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Noetherian Rings Graded by an Abelian Group

Yuji KAMOI

Tokyo Metropolitan University

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Dedicated to Professor Takeshi Ishikawa on his 60th birthday

Introduction.

Throughout this paper, all rings are assumed to be commutative with identity.

Let G be an Abelian group. We say that a ring R is a G-graded ring, if there exists a family $\{R_g\}_{g\in G}$ of additive subgroups of R such that $R=\bigoplus_{g\in G}R_g$ and $R_gR_h\subset R_{g+h}$ for every $g, h\in G$. Similarly, a G-graded R-module is an R-module M for which there is given a family $\{M_g\}_{g\in G}$ of additive subgroups of M such that $M=\bigoplus_{g\in G}M_g$ and $R_gM_h\subset M_{g+h}$ for every $g, h\in G$.

The investigation of the ring-theoretic property of graded rings started with the following question of Nagata [13].

If G is the ring of integers $\boldsymbol{Z}$, then is Cohen-Macaulay property of R determined by their local data at graded prime ideals?

As is well-known, Matijevic-Roberts [12] and Hochster-Ratliff [8] gave an affirmative answer to the conjecture as above. Similarly Aoyama-Goto [1] and Matijevic [11] showed that the same as above is also true for Gorenstein property. Furthermore Goto-Watanabe developed a theory of $\boldsymbol{Z}^n$-graded rings and modules in their papers [5] and [6] and proved the relation between Bass numbers of graded modules at nongraded prime ideals and Bass numbers at graded prime ideals.

In this paper, we study G-graded rings and G-graded modules for an arbitrary Abelian group G.

Some homological properties of a G-graded ring R depend only on their local data at graded prime ideals, when $G=\boldsymbol{Z}^n$. But, for an arbitrary Abelian group G, informations about graded prime ideals are not enough to determine homological properties. For example, the hypersurface $k[X]/(X^2-1)$ is a $\boldsymbol{Z}_2$-graded ring by $\deg(X)=1\in\boldsymbol{Z}_2$ and has no graded prime ideals. Here $\boldsymbol{Z}_2=\boldsymbol{Z}/2\boldsymbol{Z}$. Therefore we introduce the notion of G-prime ideals as follows and improve Goto-Watanabe's arguments using this notion.

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DEFINITION 1.2. A G-graded ideal $\mathfrak{p}$ of R is said to be a G-prime ideal, if it satisfies the following condition: for any G-homogeneous elements $a, b \in R$ such that $ab \in \mathfrak{p}$, eigher $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

(Of course, if G is torsion free, G-prime ideals are prime ideals (cf. chap. III, §1, no. 4 of [3]). In section 4, we give a necessary and sufficient condition for a G-prime ideal to be a prime ideal when G is an arbitrary Abelian group.)

Then we have the following.

THEOREM 2.13. *Let M be a finitely generated G-graded module over a Noetherian G-graded ring R and $\mathfrak{p}$ be a G-prime ideal of R. Then the following conditions are equivalent.*

(1) *$M_{(\mathfrak{p})}$ is a Cohen-Macaulay (resp. Gorenstein) $R_{(\mathfrak{p})}$-module.*

(2) *M_P is a Cohen-Macaulay (resp. Gorenstein) R_P-module for every $P \in \mathrm{Ass}_R(R/\mathfrak{p})$.*

(3) *M_P is a Cohen-Macaulay (resp. Gorenstein) R_P-module for some $P \in \mathrm{Ass}_R(R/\mathfrak{p})$.*

(4) *There exists $P \in \mathrm{Spec}(R)$ such that $P^* = \mathfrak{p}$ and M_P is a Cohen-Macaulay (resp. Gorenstein) R_P-module.*

Here $M_{(\mathfrak{p})}$ is the module of fractions of M with respect to the set of all homogeneous elements of $R \setminus \mathfrak{p}$ and P^* is the maximal graded ideal which is contained in P.

Furthermore, we define the i-th G-Bass number $\nu^i(\mathfrak{p}, M)$ of a G-graded module M as

$$\nu^i(\mathfrak{p}, M) = \mathrm{rank}_{R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}} \underline{\mathrm{Ext}}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})$$

where $\mathfrak{p}$ is a G-prime ideal of R (see (2.9)). The following theorem will play important roles in proving Theorem 2.13.

THEOREM 2.11. *Let M be a G-graded module over a Noetherian G-graded ring R and P be a prime ideal of R. We put $d = \dim(R_P/P^*R_P)$. Then*

$$\mu^i(P, M) = \begin{cases} \nu^{i-d}(P^*, M) & \text{if } i \geq d \\ 0 & \text{if } i < d \end{cases}$$

where $\mu^i(P, M) = \dim_{R_P/PR_P}(\mathrm{Ext}^i_{R_P}(R_P/PR_P, M_P))$ is the i-th Bass number of M at P.

1. Preliminaries.

In this section, we recall some definitions and basic facts about graded rings and graded modules (cf. [5], [6] and [14]).

Let G be an Abelian group. We say that a ring R is a G-graded ring, if there exists a family $\{R_g\}_{g \in G}$ of additive subgroups of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subset R_{g+h}$ for every $g, h \in G$. Similarly, a G-graded R-module is an R-module M for which there is given a family $\{M_g\}_{g \in G}$ of addtive subgroups of M such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subset M_{g+h}$ for every $g, h \in G$. A homomorphism $f: M \to N$ of G-graded R-modules is an R-linear map such that $f(M_g) \subset N_g$ for all $g \in G$. We denote by $M_G(R)$ the category consisting of all G-graded R-modules and their homomorphisms.

Let R be a G-graded ring and M be a G-graded R-module. For $g \in G$, we define a G-graded R-module $M(g)$ by $M = M(g)$ as the underlying R-module and graded by $[M(g)]_h = M_{g+h}$ for all $h \in G$. We say that M is free, if it is isomorphic to a direct sum of G-graded R-modules of the form $R(g)$ $(g \in G)$. The elements $\bigcup_{g \in G} M_g$ are called homogeneous elements of M, a nonzero element $x \in M_g$ is said to be homogeneous of degree g, and we denote $\deg(x) = g$. For a subset $N \subset M$, we set $h(N) = \bigcup_{g \in G}(N \cap M_g)$. Any non-zero element $x \in M$ has a unique expression as a sum of homogeneous elements, $x = \sum_{g \in G} x_g$ where $x_g \in M_g$ and $x_g = 0$ for almost all $g \in G$. With this notation, we call nonzero x_g the homogeneous component (of degree g) of x.

Let H be a subgroup of G and $g \in G$. We define $R^{(H)} = \bigoplus_{h \in H} R_h$ and $M^{(g,H)} = \bigoplus_{h \in H} M_{g+h}$. Then $R^{(H)}$ is a subring of R and $M^{(g,H)}$ is an $R^{(H)}$-submodule of M. We define a G-grading on $M^{(g,H)}$ as

$$[M^{(g,H)}]_{g'} = \begin{cases} M_{g'} & \text{if } g - g' \in H \\ (0) & \text{if } g - g' \notin H \end{cases}$$

for all $g' \in G$. If $g - g' \in H$, then we have $M^{(g,H)} = M^{(g',H)}$ as G-graded $R^{(H)}$-modules. Hence M has the following decomposition as a G-graded $R^{(H)}$-module

$$M = \bigoplus_{i \in I} M^{(g_i, H)}$$

where $\{g_i\}_{i \in I}$ is a system of representatives of $G \bmod H$. Also, we have $R^{(g_i, H)} M^{(g_j, H)} \subset M^{(g_i + g_j, H)}$ for all $i, j \in I$. Hence a G-graded ring R (resp. G-graded R-module M) can be regarded as a G/H-graded ring (resp. G/H-graded R-module).

Definition 1.1. (1) We say that R is a G-domain, if every nonzero G-homogeneous element of R is a nonzero divisor of R. That is to say, if $ab = 0$, then $a = 0$ or $b = 0$ for G-homogeneous elements $a, b \in h(R)$.

(2) We say that R is G-simple, if every nonzero G-homogeneous element is a unit of R. Or, equivalently, if R has no proper G-graded ideals except (0).

If R is a G-simple graded ring and H is a subgroup of G, then H-graded ring $R^{(H)}$ is H-simple.

Definition 1.2. (1) A G-graded ideal $\mathfrak{p}$ of R is said to be a G-prime ideal, if the G-graded ring $R/\mathfrak{p}$ is a G-domain. Or, equivalently, for any G-homogeneous elements $a, b \in h(R)$, if $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

(2) A G-graded ideal $\mathfrak{m}$ of R is said to be a G-maximal ideal, if the G-graded ring $R/\mathfrak{m}$ is G-simple.

Note that a G-prime (resp. G-maximal) ideal of R is not necessarily a prime (resp. maximal) ideal. For example, let $k[X]$ be a polynomial ring over a field k. We consider a ring $k[X]/(X^2 - 1)$ and regard it as a $\boldsymbol{Z}_2$-graded ring. Then $k[X]/(X^2 - 1)$ is a $\boldsymbol{Z}_2$-domain and also $\boldsymbol{Z}_2$-simple but it is not a domain. Thus the zero ideal of $k[X]/(X^2 - 1)$ is

Z_2-prime and not a prime ideal.

We denoty by $V_G(R)$ the set of all G-prime ideals of R. For $\mathfrak{p}\in V_G(R)$, we denote by $M_{(\mathfrak{p})}$ the module of fractions of M with respect to the multiplicatively closed subset $h(R\setminus\mathfrak{p})$ and call it the homogeneous localization of M at $\mathfrak{p}$. We set $V_G(M)=\{\mathfrak{p}\in V_G(R)\mid M_{(\mathfrak{p})}\neq(0)\}$. For an ideal P of R, we denote by P^* the maximal graded ideal of R contained in P (or the graded ideal generated by $h(P)$). If P is a prime ideal of R, then P^* is a G-prime ideal of R. Furthermore, for a G-graded R-module M and $P\in\mathrm{Spec}(R)$, $P\in\mathrm{Supp}_R(M)$ if and only if $P^*\in V_G(M)$.

DEFINITION 1.3. We say that R is a G-local graded ring, if it has the unique G-maximal ideal $\mathfrak{m}$. Often we use the notation $(R,\mathfrak{m})$ to say that R is G-local with the unique G-maximal ideal $\mathfrak{m}$.

In the rest of this section, we develop some standard techniques of G-graded rings which will be used freely in this paper.

PROPOSITION 1.4. (1) *For $g\in G$, if $a\in R_g$ is a unit of R, then $a^{-1}\in R_{-g}$ and $R_g=aR_0$.*

(2) *R is G-simple if and only if every G-graded R-module is free.*

(3) *Suppose that $(R,\mathfrak{m})$ is G-local and M is a finitely generated G-graded R-module. Then $M=\mathfrak{m}M$ implies $M=(0)$. Thus, if $x_1,\cdots,x_n\in h(M)$ and if their images in $M/\mathfrak{m}M$ form a free $R/\mathfrak{m}$-basis, then M is generated by $x_1,\cdots,x_n$.*

(4) *Let $(R,\mathfrak{m})$ be G-local and H be a subgroup of G such that $\mathfrak{m}^{(H)}R=\mathfrak{m}$. Let $\{g_i\}_{i\in I}$ be a system of representatives of G mod H. Assume that $R^{(g_i,H)}$ is a finitely generated $R^{(H)}$-module for every $i\in I$. Then the following statements hold.*

(a) *If $R^{(g_i,H)}\neq 0$ for $i\in I$, then there exists a unit $u_i\in R_{g_i+h}$ of R for some $h\in H$. Thus R is free over $R^{(H)}$ which has a free basis consisting of G-homogeneous units of R.*

(b) *For $\mathfrak{q}\in V_H(R^{(H)})$ and $\mathfrak{p}\in V_G(R)$, we have $\mathfrak{q}R\in V_G(R)$ and $\mathfrak{p}^{(H)}\in V_H(R^{(H)})$. This gives a bijective correspondence between $V_H(R^{(H)})$ and $V_G(R)$.*

(c) *For $\mathfrak{p}\in V_G(R)$, $M_{(\mathfrak{p})}=M\otimes_{R^{(H)}}(R^{(H)})_{(\mathfrak{p}^{(H)})}$.*

PROOF. Assertions (1) and (2) are the same as Theorem 1.1.4 of Goto-Watanabe [6] and the assertion (3) is a graded version of Nakayama's lemma. We only need to show the assertion (4).

(a) If $R^{(g_i,H)}\neq(0)$ $(i\in I)$, then there exists $u_i\in h(R^{(g_i,H)})$ such that $u_i\notin\mathfrak{m}^{(H)}R^{(g_i,H)}$ by (3). Since $\mathfrak{m}^{(H)}R=\mathfrak{m}$, $u_i\notin\mathfrak{m}$ and, since $(R,\mathfrak{m})$ is G-local, u_i is a unit of R.

(b) Clearly, $\mathfrak{p}^{(H)}\in V_H(R^{(H)})$ for every $\mathfrak{p}\in V_G(R)$. Let $T=\{u_i\mid i\in I,\ u_i\in R^{(g_i,H)}\neq(0)\}$ be the set of units of R which is obtained as in (a). Then we have $h(R)=\{au_i\mid a\in h(R^{(H)}),\ u_i\in T\}$. Hence we can verify that $\mathfrak{q}R\in V_G(R)$ for every $\mathfrak{q}\in V_H(R^{(H)})$.

(c) By (b), we have $h(R\setminus\mathfrak{p})=\{au_i\mid a\in h(R^{(H)}\setminus\mathfrak{p}^{(H)})\}$ for every $\mathfrak{p}\in V_G(R)$. Hence $h(R_{(\mathfrak{p}^{(H)})}/\mathfrak{p}(R_{(\mathfrak{p}^{(H)})})$ is the set of units of $R_{(\mathfrak{p}^{(H)})}$ and $M_{(\mathfrak{p})}=M\otimes_{R^{(H)}}(R^{(H)})_{(\mathfrak{p}^{(H)})}$ for every G-graded R-module M. □

EXAMPLE 1.5. Let $\mathfrak{p}$ be a finitely generated G-prime ideal of a G-graded ring R and H be a finitely generated subgroup of G which contains degrees of (finite) homogeneous generators of $\mathfrak{p}$. Then $\mathfrak{p}^{(H)}R=\mathfrak{p}$. Namely, $(R_{(\mathfrak{p})}, \mathfrak{p}R_{(\mathfrak{p})})$ and H staisfy the first assumption of (1.4), (4).

THEOREM 1.6. *Let G be a finitely generated Abelian group and R be a ring. Then the following conditions are equivalent.*

(1) *R is a G-simple graded ring.*

(2) *R contains a field k and*

$$R\cong\frac{k[X_1, \cdots, X_m, Y_1^{\pm 1}, \cdots, Y_n^{\pm 1}]}{(X_1^{q_1}-u_1, \cdots, X_m^{q_m}-u_m)}$$

where $m, n\geq 0$, $u_1, \cdots, u_m\in k^*$, $X_1, \cdots, X_m$, $Y_1, \cdots, Y_n$ *are variables and each* q_i $(i=1, \cdots, m)$ *is a power of a prime integer.*

PROOF. (2) ⇒ (1) Put $G=\bigoplus_{i=1}^{m} \boldsymbol{Z}/(q_i)\oplus\boldsymbol{Z}^n$. Then R is G-simple.

(1) ⇒ (2) It is clear that $k=R_0$ is a field. We suppose that $R\neq k$ and put $G'=\{g\in G \mid R_g\neq(0)\}$. Then G' is a nonzero subgroup of G. Thus we can write

$$G'=\bigoplus_{i=1}^{m} C(q_i)\oplus\boldsymbol{Z}^n$$

where q_i is a power of a prime number and $C(q_i)$ is a cyclic group of order q_i for $1\leq i\leq m$. Let e_i be a generator of $C(q_i)$ $(1\leq i\leq m)$ and let $e'_1, \cdots, e'_n$ be free basis of $\boldsymbol{Z}^n$. Then there exist unit elements $x_i\in R_{e_i}$ $(1\leq i\leq m)$ and $y_j\in R_{e'_j}$ $(1\leq j\leq n)$ by our choice of G'. By (1.4), (1), we have $R=k[x_1, \cdots, x_m, y_1^{\pm 1}, \cdots, y_n^{\pm 1}]$.

Next, we define a k-algebra map $\varphi: k[X_1, \cdots, X_m, Y_1^{\pm 1}, \cdots, Y_n^{\pm 1}]\to R$ by $\varphi(X_i)=x_i$ $(1\leq i\leq m)$, and $\varphi(Y_j^{\pm 1})=y_j^{\pm 1}$ $(1\leq j\leq n)$ where $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are variables. Then φ is surjective and $\ker(\varphi)=(X_1^{q_1}-u_1, \cdots, X_m^{q_m}-u_m)$ $(u_j=x_j^{q_j}\in k^*)$, by the choice of $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$.

The proof of (1.6) is now complete. □

As a consequence, we get the following.

COROLLARY 1.7. *A Noetherian G-simple graded ring R is locally a complete intersection. In particular,* $\mathrm{Ass}_R(R)=\mathrm{Min}(R)$ *and if G is a torsion group, then R is Artinian.*

PROOF. Let R be a Noetherian G-simple graded ring. We shall show that a local ring R_Q is a complete intersection for every maximal ideal Q of R. Since R is Noetherian, Q is finitely generated. Let H be the subgroup of G generated by degrees of all homogeneous components of (finite) generators of Q. By (1.4), (2), R is free over $R^{(H)}$, since $R^{(H)}$ is H-simple. Also, by the choice of H, $R/(Q\cap R^{(H)})R=R/Q$. Hence, by (1.6) and Avramov's criterion [2], R_Q is complete intersection of the same dimension as that of $(R^{(H)})_{Q\cap R^{(H)}}$. If G is torsion, then so is H. By the proof of (1.6), we have $\dim(R^{(H)})=0$.

Hence R is Artinian. □

EXAMPLE 1.8. Let A be a Noetherian ring and $R=A[G]$ be a Noetherian group ring. Then $\mathrm{Max}(A)=\{Q\cap A \mid Q\in \mathrm{Max}(R)\}$ and $V_G(R)=\{\mathfrak{p}R \mid \mathfrak{p}\in \mathrm{Spec}(A)\}$. Thus R is Cohen-Macaulay (resp. Gorenstein, locally complete intersection) if and only if so is A.

DEFINITION 1.9. R is said to be a G-Noetherian graded ring, if it satisfies the following equivalent conditions.

(1) Every strict ascending chain of G-graded ideals of R is finite.
(2) Every nonempty family of G-graded ideals of R has a maximal element.
(3) Every G-graded ideal of R is finitely generated.

REMARK 1.10. (1) Suppose that R is G-Noetherian. Then for every subgroup $H\subset G$ and every $g\in G$, $R^{(H)}$ is G-Noetherian and $R^{(g,H)}$ is finitely generated as an $R^{(H)}$-module.

(2) (Theorem 1.1 of Goto-Yamagishi [4]) Suppose that G is a finitely generated Abelian group. Then the following conditions are equivalent.

(a) R is a Noetherian graded ring.
(b) R is a G-Noetherian graded ring.
(c) R_0 is Noetherian and R is a finitely generated R_0-algebra.

In general, a G-Noetherian ring is not a Noetherian ring. For example, $Z^{(I)}$-simple graded ring $Q[\{X_i, X_i^{-1}\}_{i\in I}]$ is $Z^{(I)}$-Noetherian but it is not Noetherian, if I is infinite. Also, there exists a Noetherian graded ring R which is not a finitely generated R_0-algebra (e.g. Proposition 3.1 of Goto-Yamagishi [4]).

2. Dimension and Bass numbers of G-graded modules.

Let M be a G-graded module over a G-graded ring R. A G-prime ideal $\mathfrak{p}$ is said to be associated with M, if $\mathfrak{p}=[0:x]_R$ for some $x\in h(M)$. We denote by $\underline{\mathrm{Ass}}_R(M)$ the set of all G-prime ideals associated with M.

The followings will be proved in the same way as in the non graded case (cf. chap. IV, §1, no. 1 of [3]).

PROPOSITION 2.1. *Let M be a G-graded module over a G-graded ring R.*

(1) *If M is the union of a family $\{M_i\}_{i\in I}$ of G-graded submodules of M, then $\underline{\mathrm{Ass}}_R(M)=\bigcup_{i\in I}\underline{\mathrm{Ass}}(M_i)$.*

(2) *Every maximal element of $\{[0:x] \mid x\in h(M), x\neq 0\}$ belongs to $\underline{\mathrm{Ass}}_R(M)$. Thus $\underline{\mathrm{Ass}}_R(M)\neq\emptyset$ is equivalent to $M\neq 0$, provided R is G-Noetherian.*

(3) *Let N be a G-graded submodule of M. Then $\underline{\mathrm{Ass}}_R(N)\subset\underline{\mathrm{Ass}}_R(M)\subset\underline{\mathrm{Ass}}_R(N)\cup\underline{\mathrm{Ass}}_R(M/N)$.*

(4) *Every G-prime ideal of R containing an element of $\underline{\mathrm{Ass}}_R(M)$ belongs to $V_G(M)$. Conversely, if R is G-Noetherian, then every $\mathfrak{p}\in V_G(M)$ contains an element of $\underline{\mathrm{Ass}}_R(M)$.*

(5) *If R is G-Noetherian, then* $\underline{\mathrm{Ass}}_R(M)$ *and* $V_G(M)$ *have the same minimal elements.*

(6) *If R is G-Noetherian and M is a finitely generated R-module, then there exists a chain* $(0)=M_n\subset M_{n-1}\subset\cdots\subset M_0=M$ *of G-graded submodules of M such that, for* $1\le i\le n$, $M_i/M_{i-1}\cong(R/\mathfrak{p}_i)(g_i)$, *where* $\mathfrak{p}_i\in V_G(R)$ *and* $g_i\in G$. *In this case* $\underline{\mathrm{Ass}}_R(M)\subset\{\mathfrak{p}_1,\cdots,\mathfrak{p}_n\}\subset V_G(M)$ *and therefore* $\underline{\mathrm{Ass}}_R(M)$ *is finite.*

Next, we relate $\underline{\mathrm{Ass}}_R(M)$ to $\mathrm{Ass}_R(M)$.

PROPOSITION 2.2. *Let M be a G-graded module over a G-graded ring R.*

(1) *If* $P\in\mathrm{Ass}_R(M)$, *then* $P^*\in\underline{\mathrm{Ass}}_R(M)$.

(2) *If* $\mathfrak{p}\in V_G(R)$ *and* $P\in\mathrm{Ass}_R(R/\mathfrak{p})$, *then* $P^*=\mathfrak{p}$.

(3) $\mathrm{Ass}_R(M)=\bigcup_{\mathfrak{p}\in\underline{\mathrm{Ass}}_R(M)}\mathrm{Ass}_R(R/\mathfrak{p})$.

PROOF. (1) For $P\in\mathrm{Ass}_R(M)$, we put $P=[0:\sum_{g\in G}x_g]$ where $x_g\in M_g$ and $x_g=0$ for almost all $g\in G$. Then the G-graded ideal $\bigcap_{g\in G,x_g\neq 0}[0:x_g]$ is contained in P. Thus $\bigcap_{g\in G,x_g\neq 0}[0:x_g]\subset P^*$. Let $a\in h(P)$. Since $a\sum_{g\in G}x_g=0$, we have $ax_g=0$ for every $g\in G$. Hence $a\in[0:x_g]$ for every $g\in G$. Namely $P^*=\bigcap_{g\in G,x_g\neq 0}[0:x_g]$. Since P^* is a G-prime ideal, this implies that $P^*=[0:x_g]$ for some $g\in G$.

(2) Let $P\in\mathrm{Ass}_R(R/\mathfrak{p})$. It is clear that $\mathfrak{p}\subset P^*$. Conversely, by (1), there exists a G-homogeneous element a of $R\setminus\mathfrak{p}$ such that $P^*=[\mathfrak{p}:a]$. Hence $aP^*\subset\mathfrak{p}$. Since $\mathfrak{p}$ is a G-prime ideal and $a\notin\mathfrak{p}$, we have $P^*\subset\mathfrak{p}$.

(3) Clearly, we have $\mathrm{Ass}_R(M)\supset\bigcup_{\mathfrak{p}\in\underline{\mathrm{Ass}}_R(M)}\mathrm{Ass}_R(R/\mathfrak{p})$ and we shall show the converse inclusion.

Let $P\in\mathrm{Ass}_R(M)$ and $\mathfrak{p}=P^*$. Then, by (1), $\mathfrak{p}\in\underline{\mathrm{Ass}}_R(M)$. Thus it suffices to show that $P\in Ass_R(R/\mathfrak{p})$. We assume the contrary (i.e. $P\notin Ass_R(R/\mathfrak{p})$). By the aid of Zorn's lemma, we can show that there exists a maximal G-graded submodule $N\subset M$ such that $\underline{\mathrm{Ass}}_R(N)=\{\mathfrak{p}\}$ and $P\notin\mathrm{Ass}_R(N)$. Since $P\notin Ass_R(N)$, $P\in\mathrm{Ass}_R(M/N)$ and, by (1), $P^*=\mathfrak{p}\in\underline{\mathrm{Ass}}_R(M/N)$. Hence there exists a G-graded submodule $L\subset M$ such that $N\subset L$ and $L/N\cong(R/\mathfrak{p})(g)$ $(g\in G)$. Then, by (2.1), (3), $\underline{\mathrm{Ass}}_R(L)=\{\mathfrak{p}\}$ and $P\notin\mathrm{Ass}_R(L)$ since $\mathrm{Ass}_R(L)\subset\mathrm{Ass}_R(N)\cup\mathrm{Ass}_R(R/\mathfrak{p})$. This contradicts the maximality of N. Hence we have $P\in\mathrm{Ass}_R(R/\mathfrak{p})$. □

DEFINITION 2.3. Let M be a G-graded module over a G-graded ring R. We denote by $\underline{\dim}(M)$ the largest length of the chains of G-prime ideals in $V_G(M)$ and call it G-dimension of M.

We have the following dimension theorem for G-graded modules.

THEOREM 2.4. *Let R be a Noetherian G-graded ring and M be a G-graded R-module. If* $\mathfrak{p}\in V_G(M)$, *then we have* $\underline{\dim}(M_{(\mathfrak{p})})=\dim(M_P)$ *for every* $P\in\mathrm{Ass}_R(R/\mathfrak{p})$.

First we show a lemma.

LEMMA 2.5. *Let R be a Noetherian G-graded ring and M be a G-graded R-module.*

(1) $\mathrm{Ass}_R(R/\mathfrak{p})=\mathrm{Min}_R(R/\mathfrak{p})$ *for* $\mathfrak{p}\in V_G(R)$.

(2) *Let* $\mathfrak{p} \in V_G(R)$. *Then* $\mathfrak{p} \in V_G(M)$ *if and only if* $\mathrm{Ass}_R(R/\mathfrak{p}) \subset \mathrm{Supp}_R(M)$.

(3) *Let* $P, Q \in \mathrm{Supp}_R(M)$ *such that* $P \supset Q$. *If* $\dim(M_P) = \dim(R_P/QR_P)$, *then* $\dim(R_P/Q^*R_P) = \dim(M_P)$. *In this case,* Q^* *is a minimal element of* $V_G(M)$.

PROOF. (1) By (2.2), (2), $\mathrm{Ass}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}) = \{PR_{(\mathfrak{p})} \mid P \in \mathrm{Ass}_R(R/\mathfrak{p})\}$. Also, by (1.7), $\mathrm{Ass}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}) = \mathrm{Min}_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})$. Hence $\mathrm{Ass}_R(R/\mathfrak{p}) = \mathrm{Min}_R(R/\mathfrak{p})$.

(2) The assertion follows from (2.2), (2).

(3) It is clear that $\dim(M_P) = \dim(R_P/QR_P) \le \dim(R_P/Q^*R_P)$. Conversely, since $\mathrm{Ass}_R(R/Q^*) \subset \mathrm{Supp}_R(M)$, $\dim(M_P) \ge \dim(R_P/Q^*R_P)$. The second assertion follows from (1) and (2).

PROOF OF (2.4). Let $\mathfrak{p}$, $\mathfrak{q} \in V_G(M)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$ and $P \in \mathrm{Ass}_R(R/\mathfrak{p})$. Then, by (2.2), (2), $P^* = \mathfrak{p}$ and $P \notin \mathrm{Ass}_R(R/\mathfrak{q})$. Thus, by (2.5), (1), there exists $Q \in \mathrm{Ass}_R(R/\mathfrak{q})$ such that $Q \subsetneq P$. Proceeding in this way, we have $\underline{\dim}(M_{(\mathfrak{p})}) \le \dim(M_P)$ for every $P \in \mathrm{Ass}_R(M)$. Conversely, let $P \in \mathrm{Ass}_R(R/\mathfrak{p})$ and $Q \in \mathrm{Supp}_R(M)$ such that $\dim(M_P) = \dim(R_P/QR_P)$. We put $n = \dim(M_P)$ and show that $\underline{\dim}(M_{(\mathfrak{p})}) \ge n$ by induction on n.

If $n = 0$, then $P = Q$ and $Q^* = \mathfrak{p}$ is a minimal element of $V_G(M)$. Thus $\underline{\dim}(M_{(\mathfrak{p})}) = 0$. Therefore we assume $n > 0$ and the statement holds for $n-1$. Since $n > 0$ and by (2.5), (3), $\mathfrak{p} \ne Q^*$ and there exists $a \in h(\mathfrak{p} \setminus Q^*)$. Then $\dim(R_P/(Q^*, a)R_P) = n-1$ by (2.5), (3). Thus, by induction hypothesis, $\underline{\dim}(R_{(\mathfrak{p})}/(Q^*, a)R_{(\mathfrak{p})}) \ge n-1$. Since $V_G(R/(Q^*, a)) \subset V_G(M)$ and $Q^* \subsetneq (Q^*, a)$, we have $\underline{\dim}(M_{(\mathfrak{p})}) \ge (n-1)+1 = n$. The proof is complete. □

COROLLARY 2.6. *Let M be a G-graded module over a Noetherian G-graded ring R and* $P \in \mathrm{Supp}_R(M)$. *Then* $\dim(M_P) = \underline{\dim}(M_{(P^*)}) + \dim(R_P/P^*R_P)$.

PROOF. We put $n = \dim(M_P)$, $m = \underline{\dim}(M_{(P^*)})$ and $r = \dim(R_P/P^*R_P)$. By (2.4), we have $n \ge m + r$. We show the converse inequality by induction on m.

If $m = 0$, then P^* is a minimal element of $V_G(M)$. Then, for every $Q \in \mathrm{Supp}_R(M)$ such that $Q \subset P$, $Q^* = P^*$ (cf. (2.5)). Thus $n \le r$. Suppose that $m > 0$. Let $Q \in \mathrm{Supp}_R(M)$ such that $\dim(M_P) = \dim(R_P/QR_P)$. Then $\dim(R_P/Q^*R_P) = n$ and $\underline{\dim}(R_{(P^*)}/Q^*R_{(P^*)}) \le m$. Since P^* is not minimal, there exists an element $a \in h(P^* \setminus Q^*)$ by (2.5), (3). Then $\underline{\dim}(R_{(P^*)}/(Q^*, a)R_{(P^*)}) < \underline{\dim}(R_{(P^*)}/Q^*R_{(P^*)})$ and, by induction hypothesis, $n-1 = \dim(R_P/(Q^*, a)R_P) \le \underline{\dim}(R_{(P^*)}/(Q^*, a)R_{(P^*)}) + r < m + r$. □

COROLLARY 2.7. *Let M be a G-graded module over a G-Noetherian graded ring R. Then* $\underline{\dim}(M_{(\mathfrak{p})})$ *is finite for every* $\mathfrak{p} \in V_G(M)$.

PROOF. It suffices to show the case $M = R$. Let $\mathfrak{p} \in V_G(R)$. After the homogeneous localization at $\mathfrak{p}$, we may assume that $(R, \mathfrak{p})$ is G-local. We denote by H the subgroup of G generated by the degrees of a finite system of homogeneous generators of $\mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{p}^{(H)}R$ and, by (1.4), $\underline{\dim}(R) = \underline{\dim}(R^{(H)})$. By (1.10), $R^{(H)}$ is Noetherian and, by (2.4), $\underline{\dim}(R^{(H)})$ is finite. □

Our next goal is to establish an equality similar to (2.4) (or (2.6)) for the Bass numbers of a G-graded module over a Noetherian G-graded ring.

Let R be a G-Noetherian graded ring. For G-graded R-modules M, N, we denote by $\underline{\mathrm{Hom}}_R(M, N)_g$ the Abelian group of all the G-graded homomorphisms from M to $N(g)$. We put $\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{g \in G} \underline{\mathrm{Hom}}_R(M, N)_g$ and consider it as a G-graded R-module. We denote by $\underline{\mathrm{Ext}}^i_R(-, -)$ the i-th derived functor of $\underline{\mathrm{Hom}}_R(-, -)$. *If* M is finitely generated, then $\underline{\mathrm{Ext}}^i_R(M, N) = Ext^i_R(M, N)$ as underlying R-modules, for every $i \geq 0$.

Since R is G-Noetherian, there exists injective hull of a G-graded R-module M in $M_G(R)$ uniquely determined by M. We denote it by $\underline{E}_R(M)$.

In their papers [5] and [6], Goto-Watanabe proved that some objects of a category of $\boldsymbol{Z}^n$-graded modules can be treated as the same as in the nongraded case. The following proposition is G-graded version of one of Goto-Watanabe's arguments (cf. chap. 1, §2 of [5]).

PROPOSITION 2.8. (1) *Let M be a G-graded R-module. Then* $\underline{\mathrm{Ass}}_R(M) = \underline{\mathrm{Ass}}_R(\underline{E}_R(M))$. *In particular,* $\mathrm{Ass}_R(M) = \mathrm{Ass}_R(\underline{E}_R(M))$, *if R is Noetherian.*

(2) *A G-graded R-module E is an indecomposable injective object of $M_G(R)$ if and only if $E \cong \underline{E}_R(R/\mathfrak{p})(g)$ for some $\mathfrak{p} \in V_G(R)$ and for some $g \in G$. In this case, $\mathfrak{p}$ is uniquely determined for E.*

(3) *Every injective object E of $M_G(R)$ can be decomposed into a direct sum of indecomposable injective objects of $M_G(R)$. This decomposition is uniquely determined by E up to isomorphisms.* □

Let M be a G-graded R-module and $\mathfrak{p}$ be a G-prime ideal of R. For $i \geq 0$, a G-graded R-module $\underline{\mathrm{Ext}}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})$ can be regarded as G-graded module over a G-simple graded ring $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$. Hence it is a free $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$-module (cf. (1.4)).

DEFINITION 2.9. We set

$$\nu^i(\mathfrak{p}, M) = \mathrm{rank}\, \underline{\mathrm{Ext}}^i_{R_{(\mathfrak{p})}}(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}, M_{(\mathfrak{p})})$$

and call it the i-th G-Bass number of M at $\mathfrak{p}$.

PROPOSITION 2.10. *Let M be a G-graded R-module. We denote by*

$$0 \to M \to \underline{E}^0_R(M) \to \cdots \to \underline{E}^i_R(M) \xrightarrow{d^i} \underline{E}^{i+1}_R(M) \to \cdots$$

the minimal injective resolution of M in $M_G(R)$. Then, for every G-prime graded ideal $\mathfrak{p}$ and for every integer $i \geq 0$, $\nu^i(\mathfrak{p}, M)$ is equal to the number of the G-graded R-module of the form $\underline{E}_R(R/\mathfrak{p})(g)$ $(g \in G)$ which appears in $\underline{E}^i_R(M)$ as direct summands.

The proof is the same as Theorem 1.3.4 of Goto-Watanabe [6]. □

Finally, we describe ordinary Bass numbers in terms of G-Bass numbers.

THEOREM 2.11. *Let M be a G-graded R-module and P be a prime ideal of R. We suppose that R is Noetherian and put $d=\dim(R_P/P^*R_P)$. Then*

$$\mu^i(P, M)=\begin{cases} \nu^{i-d}(P^*, M) & \text{if } i\geq d \\ 0 & \text{if } i<d \end{cases}$$

where $\mu^i(P, M)=\dim_{R_P/PR_P}(\mathrm{Ext}^i_{R_P}(R_P/PR_P, M_P))$ is the ordinary Bass number of M at P.

PROOF. After the homogeneous localization at P^*, we may assume that (R, P^*) is G-local and put $S=R/P^*$. We consider the following spectral sequence

$$E_2^{p,q}=\mathrm{Ext}^p_{S_P}(k(P), \mathrm{Ext}^q_{R_P}(S_P, M_P)) \Rightarrow \mathrm{Ext}^{p+q}_{R_P}(k(P), M_P)$$

where $k(P)=R_P/PR_P$. Note that $\mathrm{Ext}^q_{R_P}(S_P, M_P)\simeq \underline{\mathrm{Ext}}^q_R(S, M)_P\cong(S_P)^{\oplus\nu^q(P^*,M)}$ for every $q\geq 0$. We put $\nu^q(P^*, M)=0$ for $q<0$. Then we have $E_2^{p,q}=0$ for every $p\neq d$, since S_P is a d-dimensional Gorenstein ring (cf. (1.7)). Hence we have the following isomorphism

$$\begin{aligned}\mathrm{Ext}^{d+q}_{R_P}(k(P), M_P)&\cong \mathrm{Ext}^d_{S_P}(k(P), \mathrm{Ext}^q_{R_P}(S_P, M_P))\\ &\cong \mathrm{Ext}^d_{S_P}(k(P), S_P)^{\oplus\nu^q(P^*,M)}\\ &\cong k(P)^{\oplus\nu^q(P^*,M)}.\end{aligned}$$

Thus $\mu^i(P, M)=\nu^{i-d}(P^*, M)$ for all $i\geq 0$. □

COROLLARY 2.12. *Let M be a G-graded R-module and $\mathfrak{p}$ be a G-prime graded ideal of R. If R is Noetherian, then $\nu^i(\mathfrak{p}, M)=\mu^i(P, M)$ for every $P\in \mathrm{Ass}_R(R/\mathfrak{p})$ and for every $i\geq 0$.* □

As a consequence of (2.11) and (2.12), we have the following.

THEOREM 2.13. *Let M be a finitely generated G-graded R-module and $\mathfrak{p}\in V_G(R)$. If R is Noetherian, then the following conditions are equivalent.*

(1) *$M_{(\mathfrak{p})}$ is a Cohen-Macaulay (resp. Gorenstein) $R_{(\mathfrak{p})}$-module.*

(2) *M_P is a Cohen-Macaulay (resp. Gorenstein) R_P-module for every $P\in \mathrm{Ass}_R(R/\mathfrak{p})$.*

(3) *M_P is a Cohen-Macaulay (resp. Gorenstein) R_P-module for some $P\in \mathrm{Ass}_R(R/\mathfrak{p})$.*

(4) *There exists $P\in \mathrm{Spec}(R)$ such that $P^*=\mathfrak{p}$ and M_P is a Cohen-Macaulay (resp. Gorenstein) R_P-module.* □

DEFINITION 2.14. A G-Noetherian graded ring R is said to be G-Cohen-Macaulay graded ring, if $\nu^i(\mathfrak{m}, R)=0$ for every G-maximal ideal $\mathfrak{m}$ of R and every $i<\underline{\dim}(R_{(\mathfrak{m})})$.

A G-Noetherian graded ring R is said to be G-Gorenstein graded ring, if it satisfies the condition that, for every G-maximal ideal $\mathfrak{m}$, there exists an integer $n\geq 0$ such that $\nu^m(\mathfrak{m}, R)=0$ for every $m\geq n$.

COROLLARY 2.15. *Let R be a G-Noetherian graded ring.*

(1) *R is G-Cohen-Macaulay if and only if so is $R_{(\mathfrak{p})}$ for every $\mathfrak{p}\in V_G(R)$.*

(2) *The following are equivalent.*

(a) *R is G-Gorenstein.*

(b) *$R_{(\mathfrak{p})}$ is G-Gorenstein for every $\mathfrak{p}\in V_G(R)$.*

(c) *For every G-maximal ideal $\mathfrak{m}$ of R, $\nu^i(\mathfrak{m}, R)=\delta_{id}$ where $d=\underline{\dim}(R_{(\mathfrak{m})})$.*

(d) *For every G-prime ideal $\mathfrak{p}$ of R, $\nu^i(\mathfrak{p}, R)=\delta_{id}$ where $d=\underline{\dim}(R_{(\mathfrak{p})})$.*

PROOF. Let $\mathfrak{p}\in V_G(R)$. Then there exists a finitely generated subgroup H of G such that $\mathfrak{p}^{(H)}R=\mathfrak{p}$ (cf. (1.5)). Then, by (1.4), $R_{(\mathfrak{p})}$ is free over $(R^{(H)})_{(\mathfrak{p}^{(H)})}$ and $\nu^i(\mathfrak{p}, R)=\nu^i(\mathfrak{p}^{(H)}, R^{(H)})$. Hence our assertions follow from (1.10) and (2.13). □

COROLLARY 2.16. *Let $\mathfrak{p}$ be a G-prime graded ideal of R. If R is Noetherian, then a minimal injective resolution of $\underline{E}_R(R/\mathfrak{p})$ as the underlying R-module is of the form*

$$0\to\underline{E}_R(R/\mathfrak{p})\to\bigoplus_{P\in V^0(\mathfrak{p})}E_R(R/P)\to\bigoplus_{P\in V^1(\mathfrak{p})}E_R(R/P)\to\cdots\to\bigoplus_{P\in V^n(\mathfrak{p})}E_R(R/P)\to\cdots,$$

where $V^i(\mathfrak{p})=\{P\in\operatorname{Spec}(R)\mid P^*=\mathfrak{p},\ \dim(R_P/\mathfrak{p}R_P)=i\}$.

This is a direct consequence of (2.11).

COROLLARY 2.17. *Suppose that R is Noetherian and G is torsion. Then every injective object of $M_G(R)$ is an injective module as the underlying R-module.*

PROOF. By (1.7), $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is Artinian for every $\mathfrak{p}\in V_G(R)$. Thus $P\in\operatorname{Ass}_R(R/P^*)$ for every $P\in\operatorname{Spec}(R)$ and the assertion follows from (2.16). □

3. The canonical module of a G-Noetherian graded ring.

Let $(R, \mathfrak{m})$ be a G-local G-Noetherian graded ring of $d=\underline{\dim}(R)$. In this section, we define the canonical module of R and state some properties of this module.

For every G-graded R-module M and every integer $n\geq 0$, we put

$$\underline{H}^n_{\mathfrak{m}}(M)=\varinjlim \operatorname{Ext}^n_R(R/\mathfrak{m}^t, M)$$

and call it the n-th local cohomology module of M. Note that $\underline{H}^n_{\mathfrak{m}}(M)=H^n_{\mathfrak{m}}(M)$ as underlying R-modules.

REMARK 3.1. Let us recall the following basic properties of $\underline{H}^i_{\mathfrak{m}}(-)$ (cf. [7]).

(1) $\underline{H}^0_{\mathfrak{m}}(-)$ is a left exact covariant additive functor from $M_G(R)$ to $M_G(R)$ and $\underline{H}^n_{\mathfrak{m}}(-)$ is the n-th derived functor of $\underline{H}^0_{\mathfrak{m}}(-)$.

(2) *Let* $\mathfrak{q}$ be a G-graded ideal of R such that $\sqrt{\mathfrak{q}}=\sqrt{\mathfrak{m}}$. Then, for every $n\geq 0$, there is a natural isomorphism $\underline{H}^n_{\mathfrak{q}}(-)=\underline{H}^n_{\mathfrak{m}}(-)$ of functors.

(3) Let $\varphi: R\to S$ be a ring homomorphism of G-Noetherian graded rings. Then there is a natural isomorphism $\underline{H}^n_{\mathfrak{m}}([-]_\varphi)\cong[\underline{H}^n_{\mathfrak{m}S}(-)]_\varphi$ of functors where $[M]_\varphi=M$, regarded as a G-graded R-module via φ for a G-graded S-module M.

We define a G-graded S-module structure of $\underline{H}^n_{\mathfrak{m}}([M]_\varphi)$, for a G-graded S-module M, in the following way.

Let $a \in S$. The multiplication $M \xrightarrow{a} M$ can be regarded as the R-linear map. Then we have an R-linear map $H^n_{\mathfrak{m}}(a)$: $H^n_{\mathfrak{m}}(M) \to H^n_{\mathfrak{m}}(M)$. We define the S-module structure of $H^n_{\mathfrak{m}}(M)$ by $ax = H^n_{\mathfrak{m}}(a)(x)$ for $x \in H_{\mathfrak{m}}(M)$. In particular, if $a \in S_g$, then an R-linear map $\underline{H}^n_{\mathfrak{m}}(a)$: $\underline{H}^n_{\mathfrak{m}}(M) \to H^n_{\mathfrak{m}}(M)(g)$ preserves the G-grading. Thus, since $\underline{H}^n_{\mathfrak{m}}(a) = H^n_{\mathfrak{m}}(a)$ and $\underline{H}^n_{\mathfrak{m}}(M) = H^n_{\mathfrak{m}}(M)$ as the underlying R-module, $\underline{H}^n_{\mathfrak{m}}(M)$ can be regarded as G-graded S-module. Hence, by naturality of the isomorphism in (3.1), (3), we have $\underline{H}^n_{\mathfrak{m}}(M) \cong \underline{H}^n_{\mathfrak{m}S}(M)$ as G-graded S-modules.

Proposition 3.2. *Let H be a subgroup of G with a system $\{g_i\}_{i \in I}$ of representatives of G mod H such that $\sqrt{\mathfrak{m}^{(H)}R} = \sqrt{\mathfrak{m}}$ and M be a G-graded R-module. Then, for every $n \geq 0$, we have*

$$\underline{H}^n_{\mathfrak{m}}(M) \cong \bigoplus_{i \in I} \underline{H}^n_{\mathfrak{m}^{(H)}}(M^{(g_i, H)}) \qquad \text{as } G\text{-graded } R\text{-modules, and}$$

$$\underline{H}^n_{\mathfrak{m}^{(H)}}(M^{(g_i, H)}) \cong \underline{H}^n_{\mathfrak{m}}(M)^{(g_i, H)} \qquad \text{as } G\text{-graded } R^{(H)}\text{-modules}.$$

In particular, $\underline{H}^n_{\mathfrak{m}^{(H)}}(R^{(H)}) \cong \underline{H}^n_{\mathfrak{m}}(R)^{(H)}$.

Proof. Apply (3.1), (3) to $R^{(H)} \hookrightarrow R$. □

Remark 3.3. For a subgroup $H \subset G$, if G/H is torsion, then $\sqrt{\mathfrak{m}^{(H)}R} = \sqrt{\mathfrak{m}}$.

Corollary 3.4. *If G is torsion, then $\underline{H}^n_{\mathfrak{m}}(M) \cong \bigoplus_{g \in G} \underline{H}^n_{\mathfrak{m}_0}(M_g)$, for every G-graded R-module M and every $n \geq 0$.* □

Corollary 3.5. $\underline{\dim}(R) = \sup\{n \mid \underline{H}^n_{\mathfrak{m}}(R) \neq 0)\}$ *and* $\operatorname{grade}(\mathfrak{m}, R) = \inf\{n \mid \underline{H}^n_{\mathfrak{m}}(R) \neq (0)\}$.

Proof. Since R is G-Noetherian, there exists a finitely generated subgroup H of G such that $\mathfrak{m}^{(H)}R = \mathfrak{m}$. Then $R^{(g,H)} = 0$ or $R^{(g,H)} \cong R^{(H)}$ for $g \in G$ (cf. (1.4)), and $\underline{H}^n_{\mathfrak{m}}(R) \neq (0)$ if and only if $\underline{H}^n_{\mathfrak{m}^{(H)}}(R^{(H)}) \neq (0)$. Thus we may assume that G is finitely generated. In this case, R is Noetherian (cf. (1.10)). Since $\otimes_R R_{\mathfrak{m}}$ is a faithfully flat functor on $M_G(R)$, the assertion follows from (2.4) and (2.12) (where $R_{\mathfrak{m}}$ is a ring of fractions with respect to the multiplicatively closed subset $R \setminus \bigcup_{P \in \operatorname{Ass}_R(R/\mathfrak{m})} P$). □

Corollary 3.6. *R is G-Cohen-Macaulay if and only if $\underline{H}^n_{\mathfrak{m}}(R) = (0)$ for every $n \neq d$. In particular, if G is torsion, then R is G-Cohen-Macaulay if and only if R_g is a Cohen-Macaulay R_0-module of dimension d for every $g \in G$.* □

Next, we state Matlis duality theorem for G-graded R-modules. The proof is similar to the nongraded case (cf. chap. 1, §2 of Goto-Watanabe [5]).

R is said to be G-complete, if $(R_0, \mathfrak{m}_0)$ is a complete local ring.

Proposition 3.7. *Suppose that $(R, \mathfrak{m})$ is G-complete. We denote by $M^\vee$ the G-graded R-module $\underline{\operatorname{Hom}}_{R_0}(M, E_{R_0}(R_0/\mathfrak{m}_0))$.*

(1) $(-)^\vee : M_G(R) \to M_G(R)$ *is a contravariant, faithfull, exact, additive functor.*

(2) *For every finitely generated G-graded R-module M,* $M^{\vee\vee} \cong M$.

(3) $R^{\vee} \cong \underline{E}_R(R/\mathfrak{m})$.

(4) *For every G-graded R-module M,* $M^{\vee} \cong \underline{\mathrm{Hom}}_R(M, R^{\vee})$.

(5) *A G-graded R-module M is G-Artinian if and only if there exist* $g_1, \cdots, g_n \in G$ *such that* $M \hookrightarrow \bigoplus_{i=1}^{n} R^{\vee}(g_i)$. (*We call M G-Artinian if it satisfies DCC for G-graded submodules.*)

(6) *If we denote by* $\mathscr{F}$ (*resp.* $\mathscr{A}$) *the full subcategory consisting of all finitely generated G-graded R-modules* (*resp. G-Artinian modules*) *of* $M_G(R)$, *then*

(a) *for* $M \in \mathscr{F}$ *and* $N \in \mathscr{A}$, $M^{\vee} \in \mathscr{A}$ *and* $N^{\vee} \in \mathscr{F}$,

(b) *the functor* $(-)^{\vee} : \mathscr{F} \to \mathscr{A}$ *establishes an anti-equivalence.* □

For a G-graded R-module M, we set $\hat{M} = M \otimes_{R_0} \hat{R}_0$.

DEFINITION 3.8. We call a G-graded R-module $\underline{K}_R$ a G-canonical module of R, if $(\underline{K}_R)^{\wedge} \cong \underline{H}^d_{\hat{\mathfrak{m}}}(\hat{R})^{\vee}$.

Using our previous results, we can show the following (cf. chap.2, §1 and §2 of Goto-Watanabe [5]).

PROPOSITION 3.9. (1) *If a G-canonical module* $\underline{K}_R$ *of R exists, then* $\underline{K}_R$ *is a finitely generated R-module and uniquely determined up to isomorphism.*

(2) *If* $(R, \mathfrak{m})$ *is G-complete, then* $\underline{H}^d_{\mathfrak{m}}(M)^{\vee} \cong \underline{\mathrm{Hom}}_R(M, \underline{K}_R)$ *for every finitely generated G-graded R-module M.*

(3) *If* $(R, \mathfrak{m})$ *is G-complete and* $\underline{H}^{d-n}_{\mathfrak{m}}(R) = 0$ *for* $0 < n \leq s$, *then* $\underline{H}^{d-n}_{\mathfrak{m}}(M)^{\vee} \cong \underline{\mathrm{Ext}}^n_R(M, \underline{K}_R)$ *for every finitely generated G-graded R-module M and for every* $0 \leq n \leq s$.

(4) *Let* $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a homomorphism of G-local graded ring such that* $\varphi(\mathfrak{m}) \subset \mathfrak{n}$ *and S is finitely generated as R-module. We put* $t = \underline{\dim}(R) - \underline{\dim}(S)$. *Suppose that* $\underline{H}^{d-n}_{\mathfrak{m}}(R) = 0$ *for* $0 < n \leq d - t$ *and there exists a G-canonical module* $\underline{K}_R$ *of R. Then there exists a G-canonical module* $\underline{K}_S$ *of S and* $\underline{K}_S \cong \underline{\mathrm{Ext}}^t_R(S, \underline{K}_R)$.

(5) *If* $(R, \mathfrak{m})$ *is G-Cohen-Macaulay and if* $\underline{K}_R$ *exists, then, for a nonzero divisor* $a \in R_g$ $(g \in G)$, $\underline{K}_{R/aR} \cong (\underline{K}_R / a\underline{K}_R)(g)$.

(6) *If* $(R, \mathfrak{m})$ *is G-Cohen-Macaulay and if* $\underline{K}_R$ *exists, then* $\nu^n(\mathfrak{m}, \underline{K}_R) = \delta_{id}$ *and the minimal number of homogeneous generators of* $\underline{K}_R$ *is equal to* $\nu^d(\mathfrak{m}, R)$.

(7) *The following conditions are equivalent.*

(a) *R is G-Gorenstein.*

(b) *R is G-Cohen-Macaulay and there exists a G-canonical module* $\underline{K}_R$ *of R such that* $\underline{K}_R \cong R(g)$ *for some* $g \in G$.

(8) *If R is a homomorphic image of a G-Gorenstein G-local graded ring* $(S, \mathfrak{n})$, *then there exists a G-canonical module* $\underline{K}_R$ *of R and* $\underline{K}_R \cong \underline{\mathrm{Ext}}^t_S(R, S)(g)$ *where* $t = \underline{\dim}(S) - \underline{\dim}(R)$. □

THEOREM 3.10. *Let H be a subgroup of G such that* $\sqrt{\mathfrak{m}^{(H)}R} = \sqrt{\mathfrak{m}}$.

(1) *If* $(R, \mathfrak{m})$ *is G-complete, then* $\underline{K}_R \cong \underline{\mathrm{Hom}}_{R^{(H)}}(R, \underline{K}_{R^{(H)}})$ *as G-graded R-modules.*
(2) *Then the following conditions are equivalent.*
(a) *There exists a G-canonical module* $\underline{K}_R$ *of R.*
(b) *There exists a G-canonical module* $\underline{K}_{R^{(H)}}$ *of* $R^{(H)}$.
In this case, we have

$$\underline{K}_R \cong \underline{\mathrm{Hom}}_{R^{(H)}}(R, \underline{K}_{R^{(H)}}) \quad \text{as } G\text{-graded } R\text{-modules}, \quad \text{and}$$

$$\underline{\mathrm{Hom}}_{R^{(H)}}(R^{(-g_i, H)}, \underline{K}_{R^{(H)}}) \cong (\underline{K}_R)^{(g_i, H)} \quad \text{as } G\text{-graded } R^{(H)}\text{-modules}$$

where $\{g_i\}_{i\in I}$ *is a system of representatives of G* mod *H. In particular,* $\underline{K}_{R^{(H)}} \cong (\underline{K}_R)^{(H)}$.

PROOF. (1) By (3.2) and (3.9), (2), there is the following isomorphism of G-graded R-modules:

$$\underline{\mathrm{Hom}}_{R^{(H)}}(R, \underline{K}_{R^{(H)}}) = \bigoplus_{i\in I} \underline{\mathrm{Hom}}_{R^{(H)}}(R^{(-g_i, H)}, \underline{K}_{R^{(H)}}) \cong \bigoplus_{i\in I} \underline{H}^d_{\mathfrak{m}^{(H)}}(R^{(-g_i, H)})^\vee = \underline{H}_{\mathfrak{m}}(R)^\vee .$$

(Note that it is not necessary $\underline{\mathrm{Hom}}_{R^{(H)}}(R, \underline{K}_{R^{(H)}}) = \mathrm{Hom}_{R^{(H)}}(R, K_{R^{(H)}})$.)
The assertion (2) follows from (1). □

COROLLARY 3.11. *If* R_0 *is a homomorphic image of a Gorenstein local ring, then there exists a G-canonical module* $\underline{K}_R$ *of R.*

PROOF. There exists a finitely generated subgroup H of G such that $\sqrt{\mathfrak{m}^{(H)}R} = \sqrt{\mathfrak{m}}$ (cf. (1.5)). Hence, by (3.10), we may assume that G is finitely generated. In this case, R is a finitely generated R_0-algebra by (1.10) and it is a homomorphic image of a polynomial ring S over a Gorenstein local ring R_0. Note that the G-grading on R induces a G-grading on S. (It is not necessary $S_0 = R_0$.) Then R is also homomorphic image of the Gorenstein G-local ring and the assertion follows from (3.9), (8). □

Until the end of this section, we assume that $(R_0, \mathfrak{m}_0)$ is a homomorphic image of a Gorenstein local ring.

We can show that $\underline{K}_R$ is actually a canonical module of R in usual sense.

COROLLARY 3.12. *If R is Noetherian, then* $(\underline{K}_R)_P \cong K_{(R_P)}$ *for every* $P \in \mathrm{Supp}_R(\underline{K}_R)$.

PROOF. We shall prove the assertion in the following steps.

Step (1) If G is finitely generated, then the assertion follows from (3.9). If G is not finitely generated, we need a sublemma.

SUBLEMMA. We denote $A = R_0$. Assume that $\mathfrak{m}_0 R = \mathfrak{m}$ and $\mathfrak{m} \in \mathrm{Spec}(R)$. Then we have $(\underline{K}_R)_{\mathfrak{m}} \cong K_{R_{\mathfrak{m}}}$.

PROOF OF SUBLEMMA. For every finite G-graded R-module M, the $\mathfrak{m}$-adic completion of M is equal to $\hat{M} = M \otimes_A \hat{A}$ by our assumption. Thus $(R_{\mathfrak{m}})^\wedge \cong (R \otimes_A \hat{A})_{\mathfrak{m}}$ and it is a local ring. This implies that $E_{(R_{\mathfrak{m}})^\wedge}((R\mathfrak{m})^\wedge / \mathfrak{m}(R\mathfrak{m})^\wedge) \cong \underline{E}_{\hat{R}}(\hat{R}/\mathfrak{m}\hat{R})_{\mathfrak{m}}$ (cf. (2.16)).

Hence we have the following isomorphism

$$
\begin{aligned}
[(\underline{K}_R)_{\mathfrak{m}}]^{\wedge} &\cong [[(\underline{K}_R)_{\mathfrak{m}}]^{\wedge}]^{\vee\vee} \\
&\cong [\mathrm{Hom}_{(R_{\mathfrak{m}})^{\wedge}}([(\underline{K}_R)_{\mathfrak{m}}]^{\wedge}, \quad E_{(R_{\mathfrak{m}})^{\wedge}}((R_{\mathfrak{m}})^{\wedge}/\mathfrak{m}(R_{\mathfrak{m}})^{\wedge}))]^{\vee} \\
&\cong [\mathrm{Hom}_{(\hat{R})_{\mathfrak{m}}}((\underline{K}_{\hat{R}})_{\mathfrak{m}} \underline{E}_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}})_{\mathfrak{m}})]^{\vee} \\
&\cong [\underline{\mathrm{Hom}}_{\hat{R}}(\underline{K}_{\hat{R}}, \underline{E}_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}}))_{\mathfrak{m}}]^{\vee} \\
&\cong [\underline{H}^{d}_{\hat{\mathfrak{m}}}(\hat{R})_{\mathfrak{m}}]^{\vee} \\
&\cong H^{d}_{\mathfrak{m}(R_{\mathfrak{m}})^{\wedge}}((R_{\mathfrak{m}})^{\wedge})^{\vee} .
\end{aligned}
$$

Hence $(\underline{K}_R)_{\mathfrak{m}} \cong K_{R_{\mathfrak{m}}}$. We complete the proof of Sublemma.

Step (2) Let $P \in \mathrm{Supp}_R(\underline{K}_R)$. Since P is finitely generated, there exists a finitely generated subgroup H of G such that $(P \cap R^{(H)})R = P$ (cf. the proof of (1.7)). Let $\{g_i\}_{i \in I}$ be a system of representatives of G mod H and $\mathfrak{p} = P \cap R^{(H)}$. We consider the G/H-graded ring $R_{\mathfrak{p}} = \bigoplus_{i \in I}(R^{(g_i, H)})_{\mathfrak{p}}$. Then, by Step (1), $K_{(R^{(H)})_{\mathfrak{p}}} = (\underline{K}_{R^{(H)}})_{\mathfrak{p}} = [(\underline{K}_R)^{(H)}]_{\mathfrak{p}}$ and, by (3.10), $[\underline{K}_R]_{\mathfrak{p}}$ is a G/H-canonical module of $R_{\mathfrak{p}}$. On the other hand, $(R_{\mathfrak{p}}, PR_{\mathfrak{p}})$ is G/H-local such that $\mathfrak{p}R_{\mathfrak{p}} = PR_{\mathfrak{p}}$ and $PR_{\mathfrak{p}} \in \mathrm{Spec}(R_{\mathfrak{p}})$ by the choice of H. Hence, by the Sublemma, we have $(\underline{K}_R)_P \cong [(\underline{K}_R)_{\mathfrak{p}}]_P \cong (\underline{K}_{R_{\mathfrak{p}}})_{PR_{\mathfrak{p}}} \cong K_{R_P}$. □

COROLLARY 3.13. (1) $(\underline{K}_R)_{(\mathfrak{p})} \cong \underline{K}_{R_{(\mathfrak{p})}}$ *for every* $\mathfrak{p} \in V_G(\underline{K}_R)$.

(2) $\underline{\mathrm{Ass}}_R(\underline{K}_R) = \{\mathfrak{p} \in V_G(R) \mid \underline{\dim}(R/\mathfrak{p}) = d\}$.

(3) $R \cong \underline{\mathrm{Hom}}_R(\underline{K}_R, \underline{K}_R)$ *if and only if* $\mathrm{grade}(\mathfrak{p}R_{(\mathfrak{p})}, R_{(\mathfrak{p})}) \geq \inf\{2, \underline{\dim}(R_{(\mathfrak{p})})\}$ *for every* $\mathfrak{p} \in V_G(\underline{K}_R)$.

PROOF. We can reduce to the case where G is finitely generated (cf. (1.4) and (2.15)). In this case, the proof is similar to the nongraded case. □

EXAMPLE 3.4. Let $(A, \mathfrak{m})$ be a Noetherian local normal domain with $K = Q(A)$ and L be a finite Abelian extension of K with $G = Gal(L/K)$. Let R be the integral closure of A in L and $\hat{G} = \mathrm{Hom}(G, U(A))$, where $U(A)$ is the multiplicative group of units of A. Assume that $n = |G| \in U(A)$ and A contains a primitive n-th root of unity. Then R can be regarded as $\hat{G}$-graded ring in the following sense. For $g \in \hat{G}$, we set $R_g = \{a \in R \mid \sigma(a) = g(\sigma)a \text{ for } {}^{\forall}\sigma \in G\}$. Then

(1) $R_0 = R^G = A$.

(2) $R_g R_h \subset R_{g+h}$ for every $g, h \in \hat{G}$.

(3) $R = \sum_{g \in \hat{G}} R_g = \bigoplus_{g \in \hat{G}} R_g$.

(See §2 of Itoh [9].)

Assume that A is UFD. Since R_g is isomorphic to a divisorial ideal of A, there exists $e_g \in R_g$ such that $R_g = Ae_g \cong A(g)$. Hence, by (3.6), A is Cohen-Macaulay if and only if so is R (Theorem of Roberts [15] and Corollary 3 of Itoh [9]).

We denote by $a(g, g')$ an element of A satisfying $e_g e_{g'} = a(g, g')e_{g+g'}$ for $g, g' \in \hat{G}$. Then $\underline{\mathrm{Hom}}_A(R, A) \cong R(g)$ $(g \in \hat{G})$ as G-graded R-module if and only if $a(g'+g, g'') =$

$a(-g'-g'', g'')$ for any $g', g'' \in \hat{G}$. Hence, by (3.9), R is Gorenstein if and only if A is Gorenstein and there exists $g \in \hat{G}$ such that, for any $g', g'' \in \hat{G}$, $a(g'+g, g'')= a(-g'-g'', g'')$.

4. A criterion.

In this paragraph, we consider a condition for a G-prime ideal to be a prime ideal. First, we show the following lemma.

LEMMA 4.1. *Let R be a G-graded ring and $\mathfrak{p} \in V_G(R)$. Then the following are euivalent.*

(1) *$\mathfrak{p}$ is a prime (resp. radical) ideal.*

(2) *$R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is an integral domain (resp. reduced).*

(3) *For every finitely generated subgroup $H \subset G$, $(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})^{(H)}$ is an integral domain (resp. reduced).*

(4) *For every finite subgroup $H \subset G$, $(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})^{(H)}$ is an integral domain (resp. reduced).*

PROOF. Implications $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are trivial and $(4) \Rightarrow (3)$ follows from (1.6).

$(3) \Rightarrow (2)$ Suppose that $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ is not an integral domain (resp. reduced). Let $x, y \in R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$ (resp. $z \in R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$) such that $xy=0$ (resp. $z^n=0$). Then there exists a finitely generated subgroup $H \subset G$ such that $x, y \in (R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})^{(H)}$ (resp. $z \in (R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})^{(H)}$) (cf. the proof of (1.7)). Hence $(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})^{(H)}$ is not an integral domain (resp. reduced). □

Therefore, we consider a simple graded ring R graded by a finite Abelian group. Then, by the proof of (1.6), R is isomorphic to $k[X_1, \cdots, X_m]/(X_1^{q_1}-u_1, \cdots, X_m^{q_m}-u_m)$ where $m \geq 0$, $X_1, \cdots, X_m$ are variables and each $q_1, \cdots, q_m$ is a power of a prime number.

PROPOSITION 4.2. *Let $R \cong k[X_1, \cdots, X_m]/(X_1^{q_1}-u_1, \cdots, X_m^{q_m}-u_m)$.*

(1) *R is an integral domain if and only if it satisfies the following condition* (D):

(D): *for every $1 \leq t \leq m$, $(u_t)^{1/p} \notin k[X_1, \cdots, X_{t-1}]/(X_1^{q_1}-u_1, \cdots, X_{t-1}^{q_{t-1}}-u_{t-1})$, furthermore, when $char(k) \neq 2$ and q_t is divisible by 4, $(-u_t/4)^{1/4} \notin k[X_1, \cdots, X_{t-1}]/(X_1^{q_1}-u_1, \cdots, X_{t-1}^{q_{t-1}}-u_{t-1})$.*

(2) *R is reduced if and only if it satisfies the following condition* (R):

(R): *if $char(k)=p>0$ and $\{q_{i_1}, \cdots, q_{i_t}\}=\{q_i \mid 1 \leq i \leq m, p \mid q_i\}$ then $(u_{i_s})^{1/p} \notin k[X_{i_1}, \cdots, X_{i_{s-1}}]/(X_{i_1}^{q_{i_1}}-u_{i_1}, \cdots, X_{i_{s-1}}^{q_{i_{s-1}}}-u_{i_{s-1}})$ for every $1 \leq s \leq t$.*

PROOF. (1) The assertion follows from the following fact.

(Lang, Theorem 16, §9, ch. VIII of [10]) Let K be a field and $a \in K^*$. For a prime number p and an integer $n>0$, the polynomial $X^{p^n}-a \in K[X]$ is irreducible over K if and only if $a^{1/p} \notin K$ and, furthermore, $(-a/4)^{1/4} \notin K$, $char(K) \neq 2$ and $4|p^n$.

(2) Clearly, if R does not satisfy condition (R), then it is not reduced. We will show the converse. Suppose R satisfies condition (R). If $char(k)=p>0$ and p divides $q_{i_1}, \cdots, q_{i_t}$, then, by (1), $k[X_{i_1}, \cdots, X_{i_t}]/(X_{i_1}^{q_{i_1}}-u_{i_1}, \cdots, X_{i_t}^{q_{i_t}}-u_{i_t})$ is a field. Hence we may assume that p does not divide $q_1, \cdots, q_m$, if $char(k)=p>0$.

We put $A_0=k$ and $A_i=k[X_1, \cdots, X_i]/(X_1^{q_1}-u_1, \cdots, X_i^{q_i}-u_i)$ for $1 \leq i \leq m$. We show that if A_i is reduced then so is A_{i+1} $(i<m)$.

Since A_i is Artinian, $(A_i)_P$ is a field for every $P \in \mathrm{Max}(A_i)$, and $A_i \cong \bigoplus_{P \in \mathrm{Max}(A_i)}(A_i)_P$. Thus $A_{i+1}=A_i[X_{i+1}]/(X_{i+1}^{q_{i+1}}-u_{i+1}) \cong \bigoplus_{P \in \mathrm{Max}(A_i)}(A_i)_P[X_{i+1}]/(X_{i+1}^{q_{i+1}}-u_{i+1})$. Hence it suffices to show that $(A_i)_P[X_{i+1}]/(X_{i+1}^{q_{i+1}}-u_{i+1})$ is reduced for every $P \in \mathrm{Max}(A_i)$. Since $char(k)=char((A_i)_P)$, q_{i+1} is not a multiple of $char((A_i)_P)$, if $char((A_i)_P)>0$. Thus the splitting field of $X_{i+1}^{q_{i+1}}-u_{i+1}$ over $(A_i)_P$ is a separable extension of $(A_i)_P$. This implies that $(A_i)_P[X_{i+1}]/(X_{i+1}^{q_{i+1}}-u_{i+1})$ is reduced and the proof is complete. □

Combining (4.1) and (4.2), we have the following.

THEOREM 4.3. *Let* $\mathfrak{p}$ *be a G-prime ideal of a G-graded ring R. Then* $\mathfrak{p}$ *is a prime (resp. radical) ideal if and only if* $(R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})})^{(H)}$ *satisfies condition* (D) (*resp.* (R)) *for every finite subgroup* $H \subset G$.

COROLLARY 4.4 (chap. III, §1, no. 4 of Bourbaki [3]). *If G is torsion free, then every G-prime ideal is a prime ideal.*

COROLLARY 4.5. *Let R be a G-graded ring such that* R_0 *contains a field k. Suppose that either* $char(k)=0$ *or* $char(k)=p>0$ *and G does not have a torsion of order p. Then every G-prime ideal is a radical ideal.* □

EXAMPLE 4.6. In Example (3.14), every G-prime ideal of R is a radical ideal and, thus the ramification index is determined by G-prime ideals.

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Present Address:
Department of Mathematics, Massey University,
Private Bag 11222, Palmerston North, New Zealand.
e-mail: Y.Kamoi@massey.ac.nz