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On the Bernstein-Nikolsky Inequality II *

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Abstract. Certain exact results concerning the Bernstein-Nikolsky inequality are established in this paper.

1. Introduction.

It is well-known that while trigonometric polynomials are good means of approximation for periodic functions, entire functions of exponential type may serve as a mean of approximation for nonperiodic functions. Some properties of entire functions of exponential type, bounded on the real space $\boldsymbol{R}^n$ have been considered in [5]. These results (one of them is the Bernstein-Nikolsky inequality) are very important in the imbedding theory, the approximation theory and applications. The present paper is a continuation of this direction.

Let $1\leq p\leq\infty$ and $\sigma=(\sigma_1,\cdots,\sigma_n)$, $\sigma_j>0, j=1,\cdots,n$. Denote by $M_{\sigma,p}$ the space of all entire functions of exponential type σ which as functions of a real x belong to $L_p(\boldsymbol{R}^n)$. The Bernstein-Nikolsky inequality reads as follows [5, p. 114]: Let $f(x)\in M_{\sigma,p}$. Then

$$\|D^\alpha f\|_p\leq\sigma^\alpha\|f\|_p\,,\qquad \alpha\geq 0\,. \tag{1}$$

It is natural to ask whether there is a function $f(x)\notin M_{\sigma,p}$ for which these inequalities (1) hold? We will show by a very simple proof (for a more general case) that the answer is negative. In other words, the Bernstein-Nikolsky inequality wholly characterizes the space $M_{\sigma,p}$. Further, we extend results obtained in [1, 2] for L_p-norm to Luxemburg-norm and prove one exact inequality which is dual with the Bernstein-Nikolsky inequality. Finally, we consider the corresponding results for functions defined on torus $\boldsymbol{T}^n$.

2. Results.

Let $\phi(t)$: $[0,+\infty)\to[0,+\infty]$ be an arbitrary Young function [4, 6], i.e. $\phi(0)=$

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0, $\phi(t)\geq 0$, $\phi(t)\not\equiv 0$ and $\phi(t)$ is convex. Denote by $L_\phi(\boldsymbol{R}^n)$ the space of all functions $f(x)$ measurable on $\boldsymbol{R}^n$ such that

$$\|f\|_\phi=\inf\left\{\lambda>0:\int_{\boldsymbol{R}^n}\phi(|f(x)|/\lambda)dx\leq 1\right\}<\infty\,.$$

Then $L_\phi(\boldsymbol{R}^n)$ with respect to the Luxemburg norm $\|\cdot\|_\phi$ is a Banach space. $L_\phi(\boldsymbol{R}^n)$ is called Orlicz space.

Recall that $\|\cdot\|_\phi=\|\cdot\|_p$ when $1\leq p<\infty$ and $\phi(t)=t^p$; and $\|\cdot\|_\phi=\|\cdot\|_\infty$ when $\phi(t)=0$ for $0\leq t\leq 1$ and $\phi(t)=\infty$ for $t>1$. Orlicz spaces often arise in the study of nonlinear problems.

Denote by $M_{\sigma,\phi}$ the space of all entire functions of exponential type σ which as functions of a real $x\in\boldsymbol{R}^n$ belong to $L_\phi(\boldsymbol{R}^n)$. It is easy to check that $M_{\sigma,\phi}\subset\mathscr{S}'$. Therefore, it follows from the Paley-Wiener-Schwartz theorem that

$$M_{\sigma,\phi}=\{f\in L_\phi(\boldsymbol{R}^n):\operatorname{supp}Ff\subset\Delta_\sigma\}\,,$$

where F is the Fourier transform and $\Delta_\sigma=\{\xi:|\xi_j|\leq\sigma_j, j=1,\cdots,n\}$.

We obtain the following result:

THEOREM 1. *Let $f\in\mathscr{S}'$. So that $f(x)\in M_{\sigma,\phi}$, it is necessary and sufficient that there exists a constant $C=C(f)$ such that*

$$\|D^\alpha f\|_\phi\leq C\sigma^\alpha\,,\qquad \alpha\geq 0\,. \tag{2}$$

PROOF. Necessity. In the same way as in [5] we easily get the Bernstein-Nikolsky inequality for Luxemburg norm:

$$\|D^\alpha f\|_\phi\leq\sigma^\alpha\|f\|_\phi\,,\qquad \alpha\geq 0\,.$$

Therefore, we have (2).

Sufficiency. Without loss of generality we may assume that $\phi(t)$ is left continuous. Actually, in the contrary case, there exists a point $t_0>0$ such that

$$\lim_{t\to t_0-}\phi(t)<\phi(t_0)\leq\infty\,,\qquad \text{and}\quad \phi(t)=\infty\ \text{ for } t>t_0\,.$$

We put

$$\psi(t)=\begin{cases}\phi(t)\,, & t\neq t_0\\ \lim_{t\to t_0-}\phi(t)\,, & t=t_0\,.\end{cases}$$

Then $\psi(t)$ is a left continuous Young function and $\|\cdot\|_\psi=\|\cdot\|_\phi$. Therefore, we can replace $\phi(t)$ by $\psi(t)$.

Assume that (2) holds. It is easily seen that $f(x)\in C^\infty(\boldsymbol{R}^n)$. We put

$$f_r(x)=\frac{1}{\operatorname{mes}B(0,r)}\int_{B(0,r)}f(x+t)dt\,, \tag{3}$$

where $B(0, r)$ is the ball of radius r centered at zero. Then by Jensen's inequality we get

$$\phi\left(\frac{|D^\alpha f_r(x)|}{\|D^\alpha f\|_\phi+\varepsilon}\right)\leq\frac{1}{\operatorname{mes}B(0,r)}\int_{B(0,r)}\phi\left(\frac{|D^\alpha f(x)|}{\|D^\alpha f\|_\phi+\varepsilon}\right)dt\leq\frac{1}{\operatorname{mes}B(0,r)}$$

for $\varepsilon>0$ and $\alpha\geq 0$. Therefore, taking account of the left continuity of $\phi(t)$ and (2), we have

(4) $$\sup_{x\in\boldsymbol{R}^n}|D^\alpha f_r(x)|\leq\lambda_r\|D^\alpha f\|_\phi\leq C\lambda_r\sigma^\alpha, \qquad \alpha\geq 0,$$

where $\lambda_r=\sup\{t:\phi(t)\leq 1/\operatorname{mes}B(0,r)\}$. Therefore, the Taylor series

$$\sum_{|\alpha|=0}^{\infty}\frac{1}{\alpha!}D^\alpha f_r(0)z^\alpha$$

converges for any point $z\in\boldsymbol{C}^n$ and represents $f_r(x)$ in $\boldsymbol{R}^n$. Hence, taking account of (4), we have

$$|f_r(z)|\leq C\lambda_r\exp\left(\sum_{j=1}^{n}\sigma_j|z_j|\right), \qquad z\in\boldsymbol{C}^n.$$

Therefore, $f_r(z)$ is an entire function of exponential type σ. Hence, it follows from the Paley-Wiener-Schwartz theorem that

(5) $$\operatorname{supp}Ff_r\subset\Delta_\sigma, \qquad r>0.$$

On the other hand, it obviously follows from (3) that f_r converges weakly to f in $\mathscr{S}'$ and therefore, Ff_r also converges weakly to Ff in $\mathscr{S}'$. Consequently, it follows readily from (5) that $\operatorname{supp}Ff\subset\Delta_\sigma$. The proof is complete.

To check $f(x)\in M_{\sigma,\phi}$, the following result is more convenient:

THEOREM 2. *A function $f(x)\in\mathscr{S}'$ belongs to $M_{\sigma,\phi}$ if and only if*

(6) $$\limsup_{|\alpha|\to\infty}(\sigma^{-\alpha}\|D^\alpha f\|_\phi)^{1/|\alpha|}\leq 1.$$

PROOF. The "if" part follows readily from Theorem 1. Further, we suppose that inequality (6) holds. Given $\varepsilon>0$. There exists a constant C_ε such that

$$\|D^\alpha f\|_\phi\leq C_\varepsilon(1+\varepsilon)^{|\alpha|}\sigma^\alpha, \qquad \alpha\geq 0.$$

Therefore, taking account of Theorem 1, we get

$$\operatorname{supp}Ff\subset\Delta_{(1+\varepsilon)\sigma}.$$

Therefore,

$$\operatorname{supp}Ff\subset\bigcap_{\varepsilon>0}\Delta_{(1+\varepsilon)\sigma}=\Delta_\sigma. \qquad \text{(Q.E.D.)}.$$

REMARK 1. Theorem 2 gives us ability to estimate more roughly than Theorem 1. For example, if we have

$$\|D^\alpha f\|_\phi \leq C|\alpha|^4\sigma^\alpha, \qquad \alpha \geq 0,$$

then (6) is valid although (2) does not hold. Further, we notice that the root $1/|\alpha|$ in (6) cannot be replaced by any $1/|\alpha|t(\alpha)$, where $0<t(\alpha)$, $\lim_{|\alpha|\to\infty} t(\alpha)=\infty$. Actually, let $f(x)=e^{i2\sigma x}$. Then $f(x)\in M_{2\sigma,\infty}$. At the same time,

$$\lim_{|\alpha|\to\infty} (\sigma^{-\alpha}\|D^\alpha f\|_\infty)^{1/|\alpha|t(\alpha)} = \lim_{|\alpha|\to\infty} 2^{1/t(\alpha)} = 1 .$$

In the same way as in [2] we easily get the following result:

THEOREM 3. *Let $f(x)\in M_{\sigma,\phi}$ and $\phi(t)>0$ for $t>0$. Then*

$$\lim_{|x|\to\infty} f(x)=0 .$$

REMARK 2. In order that $\lim_{|x|\to\infty} f(x)=0$, the condition $\phi(t)>0$ for $t>0$ is necessary because, in the contrary case, $M_{\sigma,\phi}$ contains all constant functions.

REMARK 3. Let $1\leq p<\infty$ and $f(x)\in M_{\sigma,p}$. It has been proved in [1] that

$$\lim_{|\alpha|\to\infty} \sigma^{-\alpha}\|D^\alpha f\|_p=0 .$$

(This property is not true if $p=\infty$.) The question arises as to what happens for $M_{\sigma,\phi}$? It is not hard to show that

$$\lim_{|\alpha|\to\infty} \sigma^{-\alpha}\|D^\alpha f\|_\phi=0$$

if $\phi(t)$ satisfies the Δ_2-condition at zero, i.e. there exist positive numbers δ, M such that $\phi(2t)\leq M\phi(t)$ for $0\leq t\leq\delta$. We omit the proof of this fact here and let us return to this question another time, when we can completely solve this problem.

Further, let $a=(a_1, \cdots, a_n)$ be an arbitrary real unit vector. Then

$$D_a f(x)=f'_a(x)=\sum_{j=1}^n a_j \frac{\partial f}{\partial x_j}(x)$$

is the derivative of f at the point x in the direction a, and

$$f_a^{(l)}(x)=D_a f_a^{(l-1)}(x)=\sum_{|\alpha|=l} a^\alpha f^{(\alpha)}(x) \qquad (l=1,2,\cdots)$$

is the derivative of order l of f at x in the direction a.

Arguing as in [1] we can prove the following theorem:

THEOREM 4. *Let $f(x)\in L_\phi(R^n)$ and $h(a)=\sup_{\xi\in \mathrm{supp} Ff}|a\xi|<\infty$. Then*

$$\|D_a^m f\|_\phi \le [h(a)]^m \|f\|_\phi\,, \qquad m \ge 0\,.$$

Let us now prove one general result, which is dual with the Bernstein-Nikolsky inequality:

THEOREM 5. *Let I be an unbounded set of integral multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\alpha_j \ge 0, j=1, \cdots, n$ and $0 \in I$. And let $f(x)$ be a nonconstant measurable function such that its generalized derivative $D^\alpha f(x)$ belongs to $L_\phi(\boldsymbol{R}^n)$, $\alpha \in I$. Then*

$$\liminf_{|\alpha| \to \infty} (|\xi^{-\alpha}| \|D^\alpha f\|_\phi)^{1/|\alpha|} \ge 1 \tag{7}$$

for any point $\xi \in \operatorname{supp} \tilde{f}$, where $\tilde{f} = Ff$.

PROOF. Let $\xi^0 \in \operatorname{supp} \tilde{f}$, $\xi_j^0 \ne 0, j=1, \cdots, n$. For the sake of convenience, we denote $\operatorname{supp} \tilde{f}$ by $\operatorname{sp}(f)$ and assume that $\xi_j^0 > 0, j=1, \cdots, n$. We fix a number $\varepsilon > 0$ such that $2\varepsilon < \min_{1 \le j \le n} \xi_j^0$ and choose a domain G with a smooth boundary Γ such that $\xi^0 \in G$ and $G \subset K$, where

$$K = \{\xi : \xi_j^0 - \varepsilon \le \xi_j \le \xi_j^0 + \varepsilon, j=1, \cdots, n\}\,.$$

It follows from $f \in L_\phi(\boldsymbol{R}^n)$ that $f \in \mathscr{S}'$. Hence,

$$\langle \tilde{f}(\xi), \varphi(\xi) \rangle = \langle f(x), \tilde{\varphi}(x) \rangle \tag{8}$$

for any function $\varphi \in \mathscr{S}$. Further, we fix a function $\tilde{v} \in C_0^\infty(G)$ such that $\xi^0 \in \operatorname{supp}(\tilde{v}\tilde{f})$. Putting $\varphi(\xi) = \tilde{v}(\xi)\tilde{w}(\xi)$ in (8), where $\tilde{w}(\xi) \in C_0^\infty(G)$ is an arbitrary function, we have

$$\langle \tilde{v}(\xi)\tilde{f}(\xi), \tilde{w}(\xi) \rangle = \langle f(x), \varphi(x) \rangle\,, \tag{9}$$

where $\varphi(x) = \check{v} * \check{w}(x)$, where $\check{u}(x) = u(-x)$. The distribution $\tilde{v}(\xi)\tilde{f}(\xi)$ has a compact support. Therefore, it can be represented in the form

$$\tilde{v}(\xi)\tilde{f}(\xi) = \sum_{|\alpha| \le m} D^\alpha h_\alpha(\xi)\,,$$

where m is a nonnegative integer and $h_\alpha(\xi)$ are ordinary functions in G. Without loss of generality we may assume that $m \ge 2n$.

It is well-known that the Dirichlet problem for the elliptic differential equation

$$L_{2m}\tilde{z}(\xi) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha(D^\alpha \tilde{z}(\xi)) = \tilde{v}(\xi)\tilde{f}(\xi)$$

has a unique solution $\tilde{z}(\xi) \in \overset{0}{W}_{m,2}(G)$ (see, for example, [3, p. 82]).

(Recall that the Sobolev space $W_{m,2}(G)$ is the completion $C^m(G)$ with respect to the norm

$$\|u\|_{m,2} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_{L^2(G)}^2 \right)^{1/2}\,.$$

And $W^0_{m,2}(G)$ is the subspace of all functions $u(x) \in W_{m,2}(G)$ such that the zero extension of $u(x)$ outside G belongs to $W_{m,2}(\boldsymbol{R}^n)$.)

From (9) we obtain

$$\langle \tilde{z}(\xi), L_{2m}\tilde{w}(\xi) \rangle = \langle f(x), \varphi(x) \rangle \tag{10}$$

for all $\tilde{w}(\xi) \in C_0^\infty(G)$. It is obvious that the left side of (10) admits a closure up to an arbitrary function $\tilde{w}(\xi) \in W^0_{m,2}(G)$. Hence, replacing $\tilde{w}(\xi)$ by $\xi^\alpha \tilde{w}(\xi)$ in (10), we get easily

$$\langle \tilde{z}(\xi), L_{2m}(\xi^\alpha \tilde{w}(\xi)) \rangle = (-i)^{|\alpha|} \langle D^\alpha f(x), \varphi(x) \rangle \tag{11}$$

for all $\tilde{w}(\xi) \in W^0_{m,2}(G)$.

Now let $\tilde{w}_0(\xi) \in W^0_{m,2}(G)$ be a solution of the equation $L_{2m}\tilde{w}_0(\xi) = \overline{\tilde{z}(\xi)}$. Then since $0 \notin G$ we get

$$L_{2m}(\xi^\alpha \tilde{w}_\alpha(\xi)) = \prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \overline{\tilde{z}(\xi)}, \tag{12}$$

where $\tilde{w}_\alpha(\xi) = \prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \xi^{-\alpha} \tilde{w}_0(\xi)$ and $\alpha \geq 0$. Therefore, it follows from (11) that

$$\prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \langle \tilde{z}(\xi), \overline{\tilde{z}(\xi)} \rangle = (-i)^{|\alpha|} \langle D^\alpha f(x), \varphi_\alpha(x) \rangle,$$

where $\varphi_\alpha(x) = \check{v} * \check{w}_\alpha(x) \in L_{\bar{\Phi}}(\boldsymbol{R}^n)$ and $\alpha \geq 0$ (the fact that $\varphi_\alpha(x) \in L_{\bar{\Phi}}(\boldsymbol{R}^n)$ will be shown later). Therefore, using the Weiss theorem [6], we get

$$\prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \langle \tilde{z}(\xi), \overline{\tilde{z}(\xi)} \rangle \leq 2 \|D^\alpha f\|_\Phi \|v\|_1 \|w_\alpha\|_{\bar{\Phi}}, \qquad \alpha \in I. \tag{13}$$

Now we prove that there exists a constant $C > 0$ such that

$$2\|v\|_1 \|w_\alpha\|_{\bar{\Phi}} \leq C, \qquad \alpha \geq 0. \tag{14}$$

Indeed, since

$$x^\beta w_\alpha(x) = (-i)^{|\beta|} \prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \int_G e^{ix\xi} D^\beta(\xi^{-\alpha} \tilde{w}_0(\xi)) d\xi,$$

from the Leibniz formula and the definition of G, we obtain for any $|\beta| \leq 2n$

$$\begin{aligned}\sup_{x \in \boldsymbol{R}^n} |x^\beta w_\alpha(x)| &\leq \prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \sum_{\gamma \leq \beta} \left\{ \frac{\beta!}{\gamma!(\beta-\gamma)!} \prod_{k=1}^n \alpha_k \cdots (\alpha_k + \gamma_k - 1) \right. \\ &\qquad \left. \times \int_G |\xi^{-(\alpha+\gamma)} D^{\beta-\gamma} \tilde{w}_0(\xi)| d\xi \right\} \\ &\leq C_1 \prod_{j=1}^n \left(\frac{\xi_j^0 - 2\varepsilon}{\xi_j^0 - \varepsilon} \right)^{\alpha_j} \sum_{\gamma \leq \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} \prod_{k=1}^n \alpha_k \cdots (\alpha_k + \gamma_k - 1),\end{aligned}$$

where

$$C_1=\max\left\{\int_G |\xi^{-\gamma}D^{\beta-\gamma}\tilde{w}_0(\xi)|d\xi : \gamma\le\beta, |\beta|\le 2n\right\}.$$

On the other hand, since

$$\prod_{k=1}^{n} \alpha_k\cdots(\alpha_k+\gamma_k-1)<(|\alpha|+2n)^{2n}$$

(because of $|\gamma|\le|\beta|\le 2n$), and

$$2^{|\beta|}=\sum_{\gamma\le\beta}\frac{\beta!}{\gamma!(\beta-\gamma)!}$$

and

$$\lim_{|\alpha|\to\infty}(|\alpha|+2n)^{2n}\prod_{j=1}^{n}\left(\frac{\xi_j^0-2\varepsilon}{\xi_j^0-\varepsilon}\right)^{\alpha_j}=0,$$

we obtain

$$\sup_{x\in\boldsymbol{R}^n}|x^\beta w_\alpha(x)|\le C_2$$

for all $|\beta|\le 2n$ and $\alpha\ge 0$. Consequently, there is an absolute constant C_3 such that

$$\sup_{x\in\boldsymbol{R}^n}(1+x_1^2)\cdots(1+x_n^2)|w_\alpha(x)|\le C_3.$$

Further, let $0<\lambda_0<\infty$ such that $\bar{\phi}(C_3/\lambda_0)\le\pi^{-n}$. Then it is easy to check that $\|w_\alpha\|_{\bar{\phi}}\le\lambda_0$ for all $\alpha\ge 0$. Thus we have proved (14) with $C=2\lambda_0\|v\|_1$. Further, combining (13) and (14), we obtain

$$\prod_{j=1}^{n}(\xi_j^0-2\varepsilon)^{\alpha_j}\langle\tilde{z}(\xi),\overline{\tilde{z}(\xi)}\rangle\le C\|D^\alpha f\|_\phi,\qquad \alpha\in I.$$

Therefore,

$$1\le\liminf_{|\alpha|\to\infty}(\|D^\alpha f\|_\phi\prod_{j=1}^{n}(\xi_j^0-2\varepsilon)^{-\alpha_j})^{1/|\alpha|}.$$

Therefore, since $\varepsilon>0$ is arbitrarily chosen and

$$\left[\prod_{j=1}^{n}\left(\frac{\xi_j^0-2\varepsilon}{\xi_j^0}\right)^{-\alpha_j}\right]^{1/|\alpha|}\le\max_{1\le j\le n}\frac{\xi_j^0}{\xi_j^0-2\varepsilon},$$

we get

$$1\le\liminf_{|\alpha|\to\infty}((\xi^0)^{-\alpha}\|D^\alpha f\|_\phi)^{1/|\alpha|}$$

by letting $\varepsilon \to 0$.

Finally, we shall prove (7) for "zero" points: Let $\xi^0 \in \mathrm{sp}(f)$, $\xi^0 \neq 0$ and $\xi_1^0 \cdots \xi_n^0 = 0$. For the sake of convenience, we assume that $\xi_j^0 > 0, j = 1, \cdots, k$ and $\xi_{k+1}^0 = \cdots = \xi_n^0 = 0$ $(1 \leq k < n)$. Then, it is enough to show (7) only for indices $\alpha \in I$ such that $\alpha_{k+1} = \cdots = \alpha_n = 0$ (we presuppose that $\lambda/0 = \infty$ for $\lambda > 0$). Then the proof is analogous to the one above after only the following modification of choosing ε: We fix a number $\varepsilon > 0$ such that $2\varepsilon < \min_{1 \leq j \leq k} \xi_j^0$ and a domain G with a smooth boundary Γ such that $\xi^0 \in G$ and $G \subset K$, where

$$K = \{\xi : \xi_j^0 - \varepsilon \leq \xi_j \leq \xi_j^0 + \varepsilon, j = 1, \cdots, n\} .$$

The proof of Theorem 5 is complete.

REMARK 4. Let $f(x) \in M_{\sigma,\phi}$ and sp(f) contains at least one vertex of the parallelepiped Δ_σ. Then, using the Bernstein-Nikolsky inequality and Theorem 5, we get easily

$$\lim_{|\alpha| \to \infty} (\sigma^{-\alpha} \| D^\alpha f \|_\phi)^{1/|\alpha|} = 1 ,$$

which shows that the bound 1 in inequality (7) cannot be improved.

REMARK 5. All the corresponding results for functions defined on torus T^n hold. We, for example, give here one result, which we can prove by a much easier way—by representing the considered function by its Fourier series:

THEOREM 6. *Let I be an unbounded set of multi-indices $\alpha \geq 0$ and $0 \in I$. And let $f(x)$ be a nonconstant measurable function such that its generalized derivative $D^\alpha f(x)$ belongs to $L_\phi(T^n)$, $\alpha \in I$. Then*

$$\liminf_{|\alpha| \to \infty} (|k^{-\alpha}| \| D^\alpha f \|_{L_\phi(T^n)})^{1/|\alpha|} \geq 1$$

for any point $k \in \mathrm{sp}(f)$.

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