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Cartan Embeddings of Compact Riemannian 3-Symmetric Spaces

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Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

Introduction.

Let G be a compact connected Lie group and σ be an automorphism on G. We put $K=\{k\in G: \sigma(k)=k\}$. A mapping $g\mapsto g\sigma(g^{-1})$ of G into G naturally induces an embedding of G/K into G. We denote the embedding by Ψ_σ and call it the *Cartan embedding*.

If we assume that σ is an involutive automorphism, then Ψ_σ is a totally geodesic embedding. The author classified the compact irreducible symmetric pairs (G, K) such that the image of the corresponding Cartan embedding is a stable minimal submanifold of G ([4]).

In this paper, we study the similar problem for the case that G is a compact simple Lie group and σ is an automorphism of order 3. In this case, the image of the Cartan embedding is not necessarily a minimal submanifold. So we study

1. Is Cartan embedding a minimal embedding?
2. If it is a minimal embedding, then is the image a stable minimal submanifold?

1. Cartan embedding.

Let G be a compact connected simple Lie group and σ be an automorphism on G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and $K=\{k\in G: \sigma(k)=k\}$ respectively. Take an $Ad(G)$-invariant and $d\sigma$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. We extend $\langle\ ,\ \rangle$ to a biinvariant Riemannian metric on G and denote it also by $\langle\ ,\ \rangle$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. We identify the subspace $\mathfrak{m}$ with the tangent space M_o of M at the origin $o=eK$ by the projection $G\to M=G/K$. A G-invariant Riemannian metric g on M is said to be a *normal homogeneous metric* if it is associated with the restriction

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of $c^2\langle\ ,\ \rangle$ to $\mathfrak{m}$ for some nonnegative constant c. The induced Riemannian metric by the Cartan embedding is not always a normal homogeneous metric.

LEMMA 1.1. *If σ is an automorphism of order 3, then we have*

$$\langle d\Psi_\sigma(X), d\Psi_\sigma(Y)\rangle = 3\langle X, Y\rangle \qquad \textit{for} \quad X, Y\in\mathfrak{m}\,.$$

PROOF. Since $d\sigma^2+d\sigma+1=0$ holds on $\mathfrak{m}$, we have

$$\begin{aligned}\langle d\Psi_\sigma(X), d\Psi_\sigma(Y)\rangle &= \langle(1-d\sigma)X, (1-d\sigma)Y\rangle\\ &= 2\langle X, Y\rangle - \langle d\sigma X, Y\rangle - \langle X, d\sigma Y\rangle\\ &= 2\langle X, Y\rangle - \langle X, (d\sigma+d\sigma^2)Y\rangle = 3\langle X, Y\rangle\,.\end{aligned}$$ □

The following endomorphism on $\mathfrak{m}$ naturally induces a G-invariant almost complex structure J on M

$$J(X)=(1/\sqrt{3})(1+2d\sigma)X \qquad \text{for} \quad X\in\mathfrak{m}\,.$$

We call J the canonical almost complex structure on M. For a vector $Z\in\mathfrak{g}$, we denote by $Z_{\mathfrak{k}}$ and $Z_{\mathfrak{m}}$ the $\mathfrak{k}$-component and $\mathfrak{m}$-component of Z respectively.

LEMMA 1.2. *Let h be the second fundamental form of the Cartan embedding Ψ_σ corresponding to an automorphism σ of order 3. Then we have*

$$h(X, Y)=(\sqrt{3}/2)[JX, Y]_{\mathfrak{k}} \qquad \textit{for} \quad X, Y\in\mathfrak{m}\,.$$

PROOF. We denote by L_g [resp. R_g] the left [resp. right] translation by $g\in G$. For an element $Y\in\mathfrak{g}$, we define a vector field Y^* on G by

$$Y^*|_g = d/dt|_0 dL_{\exp tY} dR_{\sigma(\exp(-t)Y)} g = dL_g(Ad(g^{-1})Y - d\sigma Y)\,.$$

Since $L_g R_{\sigma(g^{-1})}$ $(g\in G)$ is an isometric transformation on G and $\Psi_\sigma(g)=L_g R_{\sigma(g^{-1})}e$, Y^* is a Killing vector field on G and is tangent to $\Psi_\sigma(M)$. Put $X'=d\Psi_\sigma(X)=X-d\sigma X$ for $X\in\mathfrak{m}$. Since the parallel translation $\tau_0^t\colon T_eG\to T_{\exp tX'}G$ along the geodesic $t\mapsto \exp tX'$ is $dL_{\exp(t/2)X'} dR_{\exp(t/2)X'}$, we have

$$\begin{aligned}\nabla_{X'} dL_g Ad(g^{-1})Y &= d/dt|_0 \tau_t^0 dL_{\exp tX'} Ad((\exp tX')^{-1})Y\\ &= d/dt|_0 dL_{\exp(-t/2)X'} dR_{\exp(t/2)X'} Y = -(1/2)[X', Y]\,.\end{aligned}$$

Since $\nabla_{X'} dL_g(-d\sigma Y) = -(1/2)[X', d\sigma Y]$, we have

$$\begin{aligned}\nabla_{X'} Y^* &= -(1/2)[(1-d\sigma)X, (1+d\sigma)Y]\\ &= -(1/2)(1-d\sigma)[X, Y] - (1/2)\{[X, d\sigma Y]-[d\sigma X, Y]\}\,.\end{aligned}$$

By the definition of J

$$\nabla_{X'} Y^* = -(1/2)(1-d\sigma)[X, Y] + (\sqrt{3}/4)\{[JY, X]-[Y, JX]\}\,. \tag{1}$$

Since we have ([6, p. 136])

$$[JX, Y]_{\mathfrak{m}} = -J[X, Y]_{\mathfrak{m}} = -[JY, X]_{\mathfrak{m}}, \qquad [X, Y]_{\mathfrak{k}} = [JX, JY]_{\mathfrak{k}},$$

we get the lemma by taking the normal component of (1). □

Next we classify the pair (G, σ) of compact simple Lie group G and automorphism σ on G of order 3, for which the corresponding Cartan embedding is a minimal embedding.

Case 1. *σ is an inner automorphism.* Let G be a compact connected simple Lie group and T be a maximal torus. Let $\mathfrak{t}$ be the Lie algebra of T. We denote by $\Sigma(\mathfrak{g})$ the set of all non-zero roots of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{t}^{\mathbf{C}}$ and $\Pi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_n\}$ the set of all simple roots. In this note, we adopt the same numberings of simple roots as that used by Wolf and Gray [6]. We denote by $\alpha_0 = \sum_{j=1}^{n} m_j \alpha_j$ the highest root and put

$$\mathscr{D}_0 = \{H \in \sqrt{-1}\,\mathfrak{t} : \alpha_j(H) \geq 0 \ (1 \leq j \leq n),\ \alpha_0(H) \leq 1\}.$$

Then we may assume that $\sigma = ad(\exp 2\pi\sqrt{-1}\,H)$ for some $H \in \mathscr{D}_0$.

THEOREM A. *Let G be a compact connected simple Lie group with a biinvariant Riemannian metric and $\sigma = ad(\exp 2\pi\sqrt{-1}\,H)$ be an inner automorphism on G of order* 3. *Then the Cartan embedding*

$$\Psi_\sigma : G/K \longrightarrow G;\ gK \longmapsto g \cdot \sigma(g^{-1})$$

is a minimal embedding if and only if $(\mathfrak{g}, H, \mathfrak{k})$ is one of the following:

$\mathfrak{g}$		H	$\mathfrak{k}$
$\mathfrak{su}(3m)$	$(m \geq 1)$	$(v_m + v_{2m})/3$	$\mathfrak{su}(m) + \mathfrak{su}(m) + \mathfrak{su}(m) + \mathbf{R}^2$
$\mathfrak{so}(4m+1)$	$(m \geq 1)$	$2v_{m+1}/3$	$\mathfrak{su}(m+1) + \mathfrak{so}(2m-1) + \mathbf{R}$
$\mathfrak{sp}(3m-1)$	$(m \geq 2)$	$2v_{2m-1}/3$	$\mathfrak{su}(2m-1) + \mathfrak{sp}(m) + \mathbf{R}$
$\mathfrak{so}(6m-2)$	$(m \geq 2)$	$2v_{2m-1}/3$	$\mathfrak{su}(2m+1) + \mathfrak{so}(2m) + \mathbf{R}$
$\mathfrak{so}(8)$		$(v_3 + v_4)/3$	$\mathfrak{su}(2) + \mathbf{R}^2$
$\mathfrak{e}_6$		v_3	$\mathfrak{su}(3) + \mathfrak{su}(3) + \mathfrak{su}(3)$
$\mathfrak{e}_6$		$(v_1 + v_5)/3$	$\mathfrak{so}(8) + R^2$
$\mathfrak{e}_7$		v_3	$\mathfrak{su}(3) + \mathfrak{su}(6)$
$\mathfrak{e}_8$		v_6	$\mathfrak{su}(3) + \mathfrak{e}_6$
$\mathfrak{e}_8$		v_8	$\mathfrak{su}(9)$
$\mathfrak{f}_4$		v_3	$\mathfrak{su}(3) + \mathfrak{su}(3)$
$\mathfrak{g}_2$		v_1	$\mathfrak{su}(3)$

PROOF. Take a Weyl basis $\{X_\alpha : \alpha \in \Sigma(\mathfrak{g})\}$ of $\mathfrak{g}^{\mathbf{C}}$ mod $\mathfrak{t}^{\mathbf{C}}$ with respect to $\mathfrak{g}$ and put

$$S_\alpha = (X_\alpha - X_{-\alpha})/\sqrt{2}, \qquad T_\alpha = -(X_\alpha + X_{-\alpha})/\sqrt{-2}.$$

Then $\{S_\alpha, T_\alpha : \alpha(H)=1/3 \text{ or } 2/3\}$ is an orthonormal basis of $\mathfrak{m}$ and if $\alpha(H)=1/3$ [resp. $2/3$] then $J(S_\alpha)=T_\alpha$ [resp. $-T_\alpha$].

By Lemma 1.2, the mean curvature vector of the Cartan embedding Ψ_σ is

$$\frac{1}{2\sqrt{3}\dim M}\sum_{\alpha(H)=1/3,2/3}([J(S_\alpha), S_\alpha]+[J(T_\alpha), T_\alpha])$$
$$=\frac{1}{\sqrt{3}\dim M}\left(-\sum_{\alpha(H)=1/3}[S_\alpha, T_\alpha]+\sum_{\alpha(H)=2/3}[S_\alpha, T_\alpha]\right)$$
$$=\frac{1}{\sqrt{-3}\dim M}\left(\sum_{\alpha(H)=1/3}H_\alpha-\sum_{\alpha(H)=2/3}H_\alpha\right).$$

Thus the Cartan embedding Ψ_σ is a minimal embedding if and only if

$$\sum_{\alpha(H)=1/3}H_\alpha=\sum_{\alpha(H)=2/3}H_\alpha . \tag{2}$$

In particular, if $\alpha_0(H)=1/3$ then the Cartan embedding Ψ_σ is not a minimal embedding.

The classification of inner automorphisms of order 3 on compact simple Lie algebras are given in [6, pp. 88–89]. The theorem is proved by the inspection of each case. We give here an

EXAMPLE. $\mathfrak{g}=\mathfrak{su}(n)$, $x=(v_p+v_q)/3$ $(1\le p\le q<n)$. Taking a suitable orthonormal base $e_1, \dots, e_n$ of $\mathfrak{t}=\mathbf{R}^n$, we have

$$\Pi(\mathfrak{g})=\{\alpha_i=e_i-e_{i+1} : 1\le i\le n\},$$
$$\Sigma(\mathfrak{g})=\{\pm(e_i-e_j) : 1\le i<j\le n\},$$
$$v_i=(e_1+e_2+\cdots+e_i)-(i/n)(e_1+e_2+\cdots+e_n).$$

We have

$$\sum_{\alpha(H)=2/3}(e_i-e_j)=\sum_{i=1}^{p}\sum_{j=q+1}^{n}(e_i-e_j)$$
$$=(n-q)(e_1+e_2+\cdots+e_p)-p(e_{q+1}+e_{q+2}+\cdots+e_{n+1}),$$
$$\sum_{\alpha(H)=1/3}(e_i-e_j)=\sum_{i=1}^{p}\sum_{j=p+1}^{q}(e_i-e_j)+\sum_{i=p+1}^{q}\sum_{j=q+1}^{n}(e_i-e_j)$$
$$=(q-p)(e_1+\cdots+e_p)+(n-p-q)(e_{p+1}+\cdots+e_q)$$
$$-(q-p)(e_{q+1}+\cdots+e_n).$$

Thus $\sum_{\alpha(H)=2/3}\alpha=\sum_{\alpha(H)=1/3}\alpha$ holds if and only if $n=3p$ and $q=2p$. □

REMARK 1.3. For each pair $(\mathfrak{g}, H)$ listed in Theorem A,

$$\sum_{\alpha(H)=1/3} H_\alpha = \sum_{\alpha(H)=2/3} H_\alpha = kH$$

holds for some positive constant k.

Case 2. *σ is an outer automorphism.* Outer automorphisms of order 3 on compact simple Lie groups are classified. We denote by **Cay** the Cayley algebra. Take an orthonormal basis $\{e_0=1, e_1, \ldots, e_7\}$ of **Cay** such that

$$e_1e_2=e_3\,,\quad e_1e_4=e_5\,,\quad e_1e_6=e_7\,,\quad e_2e_5=e_7\,,$$
$$e_2e_6=e_4\,,\quad e_3e_4=e_7\,,\quad e_3e_5=e_6\,.$$

Let G_{ij} be a skew-symmetric transformation on **Cay** defined by $G_{ij}(e_k)=\delta_{jk}e_i-\delta_{ik}e_j$. Define skew-symmetric transformations F_{ij} on **Cay** as follows:

$$F_{i0}(x)=-F_{0i}(x)=(1/2)e_ix\,,\quad F_{ij}(x)=(1/2)e_j(e_ix)\qquad (i,j>0)\,.$$

Then a linear mapping τ with

$$\tau(G_{i0})=F_{0i}\,,\quad \tau(G_{ij})=F_{ij}\qquad (i,j>0)\,,$$

is an outer automorphism on $\mathfrak{so}(8)$ of order 3 (called the *triality automorphism*). We denote also by τ the induced automorphism on $Spin(8)$.

We put $\tau'=\tau\circ ad(\exp 4\pi(G_{45}-G_{67})/3)$ and denote also by τ' the induced automorphism on $Spin(8)$.

If a compact simple Lie algebra $\mathfrak{g}$ has an outer automorphism of order 3 then $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(8)$. An automorphism of order 3 on $\mathfrak{so}(8)$ is conjugate to τ or τ' in the full group of automorphisms of $\mathfrak{so}(8)$ and is conjugate to $\tau^{\pm}$ or $\tau'^{\pm}$ by an inner automorphism. Compact Lie group, whose Lie algebra is isomorphic to $\mathfrak{so}(8)$, is $Spin(8)$, $SO(8)$ or $Ad(SO(8))$. The automorphism τ (resp. τ') induces an automorphism on $Spin(8)$ and $Ad(SO(8))$. We also denote by τ (resp. τ') the induced automorphism.

THEOREM B. *Let G be either $Spin(8)$ or $Ad(SO(8))$. Then the Cartan embeddings*

$$\Psi_\tau:\ G/G_2\longrightarrow G$$
$$\Psi_{\tau'}:\ G/(SU(3)/Z_3)\longrightarrow G$$

are minimal embeddings.

PROOF. We may assume that $G=Spin(8)$.

Case τ. For each a $(1\le a\le 7)$, choose i, j, k, l, m, n such that $\{a, i, j, k, l, m, n\}=\{1, 2, 3, 4, 5, 6, 7\}$ and $e_a=e_ie_j=e_ke_l=e_me_n$ and put

$$X_a=G_{a0}\,,\quad Y_a=(G_{ij}+G_{kl}+G_{mn})/\sqrt{3}\,,\qquad 1\le a\le 7\,.$$

By a direct computation, we see that the vectors $\{X_a, Y_a: 1\le a\le 10\}$ form an orthonormal basis of $\mathfrak{m}$. Since

$$F_{01}=(G_{01}+G_{23}+G_{45}+G_{67})/2\,,\qquad F_{23}=(G_{01}+G_{23}-G_{45}-G_{67})/2\,,$$

$$F_{45}=(G_{01}-G_{23}+G_{45}-G_{67})/2\,,\qquad F_{67}=(G_{01}-G_{23}-G_{45}+G_{67})/2\,,$$

we have $J(X_1)=Y_1$ and $[J(X_1), Y_1]=0$. Similarly we have $J(X_a)=Y_a$ and $[J(X_a), Y_a]=0$ $(1\leq a\leq 7)$. Thus the Cartan embedding Ψ_τ is a minimal embedding.

Case τ'. By a direct computation, we see that the following vectors $\{X_i, Y_i : 1\leq i\leq 10\}$ form an orthonormal basis of $\mathfrak{m}$ with $J(X_i)=Y_i$ $(1\leq i\leq 10)$.

$$X_1=(G_{23}+G_{45}+G_{67})\sqrt{3}\,,\qquad Y_1=G_{01}\,,$$

$$X_2=(G_{31}+G_{64}+G_{57})/\sqrt{3}\,,\qquad Y_2=G_{02}\,,$$

$$X_3=(G_{12}+G_{47}+G_{56})/\sqrt{3}\,,\qquad Y_3=G_{03}\,,$$

$$X_4=(G_{64}-G_{57})/\sqrt{2}\,,\qquad Y_4=(G_{47}-G_{56})/\sqrt{2}\,,$$

$$X_5=(G_{14}-G_{72})/\sqrt{2}\,,\qquad Y_5=(G_{51}-G_{26})/\sqrt{2}\,,$$

$$X_6=(G_{71}-G_{42})/\sqrt{2}\,,\qquad Y_6=(G_{16}-G_{25})/\sqrt{2}\,,$$

$$X_7=(G_{14}+G_{72}-2G_{63})/\sqrt{6}\,,\qquad Y_7=(G_{51}-G_{26}-2G_{73})/\sqrt{6}\,,$$

$$X_8=(G_{71}+G_{42}-2G_{35})/\sqrt{6}\,,\qquad Y_8=(G_{16}-G_{25}-2G_{34})/\sqrt{6}\,,$$

$$X_9=G_{04}/\sqrt{2}-(G_{14}+G_{72}+G_{63})/\sqrt{6}\,,$$

$$Y_9=G_{05}/\sqrt{2}+(G_{51}+G_{26}+G_{73})/\sqrt{6}\,,$$

$$X_{10}=G_{07}/\sqrt{2}-(G_{71}+G_{42}+G_{35})/\sqrt{6}\,,$$

$$Y_{10}=G_{06}/\sqrt{2}+(G_{16}+G_{25}+G_{34})/\sqrt{6}\,.$$

With the above basis, we see that $\Psi_{\tau'}$ is a minimal embedding. □

2. Stability.

In this section, we classify the Riemannian 3-symmetric pairs to which the corresponding Cartan embeddings are stable minimal embeddings.

Let $\{M_t\}$ be a smooth variation of $M=G/K$ in G and V be its variational vector field. Take a local orthonormal frame $\{e_i\}$ of M and define a section S and A of End($N(M)$) by

$$\langle S(\xi), \eta\rangle=\sum_i\langle R^G(e_i, \xi)e_i, \eta\rangle\,,\qquad \langle A(\xi), \eta\rangle=\sum_{i,j}\langle h(e_i, e_j), \xi\rangle\langle h(e_i, e_j), \eta\rangle$$

for $\xi, \eta\in N(M)$. We denote by $\Delta^{N(M)}$ the rough Laplacian of the normal connection on $N(M)$. Then the second variational formula is given by

(3) $$d^2 Vol(M_t)/dt^2|_{t=0} = \int_M \langle(-\Delta^{N(M)} + S - A)(V^N), V^N\rangle dvol_M .$$

The differential operator $-\Delta^{N(M)} + S - A$ is called the *Jacobi differential operator* and is denoted by $\mathscr{J}$. It is easily verified that $\mathscr{J}$ is a G-homogeneous, self-adjoint, strongly elliptic linear differential operator of $\Gamma(N(M))$. Hence it has discrete eigenvalues

$$\lambda_1 < \lambda_2 < \cdots \to \infty$$

and all eigenspaces are of finite dimension. We denote by E_λ the eigenspace of $\mathscr{J}$ corresponding to λ and we call the number $i(M) = \sum_{\lambda<0} \dim E_\lambda$ the *index* of M in G. A minimal submanifold M with $i(M)=0$ is said to be a *stable* minimal submanifold of G.

Let $\tilde{G}$ be the universal covering group of G and $p: \tilde{G} \to G$ be the covering map. We denote by $\tilde{\sigma} \in \mathrm{Aut}(\tilde{G})$ the lift of σ and put $\tilde{K} = \{k \in \tilde{G} : \tilde{\sigma}(k) = k\}$. If σ is an inner automorphism then the homogeneous space G/K is simply connected, since it has a G-invariant almost complex structure ([6, p. 95]). We have the unique lift $\tilde{\Psi}_\sigma$: $G/K \to \tilde{G}$ of the Cartan embedding Ψ_σ. Furthermore, we have $G/K = \tilde{G}/\tilde{K}$ and $\tilde{\Psi}_\sigma$ coincides with the Cartan embedding $\Psi_{\tilde{\sigma}}$: $\tilde{G}/\tilde{K} \to \tilde{G}$. Thus $\Psi_\sigma(G/K)$ is a stable minimal submanifold if and only if $\Psi_{\tilde{\sigma}}(\tilde{G}/\tilde{K})$ is a stable minimal submanifold. If σ is an outer automorphism then the pair (G, σ) is either $(Spin(8), \tau)$ or $(Spin(8), \tau')$.

THEOREM C. *Let G be a compact simple Lie group and σ be an automorphism of order 3 on G. Then the image of the Cartan embedding*

$$\Psi_\sigma : G/K \longrightarrow G; \qquad g \longmapsto g \cdot \sigma(g^{-1})$$

is a stable minimal submanifold of G if and only if $(\tilde{G}, \tilde{\sigma})$ is one of the following:

$\tilde{G}$	$\tilde{\sigma}$	$\tilde{K}$
E_6	$ad(\exp 2\pi\sqrt{-1}v_3)$	$\{SU(3)\times SU(3)\times SU(3)\}/Z_3$
E_7	$ad(\exp 2\pi\sqrt{-1}v_3)$	$\{SU(3)\times SU(6)\}/Z_3$
E_8	$ad(\exp 2\pi\sqrt{-1}v_6)$	$\{SU(3)\times E_6\}/Z_3$
E_8	$ad(\exp 2\pi\sqrt{-1}v_8)$	$SU(9)/Z_3$
G_2	$ad(\exp 2\pi\sqrt{-1}v_1)$	$SU(3)$
$Spin(8)$	τ	G_2
$Spin(8)$	τ'	$SU(3)/Z_3$

where $\tilde{G}$ is the universal covering group of G and $\tilde{\sigma}$ is the lift of σ on G.

We divide the proof into two cases.

Case 1. *K has non-trivial center.*

PROPOSITION 2.1. *Let G be a compact simple Lie group and* $\sigma = ad(\exp 2\pi\sqrt{-1}H)$ *be an automorphism of order* 3. *If* $K = K_\sigma$ *has non-trivial center, then the image* $\Psi_\sigma(M)$ *of the Cartan embedding is an unstable minimal submanifold of G.*

PROOF. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. We denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Define a normal vector field $\tilde{H}$ along Σ by

$$\tilde{H}|_{gK} = dL_g(dR_{\sigma(g^{-1})}(H)) .$$

Consider a surface $(t, s) \mapsto \exp tX \exp(-s) d\sigma X$ and a vector field $\bar{H}$ along this surface defined by

$$\bar{H}(t, s) = dL_{\exp tX} dR_{\exp(-s)d\sigma X} H .$$

Then we have

$$\begin{aligned}\nabla_{d\Psi_\sigma(X)}\tilde{H} &= \nabla_{d\Psi_\sigma(X)}\bar{H} = \nabla_{X - d\sigma X}\bar{H}\\ &= \nabla_X \bar{H} - \nabla_{d\sigma_\sigma(X)}\bar{H} = (1/2)[X, H] - (1/2)[-d\sigma X, H]\\ &= (1/2)[(1 + d\sigma)(X), H] \in [\mathfrak{m}, H] \subset \mathfrak{m} .\end{aligned}$$

This implies $\nabla^\perp_{d\Psi_\sigma(X)} H = 0$ and $\nabla^\perp \tilde{H} = 0$ for $\tilde{H}$ is G-invariant. Let Ψ_t be a one parameter variation of Ψ_σ whose variational vector field is $\tilde{H}$. Since $\tilde{H}$ is a parallel normal vector field, we have

$$d^2/dt^2|_{t=0} Vol(\Psi_t(M)) = \langle S(\tilde{H}) - A(\tilde{H}), \tilde{H}\rangle|_o \cdot Vol(M) .$$

Since $R^G(X, Y)Z = (-1/4)[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{g}$, we have

$$\begin{aligned}S(H)|_o &= -(1/4) \sum_{\alpha(H) = 1/3, 2/3} ([[X_\alpha, H], X_\alpha] + [[Y_\alpha, H], Y_\alpha])\\ &= -(1/2) \sum_{\alpha(H) = 1/3, 2/3} \alpha(H) H_\alpha = -(1/2)kH\end{aligned}$$

and

$$\begin{aligned}A(H)|_o &= -(1/2) \sum_{\alpha(H) = 1/3, 2/3} (\langle [J(X_\alpha), X_\alpha], H\rangle [J(X_\alpha), X_\alpha]\\ &\quad + \langle [J(Y_\alpha), Y_\alpha], H\rangle [J(Y_\alpha), Y_\alpha])\\ &= -(1/6) \sum_{\alpha(H) = 1/3, 2/3} \alpha(H) H_\alpha = -(1/6)kH\end{aligned}$$

where we put $\sum_{\alpha(H) = 1/3} H_\alpha = kH$ by Remark 1.3. Thus we have

$$d^2/dt^2|_{t=0} Vol(\Psi_t(M)) = -(1/3)k\|H\|^2 \cdot Vol(M) < 0 .$$

Case 2. *K has trivial center.* Ikawa calculated the Jacobi differential operator under the following condition;

1. Let G [resp. U] be a compact connected Lie group with Lie algebra $\mathfrak{g}$ [resp.

u] and K [resp. L] be a closed Lie subgroup with Lie algebra $\mathfrak{k}$ [resp. $\mathfrak{l}$],
2. the Riemannian metrics on the homogeneous spaces $M=G/K$ and $P=U/L$ are normal homogeneous metrics,
3. let ρ be a homomorphism $G\to U$ with $\rho(K)\subset L$ and the mapping $f: G/K\to U/L$; $gK\mapsto\rho(g)U$ is a minimal immersion.

Let $\mathfrak{m}$ [resp. $\mathfrak{p}$] be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ [resp. $\mathfrak{l}$ in $\mathfrak{u}$]. We denote by $\mathfrak{m}^\perp$ the orthogonal complement of

$$\{(\rho_*(X))_\mathfrak{p} : X\in\mathfrak{m}\}=\{(X,-X) : X\in\mathfrak{m}\}$$

in $\mathfrak{p}$. Then $(\mathfrak{m}^\perp, (Ad\circ\rho)^\perp)$ is a representation of K and the normal bundle $N(M)$ of the immersion f is identified with the associated bundle $G\times_{(Ad\circ\rho)^\perp}\mathfrak{m}^\perp$. Let $C^\infty(G;\mathfrak{m}^\perp)$ be the space of smooth $\mathfrak{m}^\perp$-valued functions on G and $C^\infty(G;\mathfrak{m}^\perp)_K$ be the subset of elements with $f(uk)=Ad(k^{-1})f(u)$ for $u\in G$ and $k\in K$. A section ξ of $N(M)$ is identified with an element of $C^\infty(G;\mathfrak{m}^\perp)_K$ in a natural manner. Each element of the Lie algebra $\mathfrak{g}$ acts on $C^\infty(G;\mathfrak{m}^\perp)$ as a left invariant differential operator. The action of $\mathfrak{g}$ on $C^\infty(G;\mathfrak{m}^\perp)$ is extended to that of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ in a natural manner. An element $L\otimes X$ of $\mathrm{Hom}(\mathfrak{m}^\perp,\mathfrak{k})\otimes U(\mathfrak{g})$ acts, as a differential operator, on $C^\infty(G;\mathfrak{m}^\perp)$ by

$$(L\otimes X)(f)=L(Xf)\,,\qquad f\in C^\infty(G;\mathfrak{m}^\perp)\,.$$

LEMMA 2.2 ([2]). *If we assume that (U, L) is a compact symmetric pair, then the Jacobi differential operator of the minimal immersion $f: G/K\to U/L$; $gK\mapsto\rho(g)U$ is given as follows*

$$\mathscr{J}\phi=(-1\otimes C+ad_\mathfrak{u}(C)\otimes 1-2\mathscr{J}_1)\phi$$

where

$$\mathscr{J}_1\phi=\sum_{i=1}^{p}[\rho_*E_i, E_i\phi]_{\mathfrak{m}^\perp}+\sum_{i=1}^{p}[\rho_*E_i,[\rho_*E_i,\phi]]_{\mathfrak{m}^\perp}\,.$$

We put

$$U=G\times G\,,\qquad L=\{(g,g) : g\in G\}\,,$$
$$\mathfrak{l}=\{(X,X) : X\in\mathfrak{g}\}\,,\qquad \mathfrak{p}=\mathfrak{l}^\perp=\{(X,-X) : X\in\mathfrak{g}\}\,,$$

and define a homomorphism ρ of G into U by $\rho(g)=(g,\sigma(g))$. The Cartan embedding is an equivariant embedding with respect to ρ and we have

$$\mathfrak{m}^\perp=\{(X,-X) : X\in\mathfrak{k}\}\,.$$

Take an orthonormal basis $\{E_i\}_{1\le i\le p}$ of $\mathfrak{g}$ such that $\{E_i\}_{1\le i\le m}$ is an orthonormal basis of $\mathfrak{m}$ and $\{E_i\}_{m+1\le i\le p}$ is an orthonormal basis of $\mathfrak{k}$. We denote by C the Casimir element $\sum_{i=1}^{p}E_iE_i$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

We denote by $D(G)$ the set of all equivalence classes of the complex irreducible representations of G and by $(V(\lambda),\lambda)$ a representative of λ in $D(G)$. By Schur's lemma

$\lambda(C)$ is a scalar operator. We denote by a_λ the eigenvalue of $\lambda(C)$ on $V(\lambda)$.

LEMMA 2.3. *Let σ be an automorphism of order 3 on a compact simple Lie group G and Ψ_σ be the corresponding Cartan embedding. Then*
(1) *the Jacobi differential operator of the Cartan embedding is*

$$\mathscr{J}=-1\otimes C+ad_{\mathfrak{u}}(C)\otimes 1\ ,$$

(2) *the index $i(M)$ is given as follows:*

$$i(M)=\sum_{\lambda\in D(G),a_\lambda>a_{\alpha_0}}\dim \mathrm{Hom}_K(V(\lambda),(\mathfrak{k}^\perp)^{\mathbf{C}})\dim V(\lambda)\ .$$

PROOF. (1) Put $\phi=\sum_{j=m+1}^p f_j(E_j,-E_j)$, $f_j\in C^\infty(G)$. For $E_i\in\mathfrak{m}$, we have

$$[\rho_*E_i,\phi]_{\mathfrak{m}^\perp}=[(\rho_*E_i)_{\mathfrak{l}}+(\rho_*E_i)_{\mathfrak{p}},\phi]_{\mathfrak{m}^\perp}=[(\rho_*E_i)_{\mathfrak{l}},\phi]_{\mathfrak{m}^\perp}$$
$$=(1/2)\sum_{j=m+1}^{p} f_j((1+d\sigma)[E_i,E_j],-(1+d\sigma)[E_i,E_j])_{\mathfrak{m}^\perp}=0\ .$$

Similarly we have $[\rho_*E_i,E_i\phi]_{\mathfrak{m}^\perp}=0$. For $E_i\in\mathfrak{k}$, we have

$$(E_i\phi)(g)=d/dt|_0\phi(g\exp tE_i)=-[\rho_*(E_i),\phi(g)]\in\mathfrak{m}^\perp\ .$$

Thus we have

$$[\rho_*(E_i),E_i\phi(g)]_{\mathfrak{m}^\perp}+[\rho_*E_i,[\rho_*(E_i),\phi(g)]]_{\mathfrak{m}^\perp}=0\ .$$

Thus we have $J_1=0$.

(2) is a consequence of (1) and the Frobenius reciprocity theorem ([5, p. 118]). □

Let $\mathfrak{u}$ be a complex simple Lie algebra and $\mathfrak{v}$ be a complex simple Lie subalgebra. Take an $ad_{\mathfrak{u}}$-invariant [resp. $ad_{\mathfrak{v}}$-invariant] inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ [resp. $\langle\ ,\ \rangle_{\mathfrak{v}}$] on $\mathfrak{u}$ [resp. $\mathfrak{v}$] such that the square of the length of the longest root is equal to 2. Since $\langle\ ,\ \rangle_{\mathfrak{u}}$ is also $ad_{\mathfrak{v}}$-invariant, there exists a complex number j such that $\langle X,Y\rangle_{\mathfrak{u}}=j\langle X,Y\rangle_{\mathfrak{v}}$ for any $X,Y\in\mathfrak{v}$. It is known that the number j is a positive integer, and is called the *index* of $\mathfrak{v}$ in $\mathfrak{u}$ (in the sense of Dynkin).

PROPOSITION 2.4. *Let G be a compact simple Lie group and σ be an automorphism on G of order 3. If the subgroup $K=K_\sigma$ is semi-simple and each simple factor of $\mathfrak{k}^{\mathbf{C}}$ is a subalgebra of $\mathfrak{g}^{\mathbf{C}}$ of index 1, then $\Psi_\sigma(G/K)$ is a stable minimal submanifold of G.*

The proof of this proposition is the same as that of Theorem in [4]. For the sake of convenience, we shall briefly sketch the proof.

Assume that $\Sigma(G/K)$ is an unstable minimal submanifold of G. Then by Lemma 2.2, there exists $\lambda\in D(G)$ such that

$$\dim\mathrm{Hom}_K(V(\lambda),\mathfrak{k}^{\mathbf{C}})>0\quad\text{and}\quad a_\lambda>a_{\alpha_0}\ .$$

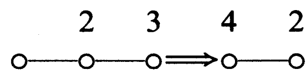
In our case $\mathfrak{k}$ is a simple Lie algebra or a direct sum of simple Lie subalgebras. We take a direct sum decomposition of $\mathfrak{k}$ by simple ideals: $\mathfrak{k}=\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_s$. Let K_i $(1\leq i\leq s)$ be the subgroup of K generated by $\mathfrak{k}_i$. Since $\dim \mathrm{Hom}_K(V(\lambda), \mathfrak{k}^{\mathbf{C}})>0$, we have $\dim \mathrm{Hom}_{K_i}(V(\lambda), \mathfrak{k}_i^{\mathbf{C}})>0$ for some i $(1\leq i\leq s)$. By the following Lemma 2.5, we have $a_\lambda\leq a_{\alpha_0}$ which is a contradiction.

Lemma 2.5. ([4]). *Let G be a compact simple Lie group and K be a closed subgroup of index* 1. *Let $\lambda\in D(G)$ and $V(\lambda)$ be its representative. If $V(\lambda)$ contains $(\mathfrak{k}^{\mathbf{C}}, Ad_K)$ as a K-irreducible component, then we have*

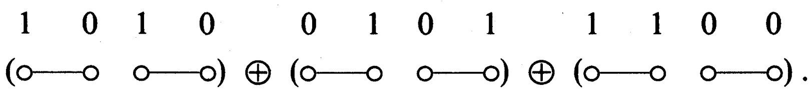
$$a_\lambda\leq a_{\alpha_0}\,.$$

Proposition 2.6. *The Cartan embedding corresponding to the inner automorphism $ad(\exp 2\pi\sqrt{-1}\,v_3)$ on F_4 is an unstable minimal embedding.*

Proof. Since the extended Dynkin diagram of $\mathfrak{f}_4$ is



the Lie algebra $\mathfrak{k}$ of the subgroup K of elements fixed by the automorphism $\sigma=Ad(\exp 2\pi\sqrt{-1}\,v_2)$ is $\mathfrak{su}(3)+\mathfrak{su}(3)$. As K-modules, the irreducible representation $V(\varpi_4)$ decomposes as follows ([3, p. 306]):



Thus we have $\dim \mathrm{Hom}_K(V(\varpi_4), \mathfrak{k}^{\mathbf{C}})=1$. On the other hand, the highest weight of the adjoint representation of $\mathfrak{f}_4$ is ϖ_1. Since we have $a_{\alpha_0}=a_{\varpi_1}=6$ and $a_{\varpi_4}=18$, the corresponding Cartan embedding is an unstable minimal embedding.

The proof of Theorem C is completed by summing up Theorem A, Theorem B, Proposition 2.1, Proposition 2.4 and Proposition 2.6.

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