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Zeta Function and Perron-Frobenius Operator of Piecewise Linear Transformations on R^k

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§1. Introduction.

In [7], we considered a piecewise linear transformation F on R^k into itself. Roughly speaking, we determined the eigenvalues of the Perron-Frobenius operator P corresponding to the transformation F by the zeros of the Fredholm determinant $\det(I-\Phi_n(z))$ (the definition of the Fredholm matrix $\Phi_n(z)$ will be given in the next section).

We denote by J^o, J^{cl} and $\Delta J = J^{cl}\setminus J^o$ be the inner, the closure and the boundary of a set J. Set

$$\xi = \liminf_{n\to\infty} \operatorname*{ess\,inf}_{x\in I} \frac{1}{n} \log|\det D(F^n)(x)|\,,$$

$$\lambda = \limsup_{n\to\infty} \sup_J \frac{1}{n} \log \#\{w\in\mathscr{W} : |w|=n,\ \langle w\rangle \cap \Delta J \neq \emptyset\}\,,$$

where $D(F^n)$ is the jacobian matrix of F^n, $\sup_J$ is the supremum over all possible convex polyhedrons J, $\mathscr{W}$ is a set of words, and $\langle w\rangle$ is a parallelopiped which corresponds to a word $w\in\mathscr{W}$ (see §2 for precise definitions). We say that F is expanding if

$$\liminf_{n\to\infty} \frac{1}{n} \operatorname*{ess\,inf}_{x\in I} \log \min|\text{the eigenvalue of } D(F^n)(x)| > 0\,.$$

Our theorem in [7] is:

THEOREM A. *Assume that F is expanding and $\xi>\lambda$.*

(i) *Then for any $\varepsilon>0$, there exists an integer n_0 and for $n\geqslant n_0$ and $|z|<e^{\xi-\lambda-\varepsilon}$, z^{-1} belongs to the spectrum of the Perron-Frobenius operator P restricted to $\mathscr{B}$ if and only if*

$$\det(I-\Phi_n(z))=0\,.$$

(ii) *The eigenfunctions of P on L^1 associated with eigenvalues modulus* 1 *belong to $\mathscr{B}$.*

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Here a family of functions $\mathscr{B}$ is defined by:

DEFINITION. Let $\mathscr{B}$ be the set of functions $f \in L^1$ for which there exists $\{C_w\}_{w \in \mathscr{W}}$ such that $f_N(x) = \sum_{|w| \le N} C_w 1_w(x)$ converges to f in L^1 and that

$$\sum_{n=1}^{\infty} e^{-\lambda' n} \sum_{|w|=n} |C_w| < \infty$$

for any $\lambda' > \lambda$.

Set for $f \in \mathscr{B}$ and $\lambda' > \lambda$

$$\|f\|_{\lambda'} = \inf \sum_{n=1}^{\infty} e^{-\lambda' n} \sum_{|w|=n} |C_w| ,$$

where inf is taken over all $\{C_w\}_{w \in \mathscr{W}}$ which satisfy the above condition.

By the norms $\| \cdot \|_{\lambda'}$, $\mathscr{B}$ becomes a locally convex linear space. This is an extension of BV, the set of functions with bounded variations, in one-dimensional cases, since for $f \in BV$ there exists $\{C_w\}_{w \in \mathscr{W}}$ such that $\sum_{n=1}^{\infty} r^n \sum_{|w|=n} |C_w| < \infty$ for any $0 < r < 1$. Note that $\mathscr{B} \supset BV$ in one dimensional case.

We will explain the transformation which we will consider in this article. Let $v_1, \cdots, v_k$ be independent vectors in $\boldsymbol{R}^k$. Let $\mathscr{A}$ be a finite set, and for each element $a \in \mathscr{A}$, which we call an alphabet, there exists a parallelopiped $\langle a \rangle$, that is, for each $a \in \mathscr{A}$ there correspond a vector p^a and constants $\alpha_1^a, \cdots, \alpha_k^a > 0$, and each $\langle a \rangle$ satisfies

$$\langle a \rangle^o = \left\{ p^a + \sum_{i=1}^{k} x_i \alpha_i^a v_i :\ 0 < x_i < 1 \right\} ,$$

$$\langle a \rangle^{cl} = \left\{ p^a + \sum_{i=1}^{k} x_i \alpha_i^a v_i :\ 0 \le x_i \le 1 \right\} ,$$

$$\langle a \rangle \cap \langle b \rangle = \varnothing \qquad (a \ne b) .$$

The piecewise linear transformation F, which we will consider in this article, is a mapping from $I = \bigcup_{a \in \mathscr{A}} \langle a \rangle$ into itself, for which there exist matrices M^a and vectors q^a $(a \in \mathscr{A})$ and the restriction F^a of F to $\langle a \rangle$ satisfies

$$F^a(x) = M^a(x - p^a) + q^a ,$$

and $\det M^a \ne 0$ for each $a \in \mathscr{A}$. Moreover, we assume

$$M^a v_i = \lambda_i^a v_i ,$$

for some constants λ_i^a, that is, v_i's are eigenvectors.

In this article, we will consider the zeta function

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{p:\, p = F^n(p)} |\det D(F^n)(p)|^{-1} \right) .$$

We will prove as in one-dimensional cases ([5], [6]):

THEOREM B. *Assume that F is expanding and $\xi > \lambda$. Then the zeta function $\zeta(z)$ satisfies* $\zeta(z) = \det(I - \Phi_n(z))^{-1}$.

Combining the results, we can show that the eigenvalues of the Perron-Frobenius operator restricted to $\mathscr{B}$ in the domain $|z| < e^{\xi - \lambda}$ are determined by the singularities of the zeta function.

To make the notations simple, we consider a transformation F which has no periodic points on $\bigcup_{a \in \mathscr{A}} \Delta\langle a\rangle$. In [7], we considered a wider class of piecewise linear transformation, for which each $\langle a\rangle$ is a polyhedron and we do not assume that each v_i is an eigenvector ($1 \le i \le k$), that is, F^a may have a rotation. As we will discuss in §3, to consider the zeta function, this general class seems too large.

2. Preliminaries.

We will summarize the notations. A finite sequence of alphabets $w = a_1 \cdots a_n$ is called a word. We denote the set of words by $\mathscr{W}$ and for a word $w = a_1 \cdots a_n$, we denote

1. $|w| = n$ (the length of w; for the empty word ε, we put $|\varepsilon| = 0$),
2. $\langle w\rangle = \begin{cases} \bigcap_{i=1}^n F^{-i+1}(\langle a_i\rangle) & \text{if } w \neq \varepsilon\,, \\ I & \text{if } \; w = \varepsilon\,, \end{cases}$
3. $w[k] = a_k$ for $1 \le k \le n$,
4. $w[k, l] = a_k \cdots a_l$ for $1 \le k \le l \le n$,
5. $F^w = F^{a_n} \cdots F^{a_1}$, ($F^{\varnothing}$ = identity map),
6. $\eta(w) = \prod_{i=1}^n |\det M^{a_i}|^{-1}$.

We denote $\mathscr{W}_n$ the set of words with length n and $\langle w\rangle \neq \varnothing$.
We define $a_1^x a_2^x \cdots (a_i^x \in \mathscr{A})$, the expansion of $x \in I$, $F^{i-1}(x) \in \langle a_i^x\rangle$, where

$$F^i(x) = \begin{cases} x & i = 0\,, \\ F(F^{i-1}(x)) & i \ge 1\,. \end{cases}$$

REMARK. For each $a \in \mathscr{A}$, the mapping F^a can be extended to $\boldsymbol{R}^k$ and we can define $(F^a)^{-1}$ from I into $\boldsymbol{R}^k$. Therefore, for any $x \in I$, we can define $(F^a)^{-1}(x) = ax \in \boldsymbol{R}^k$. In a same way, for any word w, we can define $wx \in \boldsymbol{R}^k$. If $wx \in \langle w\rangle$, then, of course, $F^{|w|}(wx) = x$, that is, the expansion of wx equals $wa_1^x a_2^x \cdots$. For this case, we say that wx exists.

In this article, we only consider parallelopiped J for which there exist vector p^J and $\alpha_1^J, \cdots, \alpha_k^J > 0$ and

$$J^o = \left\{ p^J + \sum_{i=1}^k x_i \alpha_i^J v_i :\; 0 < x_i < 1 \right\}.$$

We considered in [7] the signed symbolic dynamics with the set of faces $D = \bigcup_{p=0}^{k-1} D_p$.

For each face $\partial \in D_p$, there correspond a set of integers $1 \leq j_1 < \cdots < j_p \leq k$, a finite sequence β_j of 0 or 1 for $i \neq j_1, \cdots, j_p$, and for a parallelopiped J, a face ∂ of J is

$$\left\{p^J + \sum_{i=j_1,\cdots,j_p} x_i \alpha_i^J v_i + \sum_{i \neq j_1,\cdots,j_p} \beta_i \alpha_i^J v_i : 0 < x_i < 1, i = j_1, \cdots, j_p\right\}.$$

Put for $i \neq j_1, \cdots, j_p$

$$\pi_i^J = \left\{p^J + \beta_i \alpha_i^J v_i + \sum_{j \neq i} x_j \alpha_j^J v_j : x_j \in \boldsymbol{R}, j \neq i\right\},$$

and we say that the face ∂ of J is generated by planes π_i^J ($i \neq j_1, \cdots, j_p$). Then $\boldsymbol{R}^k$ is divided into several regions by the planes π_i^J with $i \neq j_1, \cdots j_p$, one of them contains J^o in its inside. We call this region by the interior region generated by a face ∂ of J, and we denote its boundary by J^∂ and we will simply call it a screen of J. We define $\sigma(J^\partial, x) = +1$, if $x \in \boldsymbol{R}^k$ belongs to the interior region generated by a face ∂ of J and otherwise we put $\sigma(J^\partial, x) = -1$ (Figure 1).

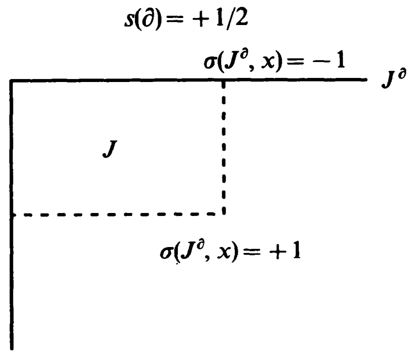


FIGURE 1

For a word w, we can naturally define $\sigma(F^w J^\partial, x) = \sigma(J^\partial, wx)$. As we mentioned before, we will prove Theorem B only for the case which has no periodic points which pass $\bigcup_{a \in \mathscr{A}} \Delta\langle a \rangle$. Hence we need not take care of $x \in J^\partial$.

For a word w, we denote w^∂ instead of $\langle w \rangle^\partial$, and we denote

$$\tilde{\mathscr{A}} = \{a^\partial : a \in \mathscr{A}, \partial \in D\}.$$

NOTATIONS. (1) For alphabets a and b, we say that a screen $\tilde{J}$ of a parallelopiped J crosses ab if $J \subset \langle a \rangle$ and $F^a(\tilde{J}) \cap \langle b \rangle \neq \varnothing$ (cf. Figure 2). Set

$$\langle ab, \tilde{J} \rangle = \begin{cases} \{\{x \in \langle b \rangle : \sigma(F^a(\tilde{J}), x) = +1\}\} & \text{if } \tilde{J} \text{ crosses } ab, \\ \{\varnothing\} & \text{otherwise}, \end{cases}$$

$$\Delta\langle ab, \tilde{J} \rangle = \Delta\left(\bigcap_{K \in \langle ab, \tilde{J} \rangle} K\right)$$

$$= \Delta\{x \in \langle b \rangle : \sigma(F^a(\tilde{J}), x) = +1\}.$$

A screen K^{∂} ($K \in \langle ab, \tilde{J}\rangle$) with a face $\partial \in D$ such that $K^{\partial} \neq b^{\partial'}$ for any $b^{\partial'} \in \tilde{\mathscr{A}}$ is called a new screen generated by $F^{a}(\tilde{J})$ in $\langle b\rangle$, and we denote by $New\langle ab, \tilde{J}\rangle$ the set of new screens of $\langle ab, \tilde{J}\rangle$. Put

$$\Delta_0\langle ab, \tilde{J}\rangle = (\Delta\langle ab, \tilde{J}\rangle \setminus \Delta\langle b\rangle) \cap F^{a}(\tilde{J}) ,$$

and $\Delta_1\langle ab, \tilde{J}\rangle$ is the union of $\tilde{K} \setminus F^{a}(\tilde{J})$ such that $\tilde{K}$ is a new screen generated by $F^{a}(\tilde{J})$ in $\langle b\rangle$ and $(\tilde{K} \setminus F^{a}(\tilde{J})) \cap \Delta\langle ab, \tilde{J}\rangle \neq \varnothing$.

$$\Delta_2\langle ab, \tilde{J}\rangle = \bigcup_{\tilde{K} \in New\langle ab, \tilde{J}\rangle} \tilde{K} \setminus (\Delta_0\langle ab, \tilde{J}\rangle \cup \Delta_1\langle ab, \tilde{J}\rangle) .$$

We also put $\Delta_0\langle a, \tilde{J}\rangle = \tilde{J}$.

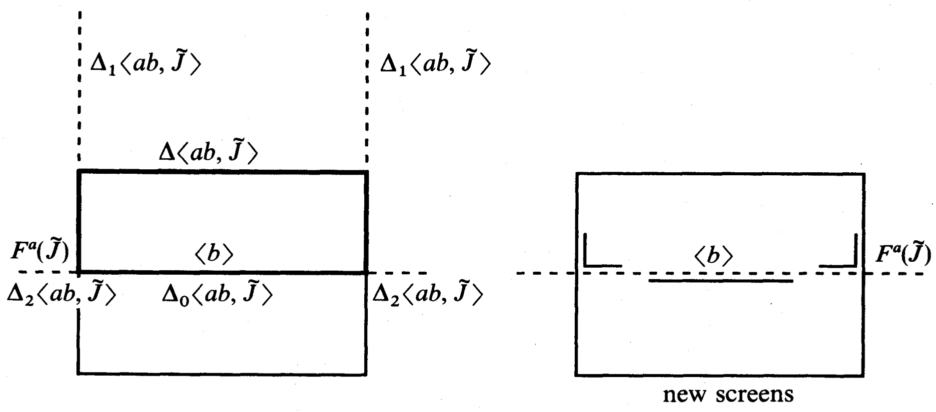


FIGURE 2

(2) For a word $w = a_1 \cdots a_n$ ($n \geq 2$) and an alphabet b, we call a screen $\tilde{J}$ of a parallelopiped J crosses wb if

$$J \subset \langle a_1\rangle ,$$
$$F^{a_i}(\Delta_0\langle w[1, i], \tilde{J}\rangle) \cap \langle a_{i+1}\rangle \neq \varnothing \qquad (1 \leq i \leq n-1) ,$$
$$F^{a_n}(\tilde{K}) \cap \langle b\rangle \neq \varnothing ,$$

for some $\tilde{K} \in New\langle w, \tilde{J}\rangle$, where $New\langle w, \tilde{J}\rangle$ is the set of new screens generated by $F^{w[1,n-1]}(\tilde{J})$ in $\langle a_n\rangle$. New screens are already defined for $|w| = 2$ in (1). For $\tilde{K} \in New\langle w, \tilde{J}\rangle$, a screen L^{∂} is called a new screen generated by $F^{w}(\tilde{J})$ in $\langle b\rangle$ if $L^{\partial} \neq b^{\partial'}$ for any $b^{\partial'} \in \tilde{\mathscr{A}}$, where $L \in \langle a_n b, \tilde{K}\rangle$, that is,

$$L = \{x \in \langle b\rangle : \sigma(F^{a_n}(\tilde{K}), x) = +1\} ,$$

and $\partial \in D$. We denote by $New\langle \omega b, \tilde{J}\rangle$ the set of new screens L^{∂} which is generated by $\tilde{K} \in New\langle w, \tilde{J}\rangle$. Set

$$\langle wb, \tilde{J}\rangle = \bigcup_{\tilde{K} \in New\langle w, \tilde{J}\rangle} \langle a_n b, \tilde{K}\rangle ,$$

that is, the union is taken over all new screens $\tilde{K}$ generated by $F^{w[1,n-1]}(\tilde{J})$ in $\langle a_n \rangle$. We call a new screen $K^{\partial} \in New\langle wb, \tilde{J} \rangle$ generated by $F^{w}(\tilde{J})$ in $\langle b \rangle$ is

(0) of type 0, if $K = \bigcap_{L \in \langle wb, J \rangle} L$,

(1) of type 1 if K^{∂} satisfies $\partial \cap (F^{a_n}(\Delta_1 \langle w, \tilde{J} \rangle) \backslash \Delta \langle b \rangle) \neq \emptyset$ and not of type 0,

(2) of type 2 otherwise.

Set

$$\Delta\langle wb, \tilde{J} \rangle = \Delta\left(\bigcap_{K \in \langle wb, J \rangle} K \right),$$

$$\Delta_0\langle wb, \tilde{J} \rangle = \Delta\left(\bigcap_{K \in \langle wb, J \rangle} K \right) \backslash \Delta\langle b \rangle ,$$

$\Delta_1\langle wb, \tilde{J} \rangle$ is the union of $\tilde{K} \backslash \Delta_0 \langle wb, \tilde{J} \rangle$ such that $\tilde{K}$ is a new screen of type 0 generated by $F^{w}(\tilde{J})$ in $\langle b \rangle$ and $(\tilde{K} \backslash F^{w}(\tilde{J})) \cap \Delta\langle wb, \tilde{J} \rangle \neq \emptyset$, and

$$\Delta_2\langle wb, \tilde{J} \rangle = \Delta\left(\bigcup_{\tilde{K} \in New\langle wb, J \rangle} \tilde{K} \right) \backslash (\Delta_0 \langle wb, \tilde{J} \rangle \cup \Delta_1 \langle wb, \tilde{J} \rangle) .$$

Set

$$s_k = \begin{cases} 1 & k \text{ is even} , \\ 0 & k \text{ is odd} , \end{cases}$$

and for $\partial \in D_i$, we also set for $0 \leq i \leq k$

$$s(\partial) = s_i = (-1)^i / 2 .$$

We also use the notation

$$s(J^{\partial}) = s(\partial) = s_i \qquad \text{if} \quad \partial \in D_i .$$

For a parallelopiped J in $J \subset \langle a \rangle$, put $\sigma^*(F\tilde{J}, \tilde{b}) = +1$, if $\sigma(F\tilde{J}, x) = +1$ holds for all $x \in \langle b \rangle$ or if $\tilde{J}$ crosses ab and $\tilde{b}$ is a screen of some $K \in \langle ab, \tilde{J} \rangle$, otherwise $\sigma^*(F\tilde{J}, \tilde{b}) = -1$.

Definition. We denote by $F\tilde{\mathscr{A}}$ the set of new screens of type 0 or 1 generated by $F^{a}(a^{\partial})$ in some $\langle b \rangle$ $(a, b \in \mathscr{A}, \partial \in D^{\langle a \rangle})$, that is, $F\tilde{\mathscr{A}} = \bigcup_{a,b \in \mathscr{A}} \bigcup_{\partial \in D^{\langle a \rangle}} New\langle ab, a^{\partial} \rangle$. For $n \geq 2$, let $F^{n}\tilde{\mathscr{A}}$ be the set of new screens of type 0 or 1 generated by $F^{a}(\tilde{J})$ in some $\langle b \rangle$ $(a, b \in \mathscr{A}, \tilde{J} \in F^{n-1}\tilde{\mathscr{A}}, J \subset \langle a \rangle)$, which does not belong to $\bigcup_{k=1}^{n-1} F^{k}\tilde{\mathscr{A}}$.

Set

$$\tilde{\phi}_1^{\tilde{a}}(\tilde{b}) = \left[\frac{s_k}{\# D} + s(\tilde{a}) \sigma^*(F\tilde{a}, \tilde{b}) \right] \eta(a) ,$$

and for $\tilde{a} \in \bigcup_{n=0}^{\infty} F^{n}\tilde{\mathscr{A}}$

$$\phi(\tilde{a}, \tilde{b}) = \begin{cases} z\tilde{\Phi}_1^{\tilde{a}}(\tilde{b}) & \text{if } \ \tilde{b} \in \tilde{\mathscr{A}}, \\ z2s(\tilde{a})\eta(a) & \text{if } \ \tilde{b} \in \bigcup_{n=0}^{\infty} F^n \tilde{\mathscr{A}} \text{ is a new screen generated by } F\tilde{a}, \\ 0 & \text{otherwise}, \end{cases}$$

and we define the Fredholm matrix $\Phi(z)$ an infinite dimensional matrix on $\bigcup_{n=0}^{\infty} F^n \tilde{\mathscr{A}}$ for which the $(\tilde{a}, \tilde{b})$ component equals $\phi(\tilde{a}, \tilde{b})$.

We also need to define finite dimensional matrices $\Phi_n(z)$. Thus we will prepare several notations.

Set $B_k\langle wb, \tilde{J}\rangle$ be the set of $K^\partial \in F^k \tilde{\mathscr{A}}$ such that $\tilde{J}$ crosses wb, and k^∂ is of type 1 generated by some $\tilde{L} \in New\langle w, \tilde{J}\rangle$. Note that if $|w| \leq k$, then $B_k\langle wb, \tilde{J}\rangle = \varnothing$, especially $B_0\langle wb, \tilde{J}\rangle = \varnothing$ for any $w \in \mathscr{W}$ and $b \in \mathscr{A}$. Let

$$C_{n,1}\langle wb, \tilde{J}\rangle = \bigcup_{k=1}^{n} B_k\langle wb, \tilde{J}\rangle,$$

$$D_{n,1}\langle wb, \tilde{J}\rangle = \bigcup_{k>n} B_k\langle wb, \tilde{J}\rangle.$$

For $l \geq 1$, set

$$C_{n,l+1}\langle wb, \tilde{J}\rangle = \bigcup_{\substack{\tilde{J}' \in D_{n,1}\langle w[1,m], \tilde{J}\rangle \\ m \geq 2}} C_{n,l}\langle w[m, |w|]b, \tilde{J}'\rangle,$$

$$D_{n,l+1}\langle wb, \tilde{J}\rangle = \bigcup_{\substack{\tilde{J}' \in D_{n,1}\langle w[1,m], \tilde{J}\rangle \\ m \geq 2}} D_{n,l}\langle w[m, |w|]b, \tilde{J}'\rangle,$$

$$C_n\langle wb, \tilde{J}\rangle = \bigcup_{l \geq 1} C_{n,l}\langle w, \tilde{J}\rangle,$$

$$D_n\langle wb, \tilde{J}\rangle = \bigcup_{l \geq 1} D_{n,l}\langle w, \tilde{J}\rangle.$$

Now we will fix $n \geq 0$ and construct the Fredholm matrices which have coefficients $\bigcup_{k=0}^{n} F^k \tilde{\mathscr{A}}$. We need not renew screens which belongs to $C_n\langle wb, \tilde{J}\rangle$. Let

$$New_n\langle w, \tilde{J}\rangle = New\langle w, \tilde{J}\rangle \cup D_n\langle w, \tilde{J}\rangle,$$

and we call a screen of type $(n, 0)$ if it is of type 0 of $New_n\langle w, \tilde{J}\rangle$ or it belongs to $D_n\langle w, \tilde{J}\rangle$, and of type $(n, 1)$ if it belongs to $C_n\langle w, \tilde{J}\rangle$.

Set $\phi_n(\tilde{J}, \tilde{L}) = \phi(\tilde{J}, \tilde{L})$ for a screen $\tilde{J} \in \bigcup_{k=0}^{n-1} F^k \tilde{\mathscr{A}}$ ($J \subset \langle a\rangle$ for some $a \in \mathscr{A}$), and for $\tilde{J} \in F^n \tilde{\mathscr{A}}$

$$\phi_n(\tilde{J}, \tilde{L}) =$$

$$\begin{cases} z\tilde{\phi}_*^{\tilde{J}}(L^\partial) + \sum\limits_{w \in \mathscr{W}} \sum\limits_{\substack{\tilde{K} \in New_n\langle w, \tilde{J}\rangle \\ \text{type}(n,0) \\ |w| \geq 2}} 2s(\tilde{J}) z^{|w|-1} \eta(w[1, |w|-1]) \tilde{\phi}_*^{\tilde{K}}(L^\partial) & \text{if } L^\partial \in \tilde{\mathscr{A}}, \\ \sum\limits_{\substack{|w| \geq 3 \\ New_n\langle w, \tilde{J}\rangle \ni L^\partial \\ \text{type}(n,1)}} 2s(\tilde{J}) z^{|w|-1} \eta(w[1, |w|-1]) & \text{if } L^\partial \in \bigcup_{k=1}^n F^k \tilde{\mathscr{A}}, \end{cases}$$

and a Fredholm matrix $\Phi_n(z)$ is the $\bigcup_{k=0}^n F^k \tilde{\mathscr{A}}$ matrix whose $(\tilde{a}, \tilde{b})$ coefficient is $\phi_n(\tilde{a}, \tilde{b})$. Using the Fredholm matrices $\Phi_n(z)$, we get Theorem A (cf. [7]).

DEFINITION. Let J and K be parallelopipeds.

(1) K^∂ is inside of J, if K^∂ is generated by $(k-1)$-dimensional planes $\pi_1^K, \cdots, \pi_l^K$ and π_i^K is between π_i^J and $\pi_i'^J$, where π_i^J and $\pi_i'^J$ are the planes which are tangent to the parallelopiped J and parallel to π_i^K (Figure 3).

(2) Set for a word $w = a_1 \cdots a_m$

$$F^{w[1,m-1]} w^\partial = F^{a_{m-1}}(\cdots F^{a_1}(a_1^\partial) \cap \langle a_2 \rangle \cdots) \cap \langle a_m \rangle,$$

and we define (cf. Figures 4 and 5)

$$\delta[K^\partial \text{ in } J] = \begin{cases} 1 & \text{if } K^\partial \text{ is inside of } J, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta[K^{\partial'} \text{ on } F^{w[1,m-1]} w^\partial] = \begin{cases} 1 & \text{if the face } \partial' \text{ of } K \text{ intersects } F^{w[1,m-1]} w^\partial, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta[K^{\partial'} \text{ on } F^{m-1}\Delta\langle w \rangle] = \begin{cases} 1 & \text{if the face } \partial' \text{ of } K \text{ intersects } F^{m-1}\Delta\langle w \rangle, \\ 0 & \text{otherwise}. \end{cases}$$

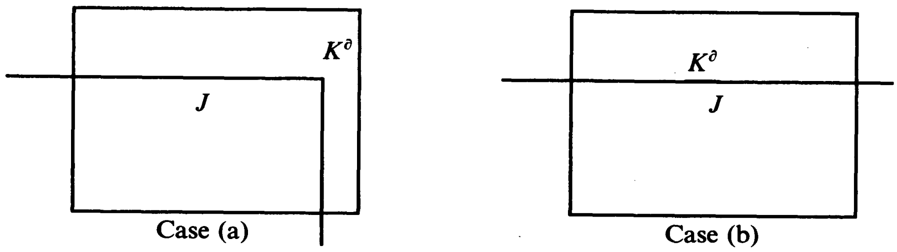
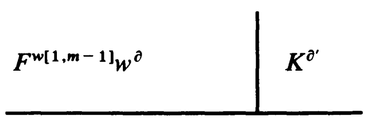
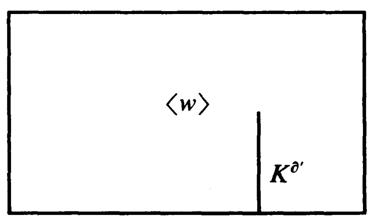


FIGURE 3



$K^{\partial'}$ is on $F^{w[1,m-1]} w^\partial$

FIGURE 4



$K^{\partial'}$ is on $F^{m-1}\Delta\langle w \rangle$

FIGURE 5

The next lemma is very simple but a key in [7] to construct a renewal equation:

LEMMA 2.1. (1) *For a parallelopiped J and a screen $\tilde{K}$, we get*

$$s_k + \sum_{\partial \in D} s(\partial)\sigma^*(FJ^\partial, \tilde{K}) = \begin{cases} 1 & \text{if } \tilde{K} \text{ is inside of } F(J)\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.1}$$

(2) *Let w be a word, $\tilde{J}$ a screen of a parallelopiped $J \subset \langle a \rangle (a \in \mathscr{A})$. For a screen $\tilde{L}$ which is generated by some $\tilde{K} \in New\langle w, \tilde{J} \rangle$ in $\langle b \rangle$ $(b \in \mathscr{A})$, we denote by $G(\tilde{L})$ the set of $\tilde{K} \in New\langle w, \tilde{J} \rangle$ which generate $\tilde{L}$. Then we get*

$$\sum_{\tilde{K} \in G(\tilde{L})} s(\tilde{K}) = \begin{cases} +1/2 & \text{if } \tilde{L} \text{ is of type } 0 \text{ or type } 1\,, \\ 0 & \text{if } \tilde{L} \text{ is of type } 2\,. \end{cases} \tag{2.2}$$

We will schematically give a proof for the case $k=2$ (cf. [7] Lemma 2.1 and Lemma 3.4 for precise proofs). A screen $\tilde{K}$ is inside of $F(J)$ either case (a) or (b) of Figure 3. Thus it is not so difficult to show (2.1) for both cases. If a face ∂ of b is between $F(A)$ and $F(B)$ of Figure 6 (type 0), then it is generated by the image of screens A, B and AB. The screens A and B give $+1/2$ and the screen AB gives $-1/2$, therefore (2.2) equals $+1/2$. If a face ∂ of b is between $F(A)$ and $F(X)$ (type 1), it is generated by the image of the screen A. Therefore (2.2) also equals $+1/2$. If a face ∂ of b is between $F(Y)$ and $F(A)$ (type 2), then it is generated by the images of screens B and AB. Therefore (2.2) equals 0. This proves (2).

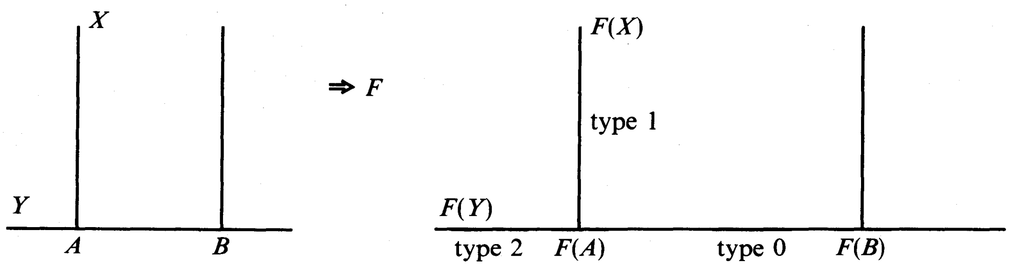


FIGURE 6

To avoid confusion, we consider the copies $D^{(i)}$ $(0 \leq i < \infty)$ of the set of faces D. Set

$$F^0\tilde{\mathscr{A}} = \tilde{\mathscr{A}} = \{a^\partial : a \in \mathscr{A}, \partial \in D^{(0)}\}\,,$$

and we denote a new screen $b^\partial \in F\tilde{\mathscr{A}}$ $(\partial \in D^{(1)})$, which is generated by some $F^a a^{\partial'}$ in $\langle b \rangle$ $(a \in \mathscr{A}, \partial' \in D^{(0)})$. Note that there may not exist b^∂ for some $\partial \in D^{(1)}$. A set of new screens in $\langle b \rangle$ which are generated by some $F^a a^\partial$ $(\partial \in D^{(1)})$ is denoted by $b^{\partial'}$ with $\partial' \in D^{(2)}$. Now recall that the screen $b^{\partial'}$ $(\partial' \in D^{(2)})$ is divided into 3 parts, Δ_0, Δ_1 and Δ_2. We denote by $c^{\partial''}$ $(\partial'' \in D^{(3)})$ if $c^{\partial''} \in New\langle bc, b^{\partial'} \rangle$ is of type 0 of some $b^{\partial'}$ $(\partial' \in D^{(2)})$. Note that new screens $c^{\partial''} \in New\langle bc, b^{\partial'} \rangle$ of type 1 belong to $F\tilde{\mathscr{A}}$, that is $\partial'' \in D^{(1)}$. We need not to consider the new screens which is of type 2 (cf. Lemma 2.1 (2)). Thus we can inductively define b^∂ $(b \in \mathscr{A}, \partial \in D^{(n)})$ which is generated in $\langle b \rangle$ by the image of Δ_0 and Δ_1 of some $F^a a^{\partial'}$ ($a \in \mathscr{A}$, $\partial' \in D^{(n-1)}$). We denote for $n \geq 2$ by $F^n \tilde{\mathscr{A}}$ the set of $b^{\partial'}$ such that $b \in \mathscr{A}$, $\partial' \in D^{(n)}$, and that $b^{\partial'} \in New\langle ab, a^{\partial}\rangle$ generated by some $a^{\partial} \in F^{n-1}\tilde{\mathscr{A}}$ in $\langle b \rangle$. For $a, b \in \mathscr{A}$ and $\partial, \partial' \in \bigcup_{i=0}^{\infty} D^{(i)}$, we denote $\psi(ab, \partial, \partial') = \phi(a^{\partial}, b^{\partial'})\eta(a)^{-1}$, and for a word $w = a_1 \cdots a_n$

$$\psi(w, \partial_1, \partial_n) = \sum_{\partial_2, \cdots, \partial_{n-1}} \prod_{i=1}^{n} \psi(a_i a_{i+1}, \partial_i, \partial_{i+1}),$$

where $a_{n+1} = a_1$, $\partial_{n+1} = \partial_1$.

3. The Proof of Theorem B.

We get as a formal expression

$$\begin{aligned}
\log[\det(I - \Phi(z))^{-1}] &= \log[\exp\{-\operatorname{tr}\log(I - \Phi(z))\}] \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \Phi(z)^k = \sum_{\tilde{a} \in \bigcup_{k=0}^{\infty} \tilde{F}^k \tilde{\mathscr{A}}} \sum_{k=1}^{\infty} \frac{1}{k} (\Phi(z)^k)_{\tilde{a}, \tilde{a}} \\
&= \sum_{\tilde{a} \in \bigcup_{k=0}^{\infty} \tilde{F}^k \tilde{\mathscr{A}}} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{b_2, \cdots, b_k \in \bigcup_{k=0}^{\infty} \tilde{F}^k \tilde{\mathscr{A}}} \sum_{\partial_2, \cdots, \partial_k \in \bigcup_{k=0}^{\infty} D^{(k)}} \prod_{j=1}^{k} \Phi(z)_{\tilde{b}_j, \tilde{b}_{j+1}},
\end{aligned}$$

where $\tilde{b}_j = b_j^{\partial_j}$ $(2 \leq j \leq k)$, $\tilde{b}_1 = \tilde{b}_{k+1} = \tilde{a}$.

Let for a word $w = a_1 \cdots a_n \in \mathscr{W}_n$, we introduce the rotation operator

$$\hat{\theta} w = a_2 \cdots a_n a_1 .$$

By this operator we consider an equivalence relation

$$w \sim w' \qquad \text{if and only if there exists some } k \text{ such that } \hat{\theta}^k w = w',$$

and we denote the factor of $\mathscr{W}_n$ by $Loop(n)$. Then, using this notation and taking $\partial_{n+1} = \partial_1$ and $w[n+1] = w[1]$, we get

$$\begin{aligned}
&\log[\det(I - \Phi(z))^{-1}] \\
&= \sum_{n=1}^{\infty} z^n \sum_{w \in Loop(n)} \sum_{\partial_1, \cdots, \partial_n \in \bigcup_{i=0}^{n} D^{(i)}} \prod_{j=1}^{n} \psi(w[j], w[j+1], \partial_j, \partial_{j+1}) \eta(a_j) \\
&= \sum_{n=1}^{\infty} z^n \sum_{w \in Loop(n)} \sum_{\partial \in \bigcup_{i=0}^{n} D^{(i)}} \psi(ww[1], \partial, \partial) \eta(w) . \qquad (3.1)
\end{aligned}$$

Note that, since $w[1]^{\partial_1} = w[n+1]^{\partial_{n+1}}$, there exists some i such that $w[i]^{\partial_i} \in \tilde{\mathscr{A}} \cup F\tilde{\mathscr{A}}$. Hence, we only need to calculate $\psi(w, \partial, \partial')$.

LEMMA 3.1. *For a word $w \in \mathscr{W}_m$ $(m \geq 3)$, we get:*

(1) *If $\partial \in D^{(n)}$ and $\partial' \in D^{(0)}$ for some n, then*

$$\psi(w, \partial, \partial') = \delta[w[m]^{\partial'} \text{ in } F^{m-2}\langle w[2, m-1]\rangle] \cdot \left[\frac{s_k}{\#D} + s(\partial)\sigma^*(F^{m-1}w[1]^{\partial}, w[m]^{\partial'})\right].$$

(2) *If* $\partial \in D^{(n)}$ *for some* n *and* $\partial' \in D^{(n+m-1)}$, *then*

$$\psi(w, \partial, \partial') = 2s(\partial)\delta[w[m]^{\partial'} \text{ on } F^{w[1,m-1]}(w[1, m-1]^{\partial})].$$

(3) *If* $\partial \in D^{(n)}$ *for some* n *and* $\partial' \in D^{(l)}$ $(l < m-1)$, *then*

$$\psi(w, \partial, \partial') = \delta[w[m]^{\partial'} \text{ on } F^{m-2}\Delta\langle w[2, m-1]\rangle] \cdot \left[\frac{s_k}{\#D} + s(\partial)\sigma^*(F^{m-1}w[1]^{\partial}, w[m]^{\partial'})\right].$$

(4) *Otherwise,*

$$\psi(w, \partial, \partial') = 0.$$

PROOF. We will prove this by induction.

(1) For $\partial \in D^{(n)}$ and $\partial' \in D^{(0)}$ and $w \in \mathscr{W}_{m+1}$, we get

$$\psi(w, \partial, \partial') = \sum_{\partial'' \in D^{(0)} \cup D^{(n+1)}} \psi(w[1, 2], \partial, \partial'')\psi(w[2, m+1], \partial'', \partial')$$

$$= \sum_{\partial'' \in D^{(0)}} \left[\frac{s_k}{\#D} + s(\partial)\sigma^*(Fw[1]^{\partial}, w[1]^{\partial''})\right]\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3, m]\rangle] \cdot \left[\frac{s_k}{\#D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$+ \sum_{\partial'' \in D^{(n+1)}} 2s(\partial)\delta[w[2]^{\partial''} \text{ on } Fw[1]^{\partial}]\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3, m]\rangle] \cdot \left[\frac{s_k}{\#D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$= \frac{s_k}{\#D}\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3, m]\rangle] \cdot \left[s_k + \sum_{\partial''} s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$+ s(\partial)\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3, m]\rangle]\Bigg\{\sum_{\partial'' \in D^{(0)}} \sigma^*(Fw[1]^{\partial}, w[2]^{\partial''}) \cdot \left[\frac{s_k}{\#D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$+\sum_{\partial''\in D^{(n+1)}} 2\delta[w[2]^{\partial''} \text{ on } Fw[1]^{\partial}]$$

$$\cdot\left[\frac{s_k}{\#D}+s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]\Bigg\}. \tag{3.2}$$

Therefore by Lemma 2.1 (1) for F^{m-1} instead of F, we get

the first term of rhs. of (3.2)

$$=\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3,m]\rangle]\delta[w[m+1]^{\partial'} \text{ in } F^{m-1}\langle w[2]\rangle]$$

$$=\frac{s_k}{\#D}\delta[w[m+1]^{\partial'} \text{ in } F^{m-1}\langle w[2,m]\rangle],$$

and

the second term of rhs. of (3.2) $=s(\partial)\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3,m]\rangle]$

$$\cdot\,\hat{\sigma}(Fw[1]^{\partial}, w[2])\left\{-\sum{}^{*}\left[\frac{s_k}{\#D}+s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]\right.$$

$$\left.+2\sum{}^{**}\left[\frac{s_k}{\#D}+s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]\right\}, \tag{3.3}$$

where for $a, b\in\mathscr{A}$ and a face $\partial\in D$

$$\hat{\sigma}(Fa^{\partial}, b)=\begin{cases}+1 & \text{if } \sigma^*(Fa^{\partial}, b^{\partial'})=+1 \text{ with some } \partial'\in D^{(0)}, \\ -1 & \text{otherwise},\end{cases}$$

and the sum $\sum^*$ expresses the sum over ∂'' for which $w[2]^{\partial''}$ is a screen of $\langle w[2]\rangle$, and $\sum^{**}$ expresses the sum over ∂'' for which $w[2]^{\partial''}$ is a screen of $L=\bigcap_{K\in\langle w[1,2],\,w[1]^{\partial}\rangle}K$ if $w[1]^{\partial}$ crosses $w[2]$, and $L=\langle w[2]\rangle$ if $w[1]^{\partial}$ does not cross $w[2]$. Then again by Lemma 2.1 (1)

(3.3) + the third term of (3.2) $=s(\partial)\delta[w[m+1]^{\partial'} \text{ in } F^{m-2}\langle w[3,m]\rangle]\hat{\sigma}(Fw[1]^{\partial}, w[2])$

$$\cdot\{-\delta[w[m+1]^{\partial'} \text{ in } F^{m-1}\langle w[2]\rangle]+2\delta[w[m+1]^{\partial'} \text{ in } F^{m-1}\langle w[1,2], w[1]^{\partial}\rangle]\}$$

$$=s(\partial)\delta[w[m+1]^{\partial'} \text{ in } F^{m-1}\langle w[2,m]\rangle]\sigma^*(F^m w[1]^{\partial}, w[m+1]^{\partial'}).$$

Therefore substituting them to (3.2), we get

$$\psi(w,\partial,\partial')=\delta[w[m+1]^{\partial'} \text{ in } F^{m-1}\langle w[2,m]\rangle]$$

$$\cdot\left[\frac{s_k}{\#D}+s(\partial)\sigma^*(F^m w[1]^{\partial}, w[m+1]^{\partial'})\right].$$

This proves (1).

(2) For $\partial \in D^{(n)}$ and $\partial' \in D^{(n+m)}$, let $w \in \mathscr{W}_{m+1}$. Then

$$\psi(w, \partial, \partial') = \sum_{\partial'' \in D^{(n+1)}} \psi(w[1, 2], \partial, \partial'')\psi(w[2, m+1], \partial'', \partial')$$

$$= \sum_{\partial'' \in D^{(n+1)}} 2s(\partial)\delta[w[2]^{\partial''} \text{ on } Fw[1]^{\partial}]2s(\partial'')\delta[w[m+1]^{\partial'} \text{ on } F^{w[1,m-1]}(w[2]^{\partial''})]\,.$$

Therefore by Lemma 2.1 (2), we get

$$\psi(w, \partial, \partial') = 2s(\partial)\delta[w[m+1]^{\partial'} \text{ on } F^{w[1,m]}(w[1]^{\partial})]\,.$$

This proves (2).

(3) For $\partial \in D^{(n)}$ and $\partial' \in D^{(l)}$ for $l < m$ and a word $w \in \mathscr{W}_{m+1}$,

$$\psi(w, \partial, \partial') = \sum_{\partial'' \in D^{(0)} \cup D^{(n+1)}} \psi(w[1, 2], \partial, \partial'')\psi(w[2, m+1], \partial'', \partial')$$

$$= \sum_{\partial'' \in D^{(0)}} \left[\frac{s_k}{\# D} + s(\partial)\sigma^*(Fw[1]^{\partial}, w[2]^{\partial''})\right]\delta[w[m+1]^{\partial'} \text{ on } F^{m-2}\Delta\langle w[3, m]\rangle]$$

$$\cdot\left[\frac{s_k}{\# D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$+ \sum_{\partial'' \in D^{(n+1)}} 2s(\partial)\delta[w[2]^{\partial''} \text{ on } Fw[1]^{\partial}]\delta[w[m+1]^{\partial'} \text{ on } F^{m-2}\Delta\langle w[3, m]\rangle]$$

$$\cdot\left[\frac{s_k}{\# D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$= \frac{s_k}{\# D}\delta[w[m+1]^{\partial'} \text{ on } F^{m-2}\Delta\langle w[3, m+1]\rangle]$$

$$\cdot\left[s_k + \sum_{\partial'' \in D^{(0)}} s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$+ s(\partial)\delta[w[m+1]^{\partial'} \text{ on } F^{m-2}\Delta\langle w[3, m]\rangle]\Bigg\{\sum_{\partial'' \in D^{(0)}} \sigma(Fw[1]^{\partial}, w[2]^{\partial''})$$

$$\cdot\left[\frac{s_k}{\# D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]$$

$$+ \sum_{\partial'' \in D^{(n+1)}} 2\delta[w[2]^{\partial''} \text{ on } Fw[1]^{\partial}]$$

$$\cdot\left[\frac{s_k}{\# D} + s(\partial'')\sigma^*(F^{m-1}w[2]^{\partial''}, w[m+1]^{\partial'})\right]\Bigg\}\,.$$

Therefore, in a similar argument as in (1), we get by Lemma 2.1 (1),

$$\psi(w, \partial, \partial') = \delta[w[m+1]^{\partial'} \text{ on } F^{m-1}\Delta\langle w[2, m+1]\rangle] \cdot \left[\frac{s_k}{\sharp D} + s(\partial)\sigma^*(F^m w[1]^{\partial}, w[m+1]^{\partial'})\right].$$

This proves the case (3). The rest of the cases never occurs, thus the proof of the lemma is completed.

Now we will prove Theorem B:

THEOREM B. *Assume that F is expanding and $\xi - \lambda > 0$. Then the zeta function $\zeta(z)$ has a meromorphic extension to the domain $|z| < e^{\xi-\lambda}$, and satisfies $\zeta(z) = \det(I - \Phi(z))^{-1}$.*

PROOF. Note that

$$\log[\det(I - \Phi(z))^{-1}] = \sum_{n=1}^{\infty} z^n \sum_{w \in Loop(n)} \sum_{\partial \in \bigcup_{i=0}^{n} D^{(i)}} \psi(ww[1], \partial, \partial)\eta(w). \tag{3.1}$$

We will calculate $\sum_{\partial \in \bigcup_{i=0}^{n} D^{(i)}} \psi(ww[1], \partial, \partial)\eta(w)$ for each $n \geq 1$ and $w \in Loop(n)$ using Lemma 3.1.

(1) For $n = 1$, $\psi(ww[1], \partial, \partial) = 0$ unless $\partial \in D^{(0)}$. Therefore,

$$\sum_{\partial \in \bigcup_{i=0}^{n} D^{(i)}} \psi(ww[1], \partial, \partial)\eta(w) = \sum_{\partial \in D^{(0)}} \left[\frac{s_k}{\sharp D} + s(\partial)\sigma^*(Fw[1]^{\partial}, w[1]^{\partial})\right]\eta(w)$$

$$= \left[s_k + \sum_{\partial} s(\partial)\sigma^*(Fw[1]^{\partial}, w[1]^{\partial})\right]\eta(w)$$

$$= \begin{cases} \eta(w) & \text{if there exists a fixed point corresponding to a word } w, \\ 0 & \text{otherwise}. \end{cases}$$

(2) Now assume that $n \geq 2$. Then

$$\sum_{\partial} \psi(ww[1], \partial, \partial)\eta(w)$$

$$= \sum_{\partial \in D^{(0)}} \psi(ww[1], \partial, \partial)\eta(w) + \sum_{l=1}^{\infty} \sum_{\partial \in D^{(l)}} \psi(ww[1], \partial, \partial)\eta(w)$$

$$= \sum_{\partial \in D^{(0)}} \delta[w[1]^{\partial} \text{ in } F^{n-1}\langle w[2, n]\rangle]\left[\frac{s_k}{\sharp D} + s(\partial)\sigma^*(F^n w[1]^{\partial}, w[1]^{\partial})\right]\eta(w)$$

$$+ \sum_{l=1}^{\infty} \sum_{\partial \in D^{(l)}} \delta[w[1]^{\partial} \text{ on } F^{n-1}\Delta\langle w[2, n]\rangle] \cdot \left[\frac{s_k}{\sharp D} + s(\partial)\sigma^*(F^n w[1]^{\partial}, w[1]^{\partial})\right]\eta(w).$$

Note that there exists a canonical correspondence between D and $D^{(l)}$. The necessary and sufficient condition for $\langle w[2, m]w[1]\rangle \neq \emptyset$ is that either $\delta[w[1]^{\partial}$ in $F^{n-1}\Delta\langle w[2, n]\rangle] = 1$ for $\partial \in D^{(0)}$ or there exists $\partial' \in D^{(l)}$ for some $l \geq 1$ which corresponds to ∂ such that $\delta[w[1]^{\partial'}$ on $F^{n-1}\Delta\langle w[2, n]\rangle] = 1$. Therefore

$$\sum_{\partial \in D} \psi(ww[1], \partial, \partial)\eta(w)$$

$$= \sum_{\partial \in D} \delta[\langle w[2, n]w[1]\rangle \neq \emptyset]\left[\frac{s_k}{\sharp D} + s(\partial)\sigma^*(F^n w[1]^{\partial}, w[1]^{\partial})\right]\eta(w)$$

$$= \begin{cases} \eta(w) & \text{if there exists a periodic orbit corresponding to } w\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where

$$\delta[\langle w\rangle \neq \emptyset] = \begin{cases} 1 & \text{if } \langle w\rangle \neq \emptyset\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Therefore,

$$\det(I - \Phi(z))^{-1} = \exp \sum_{n=1}^{\infty} \sum_{w \in Loop(n)} \eta(w)\delta[\exists\text{periodic orbit corresponding to } w]$$

$$= \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{p : F^n(p) = p} |\det D(F^n)(p)|^{-1} = \zeta(z)\,.$$

On the other hand, for any m, the coefficient of z^k of $\operatorname{tr}\Phi(z)^m$ coincide with that of $\operatorname{tr}\Phi_n(z)^m$ for $0 \leq k \leq n$, and $\zeta(z)$ is analytic in $|z| < 1$. We get

$$\zeta(z) = \lim_{n \to \infty} (I - \Phi_n(z))^{-1}\,.$$

This completes the proof.

CONJECTURE. *The zeta function has meromorphic extension to* $e^{\xi - \lambda}$.

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