|})\,.$$

Put $E=\{y_1, \cdots, y_r, x_{11}, \cdots, x_{r|m|}\} \subset R$ and $U=\mathrm{Sub}_{(\mathrm{Rings})}(B|A[E])$. Then $R \in U \subset {s^m_{B|A}}^{-1}(V)$. Thus ${s^m_{B|A}}^{-1}(V)$ is an open set. Q.E.D.

Let K be a field, A a subring of K and m a multi-index. For any object X of $\mathscr{C}_0(K|A)$, we consider the mapping s^m_X: $X \to \mathrm{Sub}_{(\mathrm{Mod})}(\Omega^m_A(K))$ defined by

$$s^m_X(x) = \mathrm{Im}\,\Omega^m_A(i_x)\,, \qquad \text{for} \quad x \in X\,, \tag{48}$$

where i_x: $\mathcal{O}_{X,x} \to K$ is the inclusion mapping. By Lemmas 3 and 7, the mapping s^m_X is continuous. Therefore we obtain the intersection sheaf Ω^m_X of X with respect to s^m_X.

LEMMA 8. *Let K, A, m and X be as above. Then*

(i) *Ω^m_X is a sheaf of $\mathcal{O}_X$-modules.*

(ii) *For any $x \in X$, $\Omega^m_{X,x} \simeq \Omega^m_A(\mathcal{O}_{X,x})$ if and only if $\Omega^m_A(\mathcal{O}_{X,x})$ is a torsion free $\mathcal{O}_{X,x}$-module.*

PROOF. (i) is easy to prove. (ii) is induced from (47).

Let f: $Y \to X$ be a morphism of $\mathscr{C}_0(K|A)$. Then by Lemma 2, we have

$$\Omega^m_X(V) \subset \Omega^m_Y(f^{-1}(V))\,, \tag{49}$$

for any open set $V \neq \varnothing$. Thus we also have

$$\Omega^m_X(X) \subset \Omega^m_Y(Y) \subset \Omega^m_Z(Z) \subset \Omega^m_A(K)\,, \tag{49'}$$

where $Z = \mathrm{Zar}(K|A)$.

Let K be a field, A a subring of K, $Z = \mathrm{Zar}(K|A)$, $N = N(K|A)$, i: $N \to Z$ the inclusion mapping and m a multi-index. Let Ω^m_N be the intersection sheaf of N with respect to the mapping $s^m_Z \circ i$. Then Ω^m_N is a sheaf of $\mathcal{O}_N$-modules and $\Omega^m_N = i^{-1}\Omega^m_Z = \Omega^m_Z|_N$. By Lemma 2, we also have

$$\Omega^m_Z(U) \subset \Omega^m_N(U \cap N)\,, \tag{50}$$

for any open set $U \neq \varnothing$ and hence

$$\Omega^m_Z(Z) \subset \Omega^m_N(N) \subset \Omega^m_A(K)\,. \tag{50'}$$

Finally, we consider regular forms on schemes. Let A be a ring and X a separated scheme over $\mathrm{Spec}\,A$. Then there exists a sheaf $\Omega^m_{X|A}$ of $\mathcal{O}_X$-modules such that

$$\Omega^m_{X|A}|_U \simeq \Omega^m_{U|A} \simeq \widetilde{\Omega^m_A(B)}\,,$$

for any affine open subset $U \simeq \mathrm{Spec}\,B$ of X. (See [4], sec.5.3 and 5.4.) Hence we obtain

$$(\Omega^m_{X|A})_x \simeq \Omega^m_A(\mathcal{O}_{X,x})\,, \tag{51}$$

for any $x \in X$.

REMARK. To avoid confusion, we denote by Ω^m_X the intersection sheaf of an object X in $\mathscr{C}_0(K|A)$ and by $\Omega^m_{X|A}$ the sheaf of regular forms on a scheme X over $\mathrm{Spec}\,A$.

THEOREM 2. *Let k be a perfect field, K a field finitely generated over k, $Z=\mathrm{Zar}(K|k)$ and $N=N(K|k)$.*

(i) *Assume that there exists a regular object X of $\mathscr{C}_1(K|k)$. Then we obtain $\Omega^m_{X|k}\simeq\Omega^m_X=\Phi_{X*}\Omega^m_Z=(\Phi_X|_N)_*\Omega^m_N$ for any multi-index m. Therefore $\Omega^m_{X|k}(X)$ is a birational invariant.*

(ii) *If $\mathscr{C}_1(K|k)$ has enough regular objects, i.e., for any object X of $\mathscr{C}_1(K|k)$, there exist a regular object Y and a morphism $f: Y\to X$ of $\mathscr{C}_1(K|k)$, then $\Omega^m_Z=i_*\Omega^m_N$ for any multi-index m. Note that $\mathscr{C}_1(K|k)$ could be replaced by $\mathscr{C}_2(K|k)$.*

PROOF. (i) Since X is regular, $\Omega^m_k(\mathscr{O}_{X,x})$ is a free $\mathscr{O}_{X,x}$-module for any $x\in X$. By Lemma 8, (ii) and (51), we obtain $\Omega^m_{X|k}\simeq\Omega^m_X$. By (49) and (50), we obtain

$$\Omega^m_X(V)\subset\Omega^m_Z(\Phi_X^{-1}(V))\subset\Omega^m_N(\Phi_X^{-1}(V)\cap N)\,,$$

for any non empty open subset V of X. Conversely, let $\omega\in\Omega^m_N(\Phi_X^{-1}(V)\cap N)$. For any $x\in V$ and $P\in\mathrm{Spec}\,\mathscr{O}_{X,x}$ such that $\mathrm{ht}\,P=1$, we have $(\mathscr{O}_{X,x})_P\in\Phi_X^{-1}(V)\cap N$. Then $\omega\in\Omega^m_k((\mathscr{O}_{X,x})_P)=\Omega^m_k(\mathscr{O}_{X,x})_P=(\Omega^m_{X,x})_P$. Since $\Omega^m_{X,x}=\bigcap_{\mathrm{ht}\,P=1}(\Omega^m_{X,x})_P$, we have $\omega\in\Omega^m_{X,x}$ and hence $\omega\in\Omega^m_X(V)$. Therefore we obtain

$$\Omega^m_X(V)=\Omega^m_Z(\Phi_X^{-1}(V))=\Omega^m_N(\Phi_X^{-1}(V)\cap N)\,. \tag{52}$$

(ii) Note that the set $\{\Phi_X^{-1}(V)\mid X$ is a regular object of $\mathscr{C}_1(K|k)$ and V is open in $X\}$ is an open base of the Zariski topology on Z. Thus by (52), we obtain $\Omega^m_Z=i_*\Omega^m_N$. Q.E.D.

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