TOKYO J. MATH.
VOL. 19, NO. 2, 1996

Generalized Unknotting Operations of Polygonal Type and Rotational Type

Kazumichi NAKAMURA

Osaka University

(Communicated by S. Suzuki)

1. Introduction.

If a set of local moves can transform every knot into a trivial knot, it is called a *generalized unknotting operation*. Note that there are many generalized unknotting operations. The n-gon moves ([A1]) and the $n(i)$-moves ([Oh1]) are typical examples. In this paper, we generalize these local moves to new local moves of "polygonal type" $P_n\binom{a}{b}$-moves and "rotational type" $R_n\binom{a}{b}$-moves (Definition 2.4), and we classify them up to "local equivalence" (Theorem 2.7). We conclude that any local move of "polygonal type" or "rotational type" is a generalized unknotting operation except for a few cases. In Section 6, we give a table of the Δ-unknotting numbers of the prime knots with ten or fewer crossings (Table A), which expands the table of [Ok2].

ACKNOWLEDGEMENT. The author wishes to thank Professors K. Kawakubo, M. Sakuma and T. Kobayashi for their help and guidance, and to thank Professors Y. Nakanishi and Y. Ohyama for their interests and valuable suggestions.

2. Fundamental results and main theorem.

Let T and T' be tangle diagrams with $\partial T=\partial T'$. By replacing T with T' on a link diagram L, we obtain a new link diagram L'. Then we say that L' is obtained from L by *a local move* $T\to T'$. Let $T\leftrightarrow T'$ denote a set of local moves $T\to T'$ and $T'\to T$.

Let $M=\{T_M^i\to T_M'^i\}_{i\in I}$ and $N=\{T_N^j\to T_N'^j\}_{j\in J}$ be sets of local moves. We say that M *induces* N if each local move $T_N^j\to T_N'^j$ in N can be realized by local moves in M, i.e., if each T_N^j can be transformed into $T_N'^j$ by performing local moves in M. If M induces N and vice versa, then we say that M and N are *locally equivalent* ([A2]). Furthermore, we say that M and N are *equivalent* if each local move in N can be realized

Received March 22, 1995
Revised November 24, 1995

by performing a local move in M once and vice versa.

If a set of local moves can transform every knot into a trivial knot, it is called a *generalized unknotting operation.* The $H(2)$-move ([HNT]), the (ordinary) unknotting operation ([W]), the unoriented #-unknotting operation ([M]), the Δ-unknotting operation ([MN]) each is a generalized unknotting operation. In [N3], Y. Nakanishi classified the known generalized unknotting operations into six types up to local equivalence; (Type T), (Type X), (Type #), (Type 2+3), (Type Δ) and (Type 6).

A generalized unknotting operation M is said to be of *Type T* (*Type X, Type #, Type Δ*, respectively) if M is locally equivalent to the $H(2)$-move (the (ordinary) unknotting operation, the unoriented #-unknotting operation, the Δ-unknotting operation, respectively). For the definitions of "Type 2+3" and "Type 6", see [N3]. Then we have the following relationship among them.

Proposition 2.1.

$$\begin{array}{ccccccccc} \textit{Type } T & \substack{\longrightarrow \\ \not\longleftarrow} & \textit{Type } X & \substack{\longrightarrow \\ \not\longleftarrow} & \textit{Type } \# & \substack{\longrightarrow \\ \not\longleftarrow} & \textit{Type } 2+3 & \substack{\longrightarrow \\ \not\longleftarrow} & \textit{Type } \Delta \\ & & & & & & \downarrow \not\uparrow & \not\swarrow \not\nearrow & \\ & & & & & & \textit{Type } 6 & & \end{array}$$

Here "Type $*\longrightarrow$ Type $**$" means that any generalized unknotting operation of Type $*$ induces that of Type $**$. And "Type $*\not\longrightarrow$ Type $**$" means that no generalized unknotting operations of Type $*$ induce that of Type $**$.

The relation "Type 2+3$\longrightarrow$Type Δ" is proved in [N1, Lemma]. The other relations "$\longrightarrow$" are easy to see.

For the proof of "Type $\Delta \;\substack{\not\longrightarrow \\ \not\longleftarrow}\;$ Type 6", we note the following facts in [N3, proof of Theorem 8.1]: (1) A 2-component trivial link and a (4,2)-torus link can be transformed into each other by a finite sequence of generalized unknotting operations of Type 6, but not by that of Type Δ. (2) A 3-component trivial link and the Borromean rings can be transformed into each other by a finite sequence of generalized unknotting operations of Type Δ, but not by that of Type 6.

The other relations "$\not\longrightarrow$" follow from [N3, Theorem 8.1].

Definition 2.2. For an integer n (≥ 2) and a sequence $\boldsymbol{a}=(a_1 a_2 \cdots a_n)$ (where $a_i \in \{+, -\}$, $1 \leq i \leq n$), $T_{\boldsymbol{a}}$ and $T_{\boldsymbol{a}}^r$ are the tangle diagrams whose projections with signs are shown in Fig. 2.1 (1), (2), respectively. Fig. 2.1 (3) illustrates the meaning of signs at crossing points. Let T and T' be tangle diagrams. If T can't be transformed into T' by performing Reidemeister moves, we denote $T \not\cong T'$.

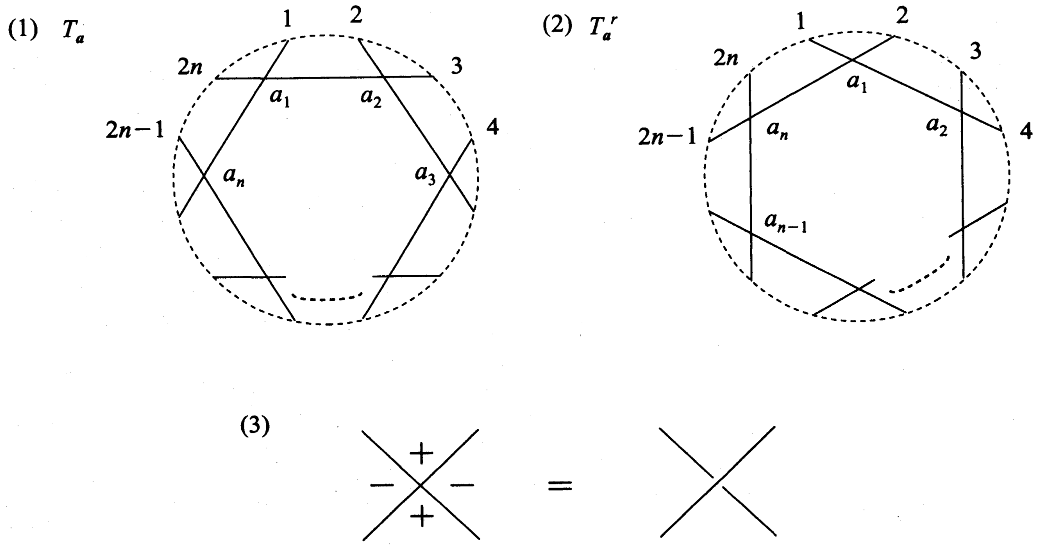

FIGURE 2.1

REMARK 2.3. (1) If $n \neq 2$, then $T_{\boldsymbol{a}} \not\cong T_{\boldsymbol{b}}$ if and only if $\boldsymbol{a} \neq \boldsymbol{b}$.
(2) Whenever $n \neq 3$, we have $T_{\boldsymbol{a}} \not\cong T_{\boldsymbol{b}}^r$.

DEFINITION 2.4. Let n be an integer (≥ 2), and let $\boldsymbol{a} = (a_1 a_2 \cdots a_n)$ and $\boldsymbol{b} = (b_1 b_2 \cdots b_n)$ be sequences of signs $a_i, b_i \in \{+, -\}$, $1 \leq i \leq n$.

(1) $I(\boldsymbol{a}, \boldsymbol{b}) = \#\{i \mid a_i \neq b_i, 1 \leq i \leq n\}$.

(2) Suppose $T_{\boldsymbol{a}} \not\cong T_{\boldsymbol{b}}$, then the $P_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move is defined to be a local move $T_{\boldsymbol{a}} \leftrightarrow T_{\boldsymbol{b}}$.

(3) Suppose $T_{\boldsymbol{a}} \not\cong T_{\boldsymbol{b}}^r$, then the $R_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move is defined to be a local move $T_{\boldsymbol{a}} \leftrightarrow T_{\boldsymbol{b}}^r$.

REMARK 2.5. (1) Any $P_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move or any $R_3\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move doesn't change the number of components of links. The number of components may be changed by any $R_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move ($n \neq 3$). In this paper, we perform any $R_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move ($n \neq 3$) without changing the number of components of links.

(2) The $P_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$-move and the $P_n\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{pmatrix}$-move are equivalent.

(3) The $P_n\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix}$-move and the $P_n\begin{pmatrix} a_i \cdots a_n a_1 \cdots a_{i-1} \\ b_i \cdots b_n b_1 \cdots b_{i-1} \end{pmatrix}$-move are equivalent.

(4) The $R_n\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix}$-move and the $R_n\begin{pmatrix} b_1 b_2 \cdots b_{n-1} b_n \\ a_2 a_3 \cdots a_n a_1 \end{pmatrix}$-move are equivalent.

(5) The $R_n\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix}$-move and the $R_n\begin{pmatrix} a_i \cdots a_n a_1 \cdots a_{i-1} \\ b_i \cdots b_n b_1 \cdots b_{i-1} \end{pmatrix}$-move are equivalent.

The following give relations between these local moves and known local moves.

PROPOSITION 2.6. (1) *If* $I(\boldsymbol{a}, \boldsymbol{b})=1$, *then the* $P_2\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$*-move is the (ordinary) unknotting operation.*

(2) *The* $P_2\begin{pmatrix} + + \\ - - \end{pmatrix}$*-move and the local move as indicated in Fig.* 2.2 (1) *are equivalent.*

(3) *The* $P_4\begin{pmatrix} + - + - \\ - + - + \end{pmatrix}$*-move is the unoriented* $\sharp$*-unknotting operation.*

(4) *For any integer* n (≥ 3), *the* $P_n\begin{pmatrix} + + \cdots + \\ - - \cdots - \end{pmatrix}$*-move is the n-gon move, and hence it is of Type* $\sharp$ ([A1], [N3, Theorem 4.3]).

(5) *If* $a_1 \neq a_2$ *or* $b_1 \neq b_2$, *then the* $R_2\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$*-move is the* $H(2)$*-move.*

(6) *The* $R_2\begin{pmatrix} + + \\ - - \end{pmatrix}$*-move and the local move as indicated in Fig.* 2.2 (2) *are equivalent.*

(7) *The* $R_3\begin{pmatrix} + + + \\ + + + \end{pmatrix}$*-move and the* $R_3\begin{pmatrix} - - - \\ - - - \end{pmatrix}$*-move each is the* Δ*-unknotting operation* ([MN]). *The* $R_3\begin{pmatrix} + + + \\ - - - \end{pmatrix}$*-move is the* Δ_{12}*-move, and hence it is of Type* $\sharp$ ([N2], [N3, Theorem 4.2]). *Another* $R_3\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$*-move is equivalent to some* $P_3\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}$*-move.*

(8) *For any integer* n (≥ 4), *the* $R_n\begin{pmatrix} + + \cdots + \\ + + \cdots + \end{pmatrix}$*-move (i.e.,* $n(1)$*-move), the* $R_n\begin{pmatrix} - - \cdots - \\ - - \cdots - \end{pmatrix}$*-move (i.e.,* $n(1)'$*-move) and the* $R_n\begin{pmatrix} + + \cdots + \\ - - \cdots - \end{pmatrix}$*-move (i.e.,* $n(2)$*-move) each is of Type* T ([N3, Theorem 2.2], [Oh1]).

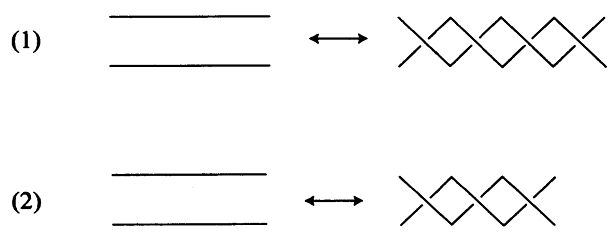

FIGURE 2.2

The following is the main theorem of this paper.

THEOREM 2.7. (1) *For any integer* n (≥ 3) *and any pair of* $\boldsymbol{a}=(a_1 \cdots a_n)$ *and* $\boldsymbol{b}=(b_1 \cdots b_n)$,

(a) *if* $I(\boldsymbol{a}, \boldsymbol{b})=n$, *then the* $P_n\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$*-move is of Type* $\sharp$,

(b) *otherwise, the* $P_n\binom{\boldsymbol{a}}{\boldsymbol{b}}$*-move is of Type X.*

(2) *For any integer* n (≥ 4) *and any pair of* $\boldsymbol{a}=(a_1 \cdots a_n)$ *and* $\boldsymbol{b}=(b_1 \cdots b_n)$, *the* $R_n\binom{\boldsymbol{a}}{\boldsymbol{b}}$*-move is of Type T.*

The proof will be given in Section 3.

The auther doesn't know whether the $P_2\binom{++}{--}$-move is a generalized unknotting operation. If we perform the $R_2\binom{++}{--}$-move $\left(R_2\binom{++}{++}\text{-move, } R_2\binom{--}{--}\text{-move}\right)$ without changing the number of components of links, the local move isn't a generalized unknotting operation ([N3, Section 9]).

3. Proof of Theorem 2.7.

For $\varepsilon=+(-$, respectively), let $\bar{\varepsilon}=-(+$, respectively). For $\boldsymbol{a}=(a_1a_2 \cdots a_n)$, let $\bar{\boldsymbol{a}}$ denote $(\bar{a}_1\bar{a}_2 \cdots \bar{a}_n)$. Theorem 2.7 (1) (a) follows from the following lemma:

LEMMA 3.1. (1) *Any* $P_3\binom{\boldsymbol{a}}{\bar{\boldsymbol{a}}}$*-move is of Type* ♯.

(2) *For any integer* n (≥ 3), *any* $P_n\binom{a_1 \cdots a_n}{\bar{a}_1 \cdots \bar{a}_n}$*-move can be realized by one* $P_{n+1}\binom{a_1 \cdots a_na_{n+1}}{\bar{a}_1 \cdots \bar{a}_n\bar{a}_{n+1}}$*-move.*

(3) *For any integer* n (≥ 3), *any* $P_{n+1}\binom{a_1 \cdots a_na_{n+1}}{\bar{a}_1 \cdots \bar{a}_n\bar{a}_{n+1}}$*-move can be realized by one* $P_3\binom{a_1a_1a_{n+1}}{\bar{a}_1\bar{a}_1\bar{a}_{n+1}}$*-move and one* $P_n\binom{a_1 \cdots a_n}{\bar{a}_1 \cdots \bar{a}_n}$*-move.*

PROOF. By Remark 2.5 (2) and (3), any $P_3\binom{\boldsymbol{a}}{\bar{\boldsymbol{a}}}$-move is equivalent to the $P_3\binom{+++}{---}$-move or the $P_3\binom{-+-}{+-+}$-move. By Proposition 2.6 (4), the $P_3\binom{+++}{---}$-move is of Type ♯. Fig. 3.1 (1) shows that the $P_3\binom{+++}{---}$-move and the $P_3\binom{-+-}{+-+}$-move are locally equivalent. Hence the proof of Lemma 3.1 (1) is completed. Lemma 3.1 (2) and (3) are proved by Fig. 3.1 (2) and (3), respectively. □

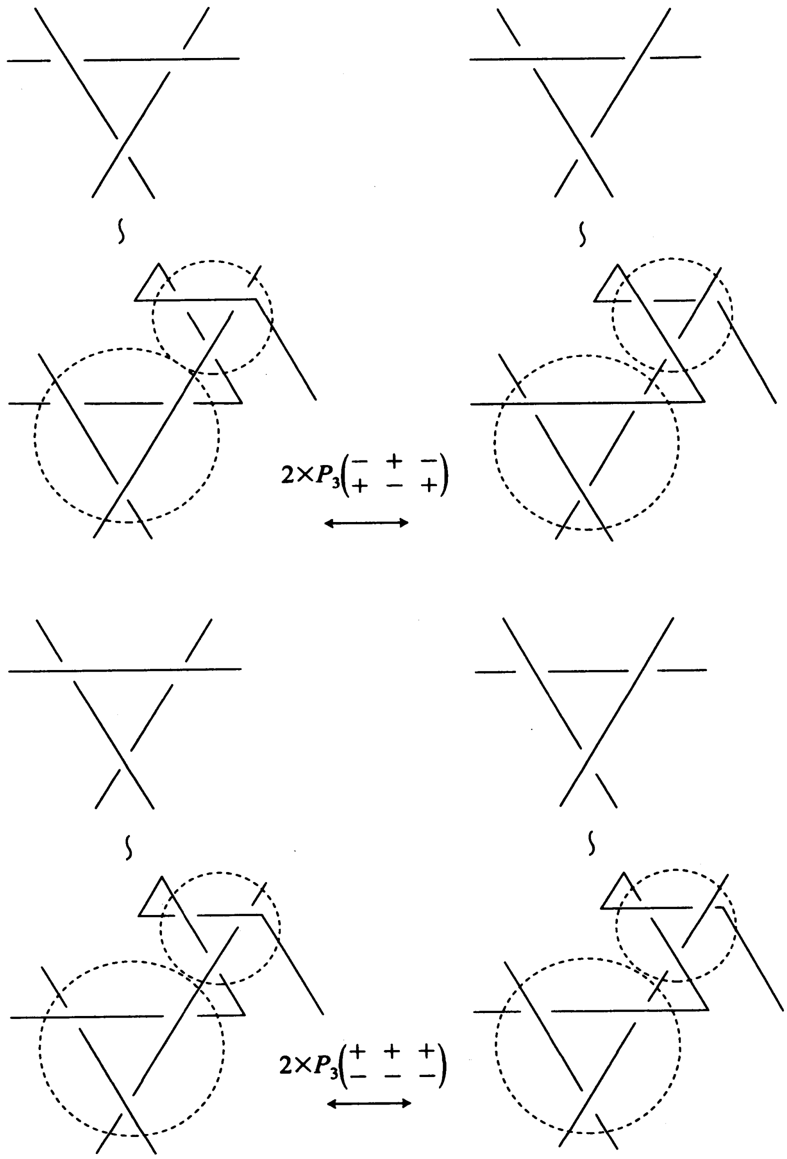
$2\times P_3\begin{pmatrix} - & + & - \\ + & - & + \end{pmatrix}$
$2\times P_3\begin{pmatrix} + & + & + \\ - & - & - \end{pmatrix}$

FIGURE 3.1 (1)

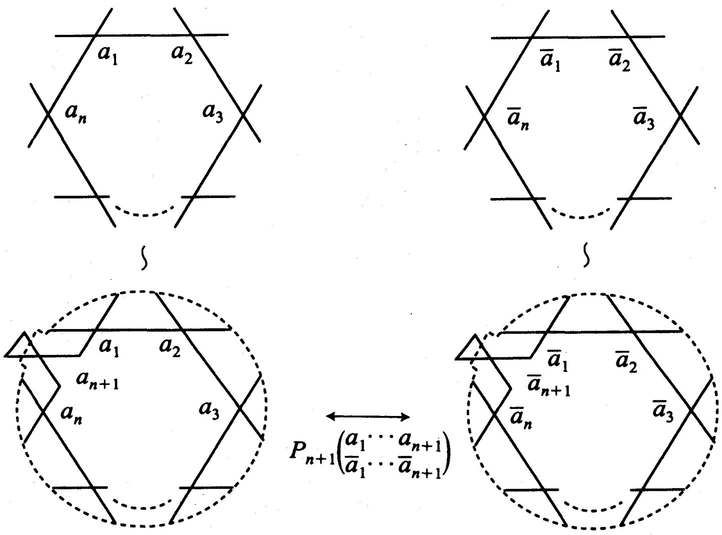

FIGURE 3.1 (2)

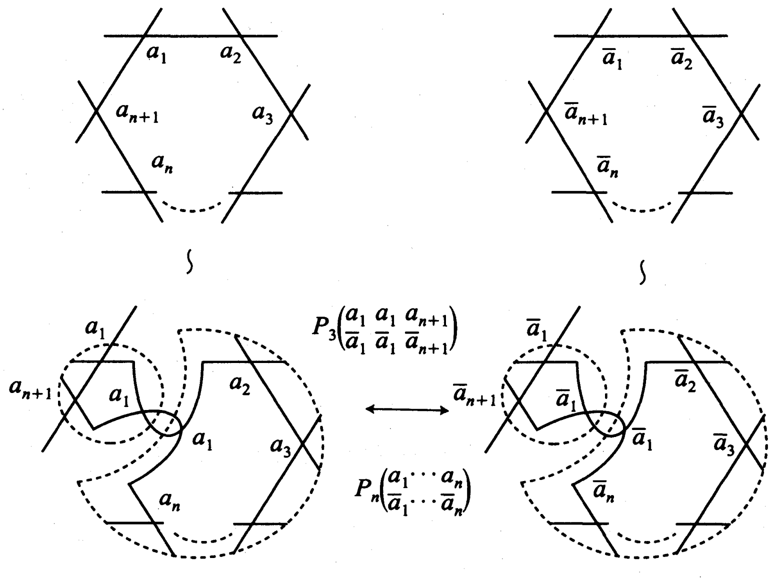

FIGURE 3.1 (3)

PROOF OF THEOREM 2.7 (1) (b). It is obvious that any $P_n\binom{\boldsymbol{a}}{\boldsymbol{b}}$-move can be realized by $I(\boldsymbol{a}, \boldsymbol{b})$ (ordinary) unknotting operations. Fig. 3.2 shows that the (ordinary) unknotting operation can be realized by one $P_n\binom{\boldsymbol{a}}{\boldsymbol{b}}$-move (where $a_i \neq b_i$, $a_{i+1} = b_{i+1}$). □

Theorem 2.7 (2) follows from Proposition 2.6 (8) and the following lemma:

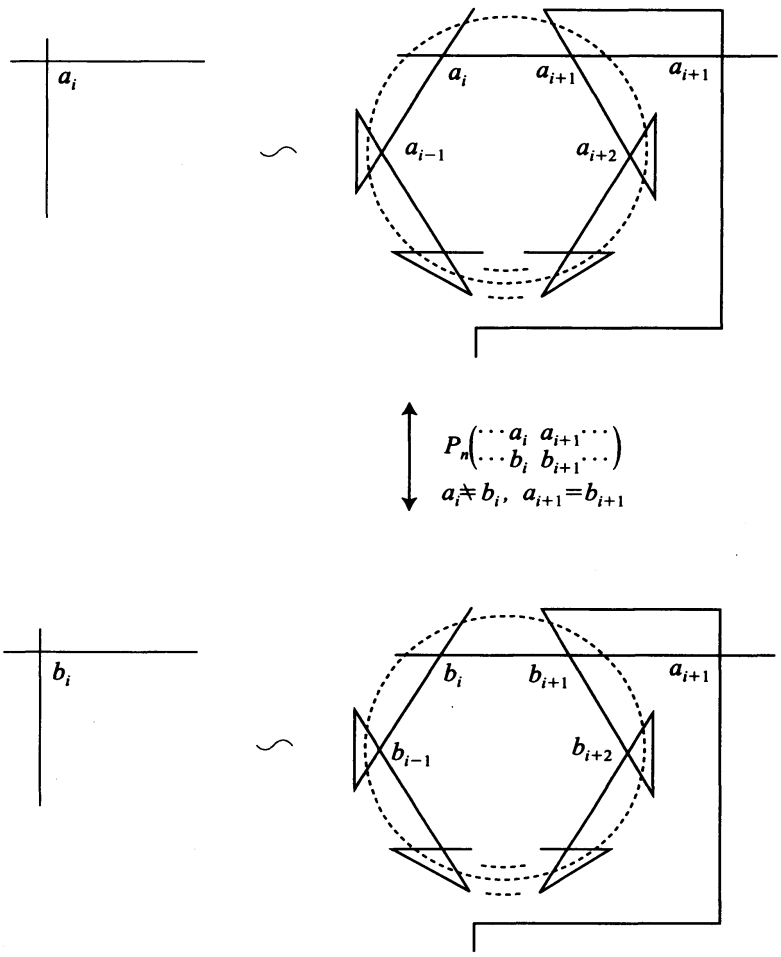

FIGURE 3.2

LEMMA 3.2. (1) *Any* $R_4\begin{pmatrix} a \\ b \end{pmatrix}$*-move induces the* $H(2)$*-move.*

(2) *Any* $R_5\begin{pmatrix} a \\ b \end{pmatrix}$*-move induces the* $H(2)$*-move.*

(3) *For any integer* n (≥ 4), *any* $R_n\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix}$*-move can be realized by one* $R_{n+2}\begin{pmatrix} a_1 \cdots a_n a_{n+1} a_{n+2} \\ b_1 \cdots b_n b_{n+1} \overline{b_{n+1}} \end{pmatrix}$*-move.*

(4) *For any integer* n (≥ 4), *any* $R_n\begin{pmatrix} a \\ b \end{pmatrix}$*-move can be realized by three* $H(n)$*-moves.*

(5) *For any integer* n (≥ 2), *the* $H(n)$*-move is of Type* T ([N3, Theorem 2.1]).

PROOF OF LEMMA 3.2. (1): Let $\mathfrak{T}_1 = \left\{ R_4\left(\begin{smallmatrix} a_1\bar{a}_1a_3a_4 \\ b_1b_2\bar{b}_2b_4 \end{smallmatrix}\right)\text{-move} \mid a_i,\, b_i \in \{+,\,-\}\right\}$, $\mathfrak{T}_2 = \left\{ R_4\left(\begin{smallmatrix} a_1\bar{a}_1a_3\bar{a}_3 \\ b_1b_2b_3b_4 \end{smallmatrix}\right)\text{-move} \mid a_i,\, b_i \in \{+,\,-\}\right\}$, $\mathfrak{T}_3 = \left\{ R_4\left(\begin{smallmatrix} a\bar{a}aa \\ bbbb \end{smallmatrix}\right)\text{-move} \mid a,\, b \in \{+,\,-\}\right\}$ and $\mathfrak{T}_4 = \left\{ R_4\left(\begin{smallmatrix} aaaa \\ bbbb \end{smallmatrix}\right)\text{-move} \mid a,\, b \in \{+,\,-\}\right\}$.

CLAIM 3.3. *Any local move in any $\mathfrak{T}_j$ $(1 \leq j \leq 4)$ induces the $H(2)$-move.*

PROOF. See Fig. 3.3 and Proposition 2.6 (8). Then the proof is completed.

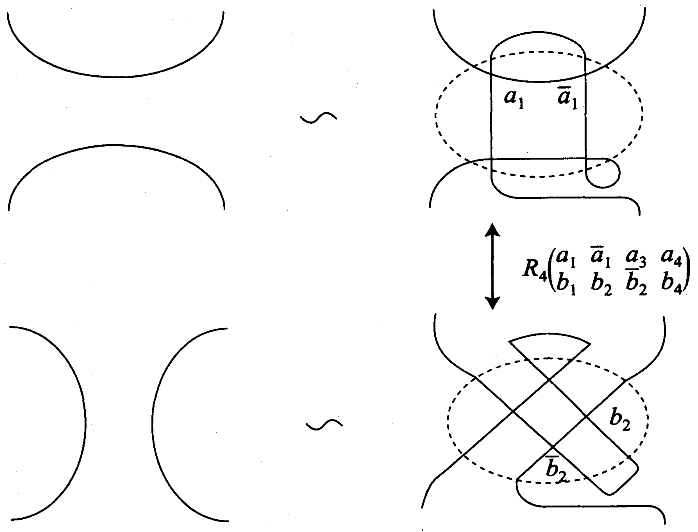

FIGURE 3.3 (1)

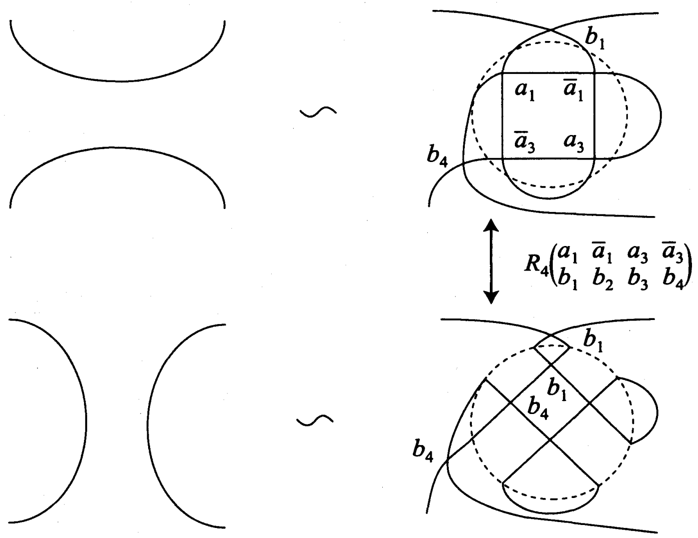

FIGURE 3.3 (2)

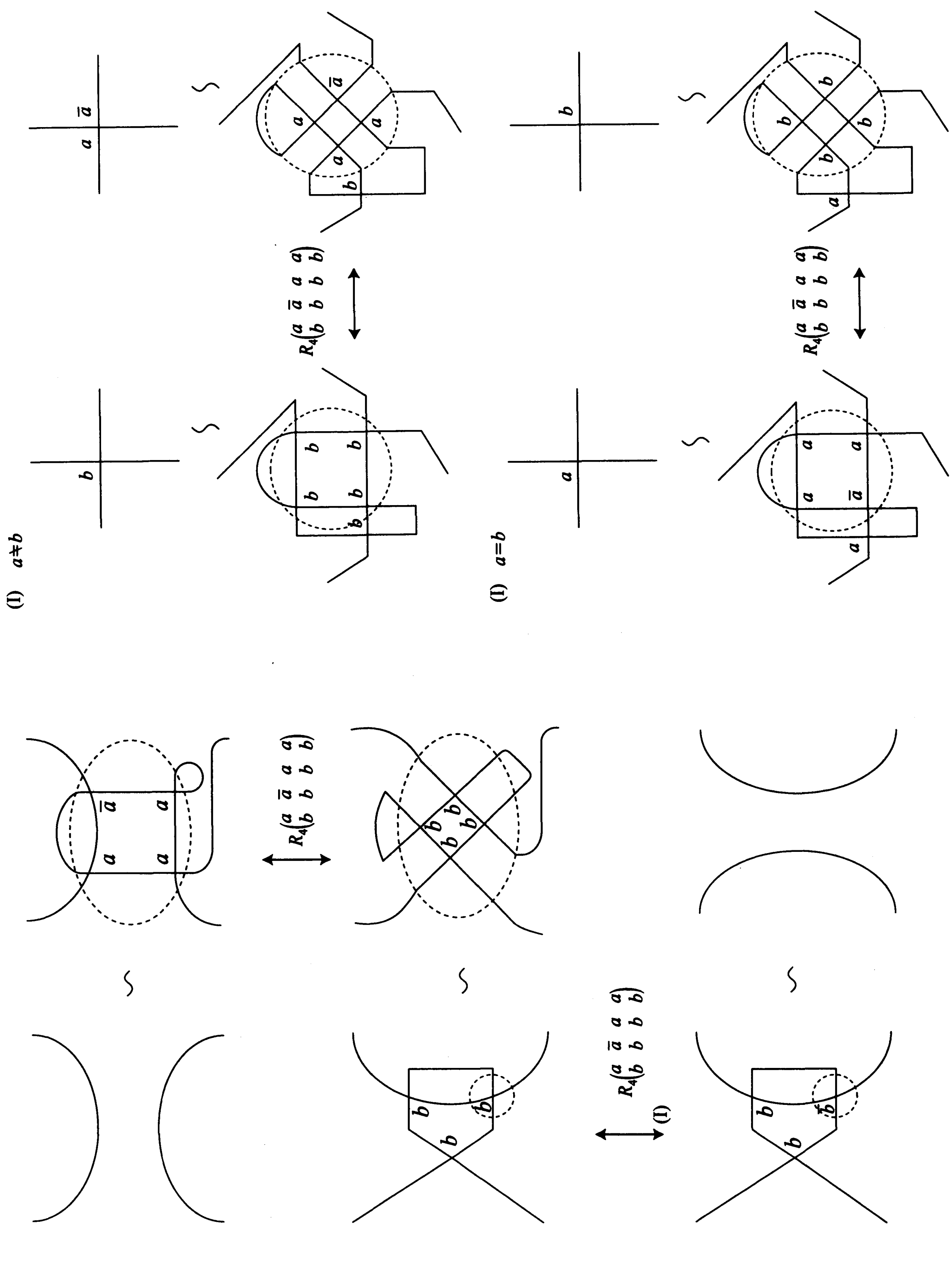

FIGURE 3.3 (3)

CLAIM 3.4. *Any $R_4\binom{a}{b}$-move belongs to some $\mathfrak{T}_j$ $(1 \leq j \leq 4)$.*

PROOF. Let $\boldsymbol{a}=(a_1a_2a_3a_4)$ and $\boldsymbol{b}=(b_1b_2b_3b_4)$ be arbitrary sequences of 4 signs, and consider the $R_4\binom{a}{b}$-move. If $a_i=a_j$ and $b_i=b_j$ for any pair of i, j $(1 \leq i < j \leq 4)$, then the $R_4\binom{a}{b}$-move belongs to $\mathfrak{T}_4$. So we may assume $a_i \neq a_j$ or $b_i \neq b_j$ for some pair of

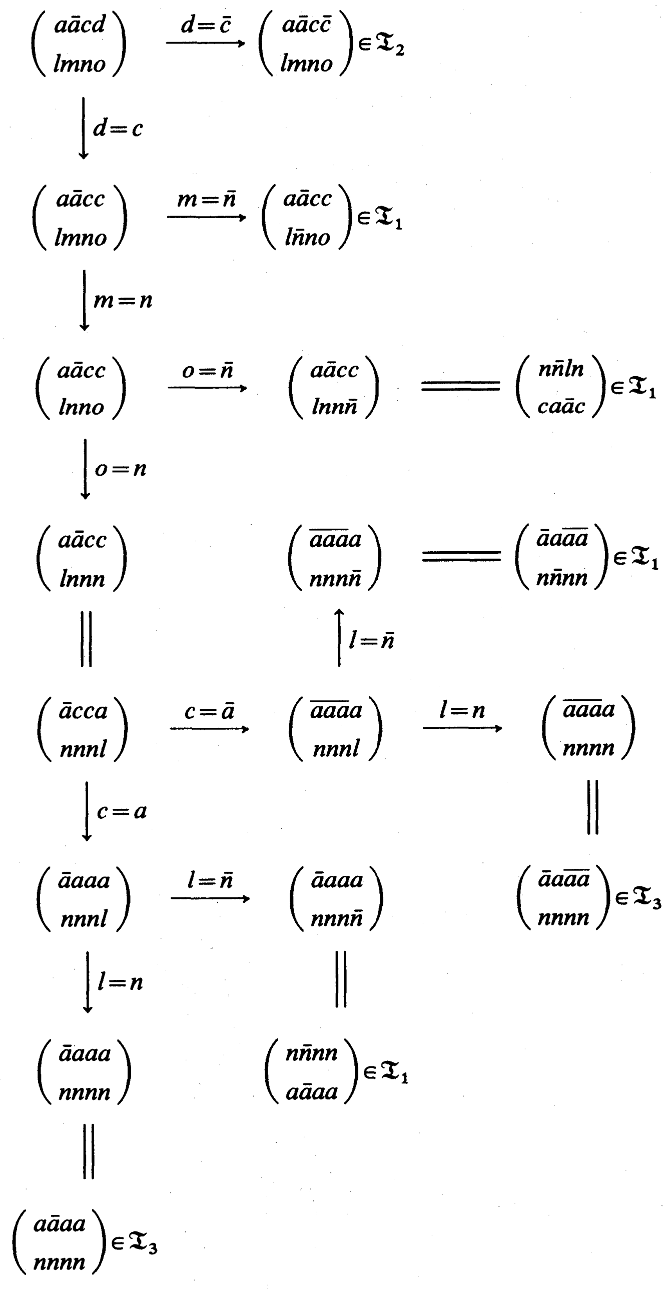

DIAGRAM 3.1

i, j. By Remark 2.5, we may assume $\binom{a}{b}=\binom{a\bar{a}cd}{lmno}$. See Diagram 3.1. The first branch has the following meaning: if $d=\bar{c}\left(i.e., \binom{a\bar{a}cd}{lmno}=\binom{a\bar{a}c\bar{c}}{lmno}\right)$, then this move belongs to $\mathfrak{T}_2$. Therefore we may assume $d=c$, and hence $\binom{a\bar{a}cd}{lmno}=\binom{a\bar{a}cc}{lmno}$. The other branches have similar meanings. Then the proof is completed.

Hence the proof of Lemma 3.2 (1) is completed.

(2): The proof is similar to that of Lemma 3.2 (1). Let $\mathfrak{R}_1=\left\{R_5\binom{a_1a_2a_3a_4a_5}{b_1\bar{a}_2b_3\bar{a}_4b_5}\text{-move} \,\middle|\, a_i, b_i\in\{+,-\}\right\}$, $\mathfrak{R}_2=\left\{R_5\binom{a_1a_2a_3a_4a_5}{\bar{a}_2b_2b_3\bar{a}_4b_5}\text{-move} \,\middle|\, a_i, b_i\in\{+,-\}\right\}$, $\mathfrak{R}_3=\left\{R_5\binom{a_1a_2\bar{a}_2a_4a_5}{b_1a_2b_3\bar{a}_4b_5}\text{-move} \,\middle|\, a_i, b_i\in\{+,-\}\right\}$, $\mathfrak{R}_4=\left\{R_5\binom{aaa\bar{a}a}{aabaa}\text{-move} \,\middle|\, a, b\in\{+,-\}\right\}$ and $\mathfrak{R}_5=\left\{R_5\binom{aaaaa}{aaaaa}\text{-move} \,\middle|\, a\in\{+,-\}\right\}$.

CLAIM 3.5. *Any local move in any $\mathfrak{R}_j$ $(1\leq j\leq 5)$ induces the $H(2)$-move.*

PROOF. See Fig. 3.4 and Proposition 2.6 (8). Then the proof is completed.

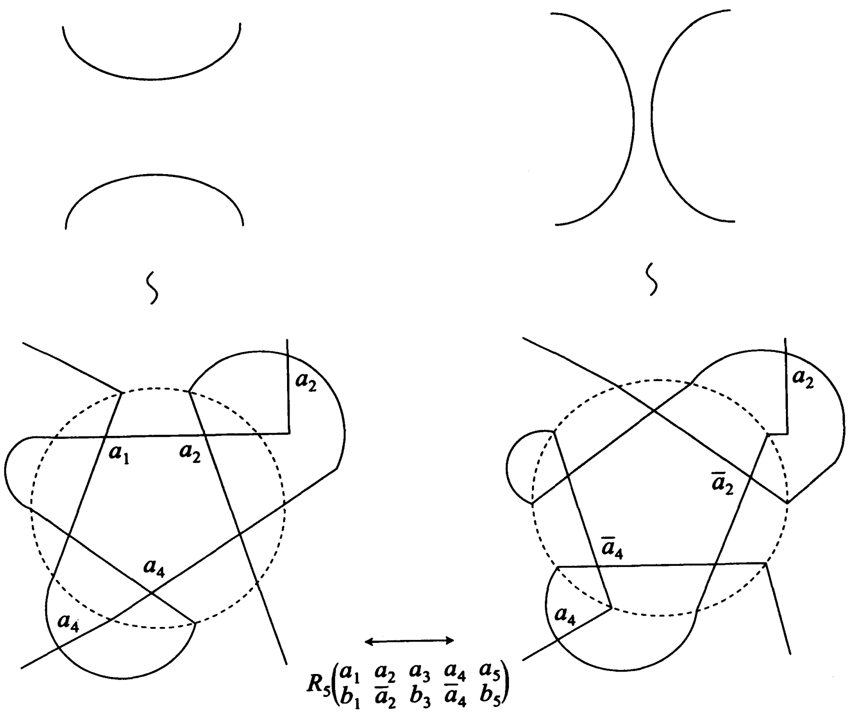

FIGURE 3.4 (1)

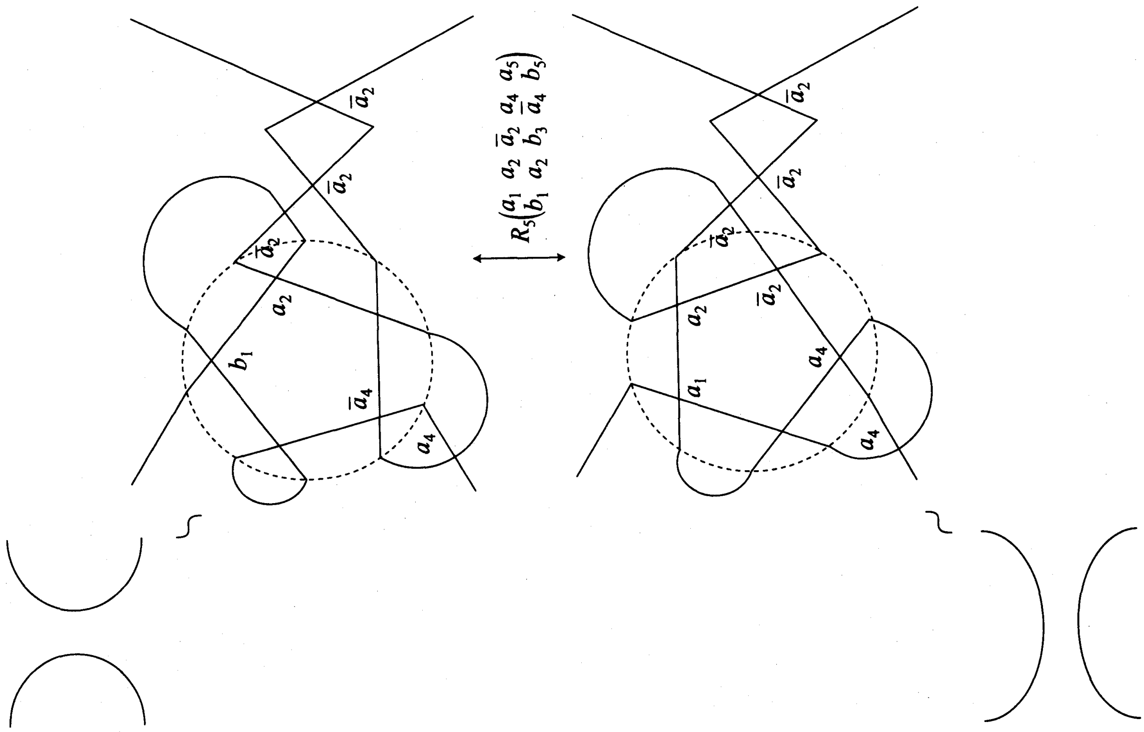

FIGURE 3.4 (3)

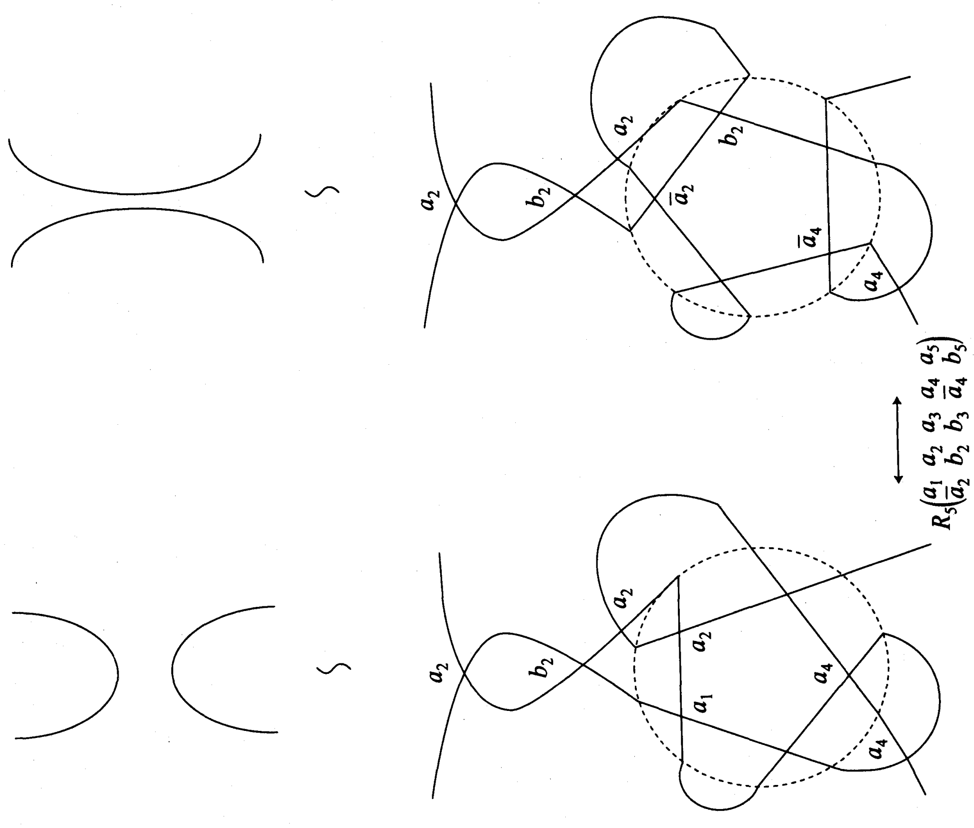

FIGURE 3.4 (2)

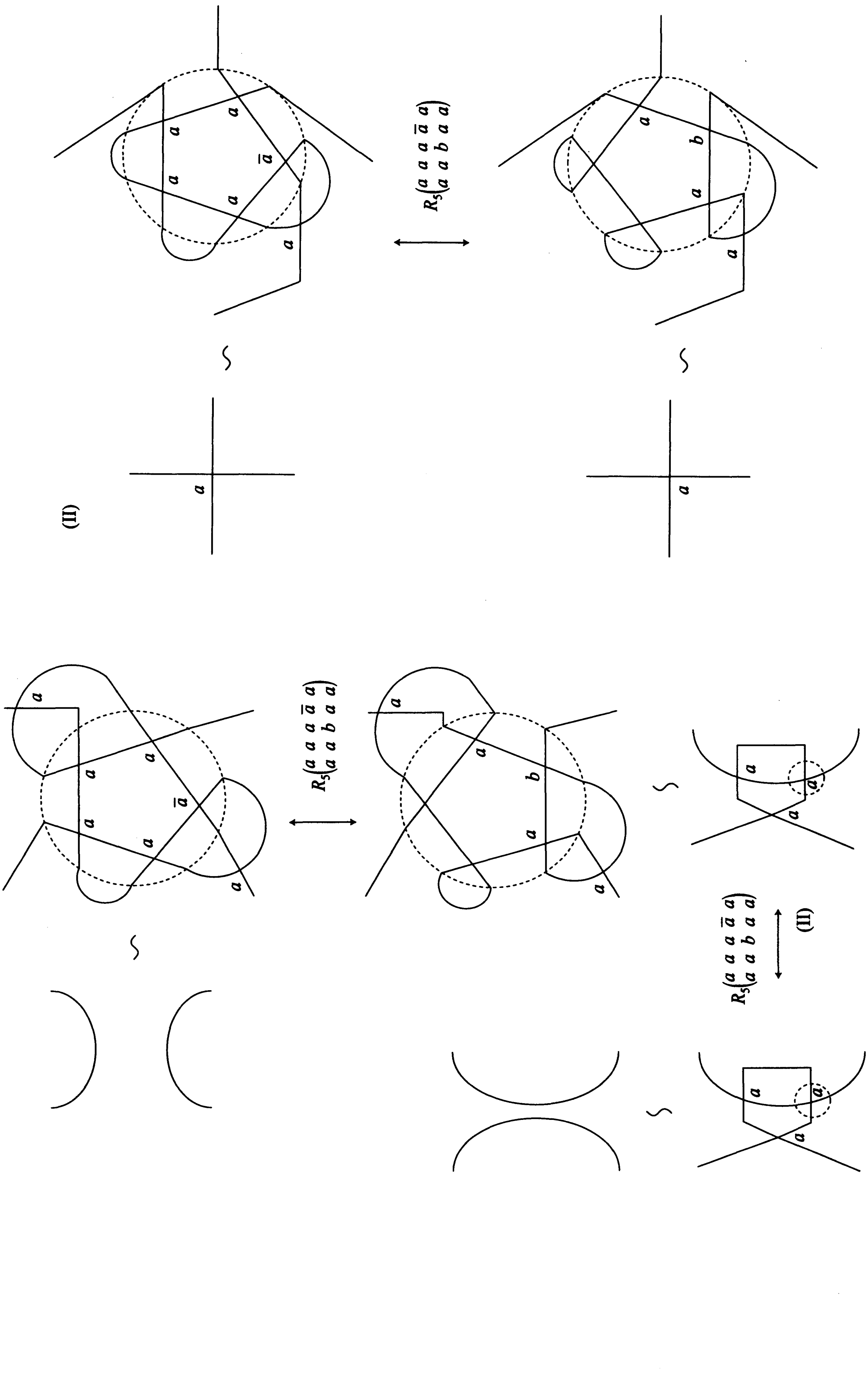
(II)
$R_5\binom{a\ a\ a\ \bar{a}\ a}{a\ a\ b\ a\ a}$
$R_5\binom{a\ a\ a\ \bar{a}\ a}{a\ a\ b\ a\ a}$
$R_5\binom{a\ a\ a\ \bar{a}\ a}{a\ a\ b\ a\ a}$
(II)

FIGURE 3.4 (4)

$$
\begin{array}{ccccccc}
\begin{pmatrix} abcde \\ lmn\bar{d}p \end{pmatrix} & \xrightarrow{m=\bar{b}} & \begin{pmatrix} abcde \\ l\bar{b}n\bar{d}p \end{pmatrix} \in \mathfrak{N}_1 & & & & \\
\Big\downarrow{\scriptstyle m=b} & & & & & & \\
\begin{pmatrix} abcde \\ lbn\bar{d}p \end{pmatrix} & \xrightarrow{l=\bar{b}} & \begin{pmatrix} abcde \\ \bar{b}bn\bar{d}p \end{pmatrix} \in \mathfrak{N}_2 & & & & \\
\Big\downarrow{\scriptstyle l=b} & & & & & & \\
\begin{pmatrix} abcde \\ bbn\bar{d}p \end{pmatrix} & \xrightarrow{c=\bar{b}} & \begin{pmatrix} ab\bar{b}de \\ bbn\bar{d}p \end{pmatrix} \in \mathfrak{N}_3 & & & & \\
\Big\downarrow{\scriptstyle c=b} & & & & & & \\
\begin{pmatrix} abbde \\ bbn\bar{d}p \end{pmatrix} & & & & & & \\
\Big\| & & & & & & \\
\begin{pmatrix} bdeab \\ n\bar{d}pbb \end{pmatrix} & \xrightarrow{a=\bar{b}} & \begin{pmatrix} bde\bar{b}b \\ n\bar{d}pbb \end{pmatrix} \in \mathfrak{N}_1 & & & & \\
\Big\downarrow{\scriptstyle a=b} & & & & & & \\
\begin{pmatrix} bdebb \\ n\bar{d}pbb \end{pmatrix} & \xrightarrow{n=d} & \begin{pmatrix} bdebb \\ d\bar{d}pbb \end{pmatrix} & \xrightarrow{d=\bar{b}} & \begin{pmatrix} b\bar{b}ebb \\ \bar{b}bpbb \end{pmatrix} & & \\
\Big\downarrow{\scriptstyle n=\bar{d}} & & \Big\downarrow{\scriptstyle d=b} & & \Big\| & & \\
{*}\begin{pmatrix} bdebb \\ \bar{d}\bar{d}pbb \end{pmatrix} & & \begin{pmatrix} bbebb \\ b\bar{b}pbb \end{pmatrix} & & \begin{pmatrix} bb\bar{b}eb \\ b\bar{b}bpb \end{pmatrix} & \xrightarrow{p=\bar{e}} & \begin{pmatrix} bb\bar{b}eb \\ b\bar{b}b\bar{e}b \end{pmatrix} \in \mathfrak{N}_1 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle p=e} & & \\
\mathfrak{N}_3 \ni \begin{pmatrix} bb\bar{b}\bar{b}b \\ bbebb \end{pmatrix} & \xleftarrow{p=\bar{b}} & \begin{pmatrix} bb\bar{b}pb \\ bbebb \end{pmatrix} & & \begin{pmatrix} bb\bar{b}eb \\ b\bar{b}beb \end{pmatrix} & \xrightarrow{e=\bar{b}} & \begin{pmatrix} bb\bar{b}\bar{b}b \\ b\bar{b}b\bar{b}b \end{pmatrix} \\
 & & \Big\downarrow{\scriptstyle p=b} & & \Big\downarrow{\scriptstyle e=b} & & \Big\| \\
\begin{pmatrix} bb\bar{b}bb \\ bbbbb \end{pmatrix} & \xleftarrow{e=b} & \begin{pmatrix} bb\bar{b}bb \\ bbebb \end{pmatrix} & & \begin{pmatrix} bb\bar{b}bb \\ b\bar{b}bbb \end{pmatrix} & & \begin{pmatrix} b\bar{b}b\bar{b}b \\ b\bar{b}\bar{b}bb \end{pmatrix} \in \mathfrak{N}_3 \\
\Big\| & & \Big\downarrow{\scriptstyle e=\bar{b}} & & \Big\| & & \\
\mathfrak{N}_4 \ni \begin{pmatrix} bbb\bar{b}b \\ bbbbb \end{pmatrix} & & \begin{pmatrix} bb\bar{b}bb \\ bb\bar{b}bb \end{pmatrix} & = & \begin{pmatrix} bbb\bar{b}b \\ bb\bar{b}bb \end{pmatrix} \in \mathfrak{N}_4 & &
\end{array}
$$

DIAGRAM 3.2 (1)

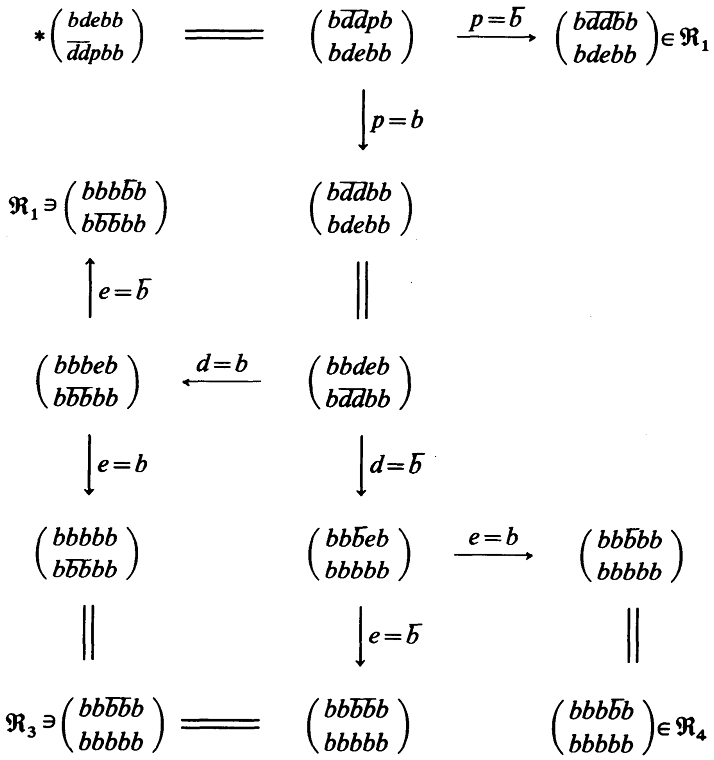

Diagram 3.2 (2)

Claim 3.6. *Any* $R_5\binom{\boldsymbol{a}}{\boldsymbol{b}}$*-move belongs to some* $\mathfrak{R}_j$ $(1 \le j \le 5)$.

Proof. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be arbitrary sequences of 5 signs, and consider the $R_5\binom{\boldsymbol{a}}{\boldsymbol{b}}$-move. If the $R_5\binom{\boldsymbol{a}}{\boldsymbol{b}}$-move doesn't belong to $\left\{R_5\binom{abcde}{lmn\bar{d}p}\text{-move}\right\}$, by Remark 2.5, the $R_5\binom{\boldsymbol{a}}{\boldsymbol{b}}$-move belongs to $\mathfrak{R}_5$. So we may assume $\binom{\boldsymbol{a}}{\boldsymbol{b}}=\binom{abcde}{lmn\bar{d}p}$. See Diagram 3.2. Then the proof is completed.

Hence the proof of Lemma 3.2 (2) is completed.

(3), (4): Lemma 3.2 (3) and (4) are proved by Fig. 3.5 and 3.6, respectively. □

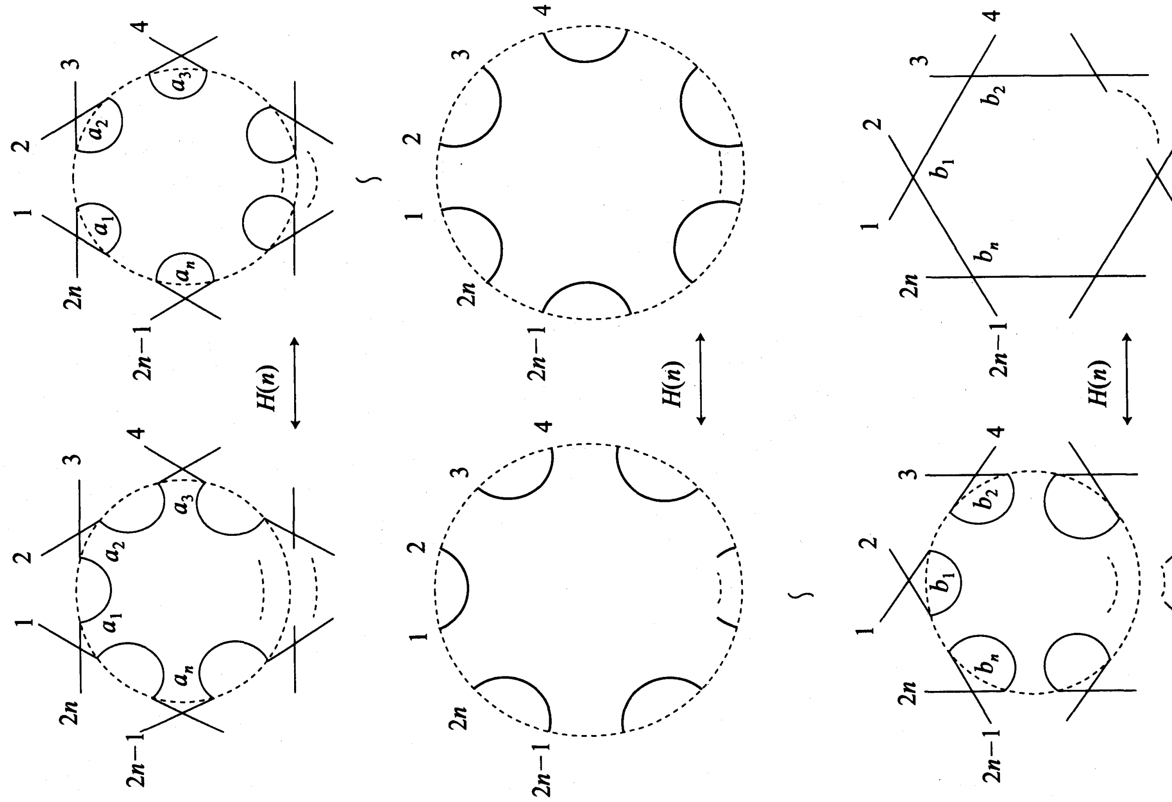

FIGURE 3.6

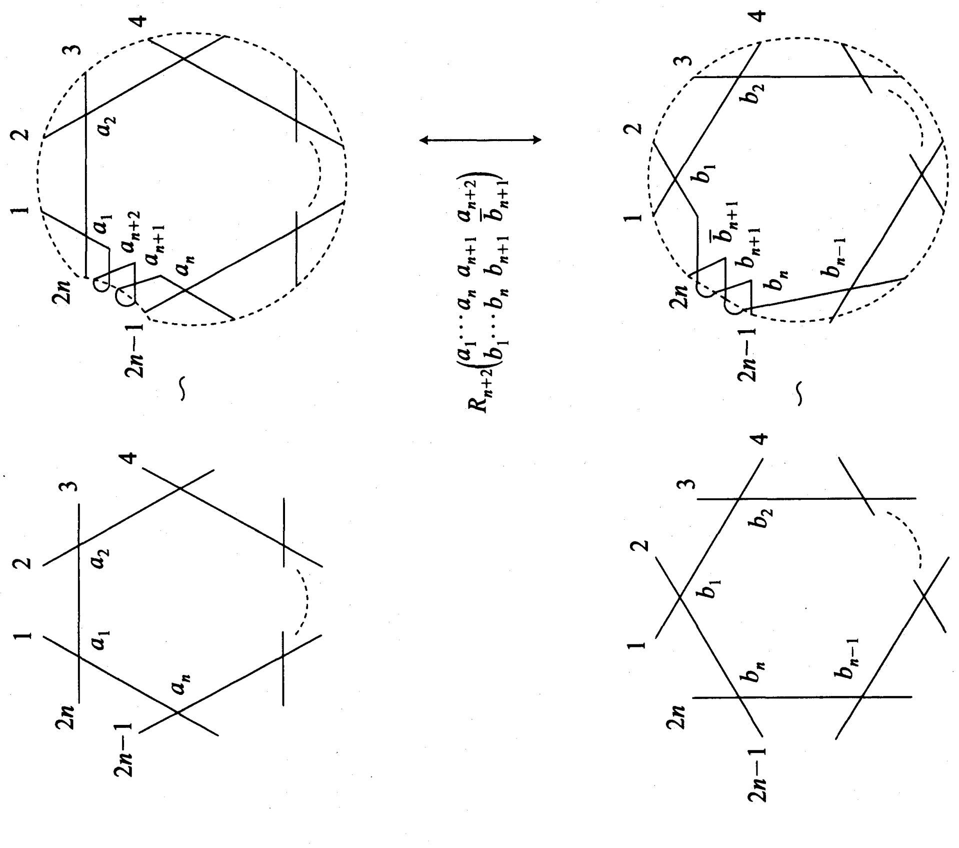

FIGURE 3.5

4. Oriented versions.

In the previous sections, we didn't consider the link orientation. For oriented link diagrams, we can also define the conception of local moves, generalized unknotting operations, local equivalence. In this section, we consider two special oriented versions of $P_n\binom{\boldsymbol{a}}{\bar{\boldsymbol{a}}}$-moves.

DEFINITION 4.1. For an integer n (≥ 3) and a sequence $\boldsymbol{a}=(a_1 a_2 \cdots a_n)$ (where $a_i \in \{+, -\}$, $1 \leq i \leq n$), the $P_n^{\circ}(\boldsymbol{a})$-move and the $P_n^{\wedge}(\boldsymbol{a})$-move are local moves on oriented link diagrams as indicated in Fig. 4.1 (1) and (2), respectively.

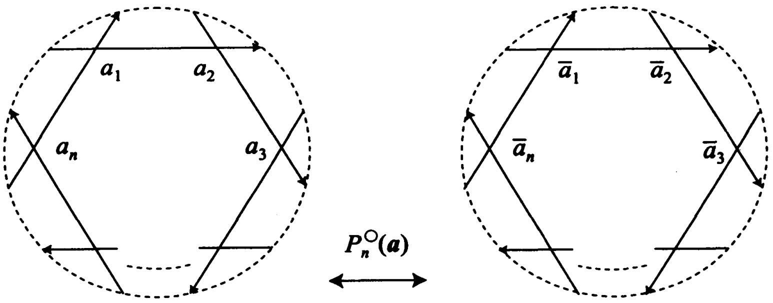

FIGURE 4.1 (1)

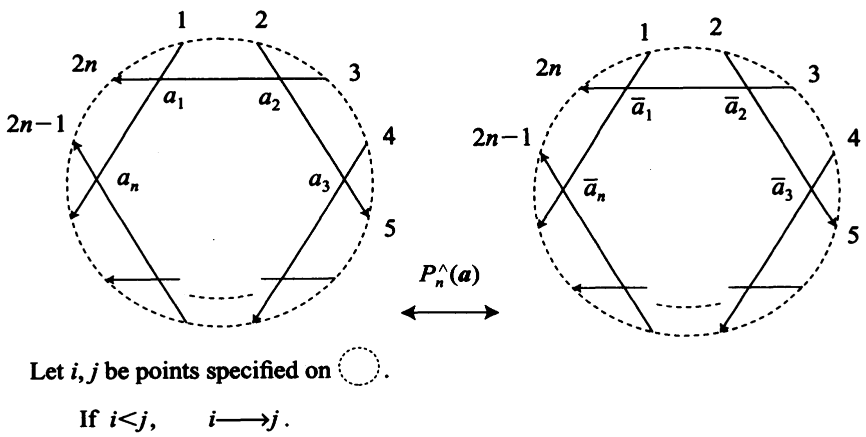

FIGURE 4.1 (2)

The $P_4^{\circ}(+-+-)$-move is the pass-move, hence it isn't a generalized unknotting operation. The $P_4^{\wedge}(+-+-)$-move is the #-unknotting operation, hence it is a generalized unknotting operation. Furthermore, the #-unknotting operation induces the pass-move ([MN, Appendix]). The following theorem is proved by H. Aida.

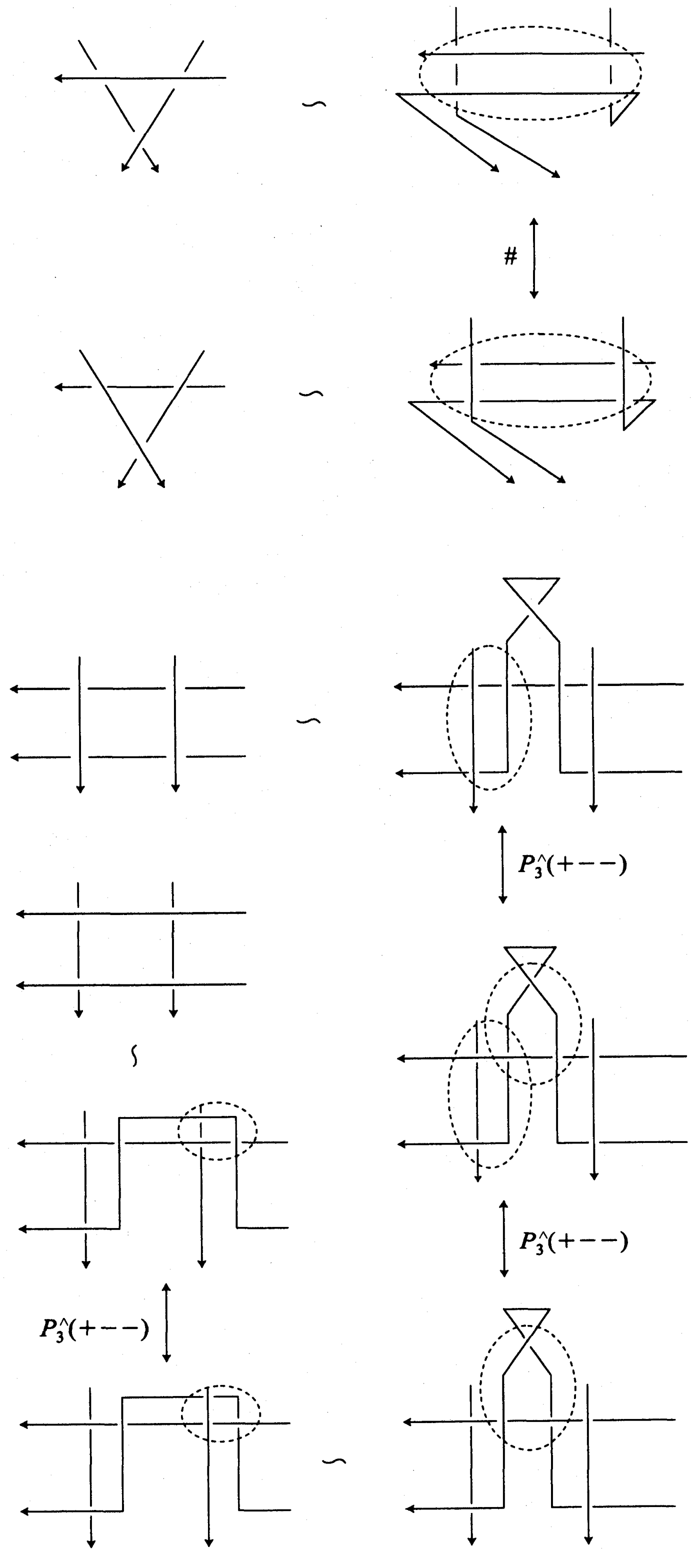

FIGURE 4.2

THEOREM 4.2. ([A2, Theorem 3]). (1) *The Δ°_{13}-move (i.e., $P^{\circ}_3(+++)$-move) and the pass-move are locally equivalent.*

(2) *The $\Delta^{\wedge}_{13}$-move (i.e., $P^{\wedge}_3(+++)$-move) and the $\sharp$-unknotting operation are locally equivalent.*

The following theorem generalizes the above results.

THEOREM 4.3. (1) *For any integer n (≥ 3) and any sequence $\boldsymbol{a}=(a_1a_2\cdots a_n)$, the $P^{\circ}_n(\boldsymbol{a})$-move and the pass-move are locally equivalent.*

(2) *For any integer n (≥ 3), if $\boldsymbol{a}=(+-\underbrace{++\cdots+}_{n-3}-)$ or $(++\cdots+)$, then the $P^{\wedge}_n(\boldsymbol{a})$-move and the $\sharp$-unknotting operation are locally equivalent.*

PROOF. (1): By Theorem 4.2 (1) and by suitably orienting strings in Fig. 3.1, the proof is similar to that of Theorem 2.7 (1) (a).

(2): The $P^{\wedge}_3(+++)$-move and the $\sharp$-unknotting operation are locally equivalent (Theorem 4.2 (2)), and so is the $P^{\wedge}_3(+--)$-move as indicated in Fig. 4.2.

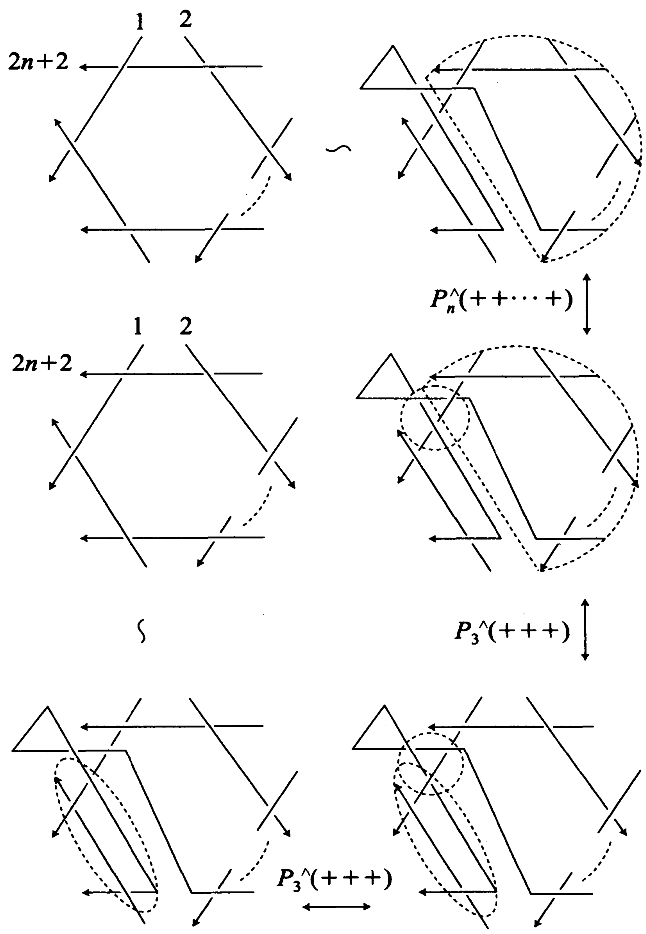

FIGURE 4.3

By suitably rotating and by suitably orienting strings (i.e., by letting $a_1=b_n$, $a_2=b_1$, $a_3=b_2$, $\cdots$, $a_n=b_{n-1}$, $a_{n+1}=b_{n+1}$) in Fig. 3.1 (2), we obtain the following lemma:

LEMMA 4.4. *For any integer n* (≥ 3), *any* $P_n^\wedge(b_1\cdots b_{n-1}b_n)$*-move can be realized by one* $P_{n+1}^\wedge(b_1\cdots b_{n-1}b_{n+1}b_n)$*-move.*

Fig. 4.3 shows that one $P_{n+1}^\wedge(+++\cdots+)$-move can be realized by one $P_n^\wedge(++\cdots+)$-move and two $P_3^\wedge(+++)$-moves. Similarly, we obtain that one $P_{n+1}^\wedge(+-++\cdots+++-)$-move can be realized by one $P_n^\wedge(+-++\cdots++-)$-move and two $P_3^\wedge(+--)$-moves. Hence the proof is completed. □

REMARK 4.5. Every $P_n^\wedge(\boldsymbol{a})$-move isn't locally equivalent to the ♯-unknotting operation. Fig. 4.4 shows that one $P_3^\wedge(--+)$-move can be realized by one pass-move. Therefore the $P_3^\wedge(--+)$-move isn't even a generalized unknotting operation. The author doesn't know whether another $P_n^\wedge(\boldsymbol{a})$-move is locally equivalent to the ♯-unknotting operation.

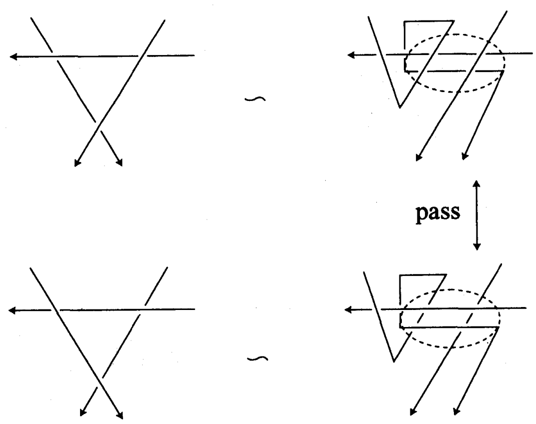

FIGURE 4.4

5. *M*-unknotting number one.

Let K be (oriented) knot in S^3. The *unknotting number of K*, denoted by $u(K)$, is the minimum number of (ordinary) unknotting operations which are necessary to transform K into a trivial knot. Let M-move be a generalized unknotting operation. We define the *M-unknotting number of K*, denoted by $u^M(K)$, using the M-moves instead of the (ordinary) unknotting operations.

The unknotting number one knots are prime ([Sc]). On the other hand, there exists a knot with the $P_n\begin{pmatrix} ++\cdots+ \\ --\cdots- \end{pmatrix}$-unknotting number one and with arbitrary many prime factors. For example, the connected sum of n copies of 3_1 has $P_{2n}\begin{pmatrix} ++\cdots+ \\ --\cdots- \end{pmatrix}$-unknotting number one.

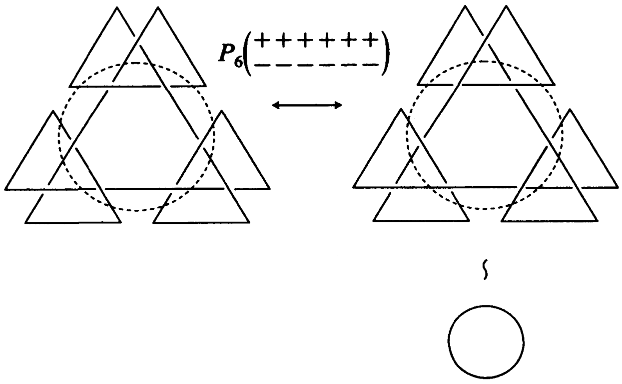

FIGURE 5.1

Besides, there are infinitely many oriented composite knots with "#-unknotting number" (i.e., $P_4^{\wedge}(+ - + -)$-unknotting number) one ([Sa, Proposition 1.1]). By modifying the method in [Sa, proof of Proposition 1.1], we obtain the following results.

PROPOSITION 5.1. *For any integer n (≥ 4) and any $\boldsymbol{a}=(a_1 \cdots a_n)$,*

(1) *There are infinitely many composite knots with $P_n\begin{pmatrix}\boldsymbol{a}\\ \bar{\boldsymbol{a}}\end{pmatrix}$-unknotting number one.*

(2) *There are infinitely many oriented composite knots with $P_n^{\wedge}(\boldsymbol{a})$-unknotting number one ($\boldsymbol{a}=(+ - \underbrace{+ + \cdots +}_{n-3} -)$ or $(+ + \cdots +)$).*

PROOF. By Lemma 3.1 (2) and Lemma 4.4, we obtain the following lemma:

LEMMA 5.2. *For any (oriented) knot K and any integer n (≥ 3),*

(1) $u^{P_n\begin{pmatrix}a_1 \cdots a_n\\ \bar{a}_1 \cdots \bar{a}_n\end{pmatrix}}(K) \geq u^{P_{n+1}\begin{pmatrix}a_1 \cdots a_n a_{n+1}\\ \bar{a}_1 \cdots \bar{a}_n \bar{a}_{n+1}\end{pmatrix}}(K)$

(2) $u^{P_n^{\wedge}(+ - + + \cdots + -)}(K) \geq u^{P_{n+1}^{\wedge}(+ - + + \cdots + + -)}(K)$

(3) $u^{P_n^{\wedge}(+ \cdots + +)}(K) \geq u^{P_{n+1}^{\wedge}(+ \cdots + + +)}(K)$.

We consider the knots

$$K_m^1 = C(-3)\#C(-3)\#C(2, 2, 2, 2m)$$

$$K_m^2 = C(-2, -2)\#C(-3)\#C(2, 2, 2, 2m)$$

$$K_m^3 = C(-2, -2)\#C(2, 2, 2, 2m)$$

$$K_m^4 = C(2, -2m)\#C(2, 2m+2)$$

described in Fig. 5.2. Here the symbol of $C(*)$ means the Conway notation for two-bridge knots ([C]). Fig. 5.2 illustrates

$$u^{P_4\begin{pmatrix}+ - + -\\ - + - +\end{pmatrix}}(K_m^1)=1 \quad u^{P_4\begin{pmatrix}+ + - +\\ - - + -\end{pmatrix}}(K_m^2)=1 \quad u^{P_4\begin{pmatrix}+ - - +\\ - + + -\end{pmatrix}}(K_m^3)=1 \quad u^{P_4\begin{pmatrix}+ + + +\\ - - - -\end{pmatrix}}(K_m^4)=1 .$$

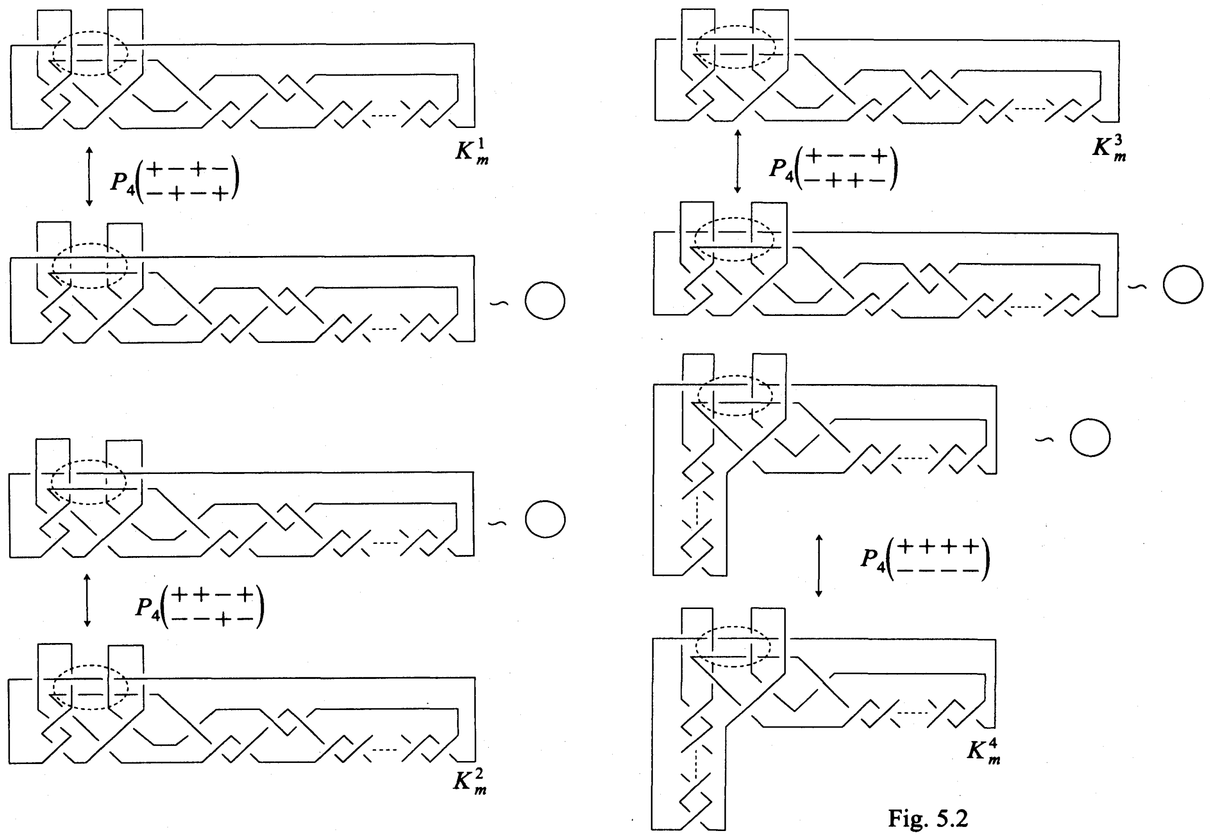

FIGURE 5.2

(1): By Remark 2.5 (2) and (3), any $P_4\begin{pmatrix} a \\ \bar{a} \end{pmatrix}$-move is equivalent to the $P_4\begin{pmatrix} +-+- \\ -+-+ \end{pmatrix}$-move, the $P_4\begin{pmatrix} ++-+ \\ --+- \end{pmatrix}$-move, the $P_4\begin{pmatrix} +--+ \\ -++- \end{pmatrix}$-move or the $P_4\begin{pmatrix} ++++ \\ ---- \end{pmatrix}$-move. By Lemma 5.2 (1), the proof is completed.

(2): By suitably orienting strings and rotating and turning over the figures, the proof is similar to that of Proposition 5.1 (1). □

6. Appendix.

The $u(K)$ for approximately a quarter of the prime knots with ten or fewer crossings remains undetermined. Though the Δ-unknotting operation seems to be more complicated than the (ordinary) unknotting operation, we can determine the Δ-unknotting numbers $u^\Delta(K)$ for the prime knots with ten crossings except for 14 knots as in Table A. Here "N" means 1 or 3, and "M" means 2 or 4. (In [Ok2], the $u^\Delta(K)$ for the prime knots with nine or fewer crossings was determined.) For the diagrams of these prime knots, we refer to [R].

TABLE A

K	u^{Δ}	K	u^{Δ}	K	u^{Δ}	K	u^{Δ}	K	u^{Δ}
3_1	1	9_{16}	6	10_{17}	2	10_{67}	2	10_{117}	2
4_1	1	9_{17}	2	10_{18}	2	10_{68}	2	10_{118}	2
5_1	3	9_{18}	6	10_{19}	1	10_{69}	2	10_{119}	1
5_2	2	9_{19}	2	10_{20}	3	10_{70}	3	10_{120}	6
6_1	2	9_{20}	2	10_{21}	3	10_{71}	1	10_{121}	1
6_2	1	9_{21}	3	10_{22}	4	10_{72}	M	10_{122}	2
6_3	1	9_{22}	1	10_{23}	3	10_{73}	1	10_{123}	2
7_1	6	9_{23}	5	10_{24}	2	10_{74}	2	10_{124}	8
7_2	3	9_{24}	1	10_{25}	2	10_{75}	2	10_{125}	3
7_3	5	9_{25}	2	10_{26}	3	10_{76}	M	10_{126}	5
7_4	4	9_{26}	2	10_{27}	2	10_{77}	4	10_{127}	3
7_5	4	9_{27}	2	10_{28}	3	10_{78}	3	10_{128}	7
7_6	1	9_{28}	1	10_{29}	4	10_{79}	5	10_{129}	2
7_7	1	9_{29}	1	10_{30}	N	10_{80}	6	10_{130}	4
8_1	3	9_{30}	1	10_{31}	2	10_{81}	3	10_{131}	2
8_2	2	9_{31}	2	10_{32}	1	10_{82}	2	10_{132}	3
8_3	4	9_{32}	1	10_{33}	2	10_{83}	1	10_{133}	N
8_4	3	9_{33}	1	10_{34}	3	10_{84}	2	10_{134}	6
8_5	3	9_{34}	1	10_{35}	4	10_{85}	2	10_{135}	3
8_6	2	9_{35}	7	10_{36}	N	10_{86}	1	10_{136}	2
8_7	2	9_{36}	3	10_{37}	3	10_{87}	2	10_{137}	2
8_8	2	9_{37}	3	10_{38}	N	10_{88}	1	10_{138}	3
8_9	2	9_{38}	6	10_{39}	3	10_{89}	1	10_{139}	9
8_{10}	3	9_{39}	2	10_{40}	3	10_{90}	3	10_{140}	2
8_{11}	1	9_{40}	1	10_{41}	2	10_{91}	2	10_{141}	1
8_{12}	3	9_{41}	2	10_{42}	2	10_{92}	M	10_{142}	8
8_{13}	1	9_{42}	2	10_{43}	2	10_{93}	N	10_{143}	3
8_{14}	2	9_{43}	3	10_{44}	2	10_{94}	2	10_{144}	2
8_{15}	4	9_{44}	2	10_{45}	2	10_{95}	3	10_{145}	5
8_{16}	1	9_{45}	2	10_{46}	4	10_{96}	3	10_{146}	2
8_{17}	1	9_{46}	2	10_{47}	6	10_{97}	M	10_{147}	1
8_{18}	1	9_{47}	1	10_{48}	4	10_{98}	M	10_{148}	4
8_{19}	5	9_{48}	3	10_{49}	7	10_{99}	4	10_{149}	M
8_{20}	2	9_{49}	6	10_{50}	3	10_{100}	4	10_{150}	3
8_{21}	2	10_1	4	10_{51}	5	10_{101}	7	10_{151}	3
9_1	10	10_2	4	10_{52}	3	10_{102}	2	10_{152}	7
9_2	4	10_3	6	10_{53}	6	10_{103}	3	10_{153}	4
9_3	9	10_4	5	10_{54}	4	10_{104}	1	10_{154}	5
9_4	7	10_5	4	10_{55}	5	10_{105}	1	10_{155}	2
9_5	6	10_6	3	10_{56}	M	10_{106}	1	10_{156}	1
9_6	7	10_7	1	10_{57}	4	10_{107}	1	10_{157}	4
9_7	5	10_8	3	10_{58}	4	10_{108}	2	10_{158}	3
9_8	2	10_9	2	10_{59}	1	10_{109}	3	10_{159}	2
9_9	8	10_{10}	1	10_{60}	1	10_{110}	3	10_{160}	3
9_{10}	8	10_{11}	5	10_{61}	4	10_{111}	3	10_{161}	7
9_{11}	4	10_{12}	4	10_{62}	5	10_{112}	2	10_{162}	7
9_{12}	1	10_{13}	5	10_{63}	6	10_{113}	2	10_{163}	3
9_{13}	7	10_{14}	M	10_{64}	3	10_{114}	1	10_{164}	1
9_{14}	1	10_{15}	3	10_{65}	4	10_{115}	1	10_{165}	1
9_{15}	2	10_{16}	4	10_{66}	7	10_{116}	2	10_{166}	M

References

[A1] H. AIDA, Unknotting operations of polygonal type, Tokyo J. Math. **15** (1992), 111–121.

[A2] H. AIDA, The oriented Δ_{ij}-moves on links, Kobe J. Math. **9** (1992), 163–170.

[C] J. H. CONWAY, An enumeration of knots and links and some of their algebraic properties, *Computational Problems in Abstract Algebra*, Pergamon (1970), 329–358.

[HNT] J. HOSTE, Y. NAKANISHI and K. TANIYAMA, Unknotting operations involving trivial tangle, Osaka J. Math. **27** (1990), 555–566.

[K] A. KAWAUCHI (ed.), *Musubimeriron*, Springer (1990) (in Japanese).

[M] H. MURAKAMI, Some metrics on classical knots, Math. Ann. **270** (1985), 35–45.

[MN] H. MURAKAMI and Y. NAKANISHI, On a certain move generating link-homology, Math. Ann. **284** (1989), 75–89.

[N1] Y. NAKANISHI, On Fox's congruence classes of knots, II, Osaka J. Math. **27** (1990), 207–215.

[N2] Y. NAKANISHI, Replacements in the Conway third identity, Tokyo J. Math. **14** (1991), 197–203.

[N3] Y. NAKANISHI, On generalized unknotting operations, J. Knot Theory Ramification **3** (1994), 197–209.

[Oh1] Y. OHYAMA, Unknotting operations of rotation type, Tokyo J. Math. **15** (1992), 357–363.

[Oh2] Y. OHYAMA, Twisting and unknotting operations, Rev. Mat. Univ. Complut. Madrid **7** (1994), 289–305.

[Ok1] M. OKADA, Delta-unknotting operation and the second coefficient of the Conway polynomial, J. Math. Soc. Japan **42** (1990), 713–717.

[Ok2] M. OKADA, Master thesis, Osaka Univ. (1991) (in Japanese).

[R] D. ROLFSEN, *Knots and Links*, Publish or Perish (1976).

[Sa] S. SAKAI, Master thesis, Osaka Univ. (1992) (in Japanese).

[Sc] M. G. SCHARLEMANN, Unknotting number one knots are prime, Invent. Math. **82** (1985), 37–55.

[U] Y. UCHIDA, On delta-unknotting operation, Osaka J. Math. **30** (1993), 753–757.

[W] H. WENDT, Die gordische Auflösung von Knoten, Math. Z. **42** (1937), 680–696 (in German).

Present Address:
8–11 CHOFUTOYOURA-CHO, SHIMONOSEKI-SHI, YAMAGUCHI, 752 JAPAN.