TOKYO J. MATH.
VOL. 20, NO. 1, 1997

Covariant Representations Associated with Chaotic Dynamical Systems

Shinzo KAWAMURA

Yamagata University

(Communicated by T. Ishikawa)

Dedicated to Professor Tsuyoshi Ando
on the occasion of his retirement from Hokkaido University

Abstract. We define a property of strong-mixing in the theory of C^*-dynamical systems and show that the property follows from the existence of a base of Walsh type. Moreover as applications we analyze covariant representations of chaotic dynamical systems.

1. Introduction.

One of the most important property in the chaotic dynamical theory is to be sensitive dependence on initial conditions. In the case of continuous map φ on a metric space X with metric d, this means that there exists $\delta>0$ such that, for any x in X and neighbourhood $U(x)$ of x, there exist y in $U(x)$ and $n\geq 0$ such that $d(\varphi^n(x), \varphi^n(y))>\delta$. Concerning chaotic dynamical systems in the nature, from this property we can know the difficulty in expecting the future $\varphi^n(x)$ of a given *point* x, because we cannot get the exact value of initial point x. On the other hand, the property of sensitive dependence on initial conditions is induced from that of topological-mixing. This property says for the future of an open *set*, that is, for any pair of open sets U and V there exists a large integer N such that $\varphi^n(U)\cap V\neq\varnothing$ for any $n\geq N$. Moreover in the case where a map φ is topological-mixing and there exists φ-invariant measure m whose support is the whole space X, the map φ is strong-mixing on the measure space (X, m), that is,

$$\lim_{n\to\infty}\int_X f(\varphi^n(x))g(x)dm=\int_X f(x)dm$$

for any continuous function f on a metric space X and L^1-function g on the measurable space (X, m) with $\int_X g(x)dm=1$. This property says that, in the sense of probability, we can expect the future of the sequence $\{\varphi^n(E)\}$ for a set E of initial points (cf. Example 3.7.3).

Received October 9, 1995

Two properties: sensitive dependence on initial conditions and being strong-mixing are seemed to be confronted with each other. However, we do not see these phenomena simultaneously, because the first one occurs as the orbit of a *point* x in a metric space but the second one as that of a *density function* g in the L^1-space. Our purpose is to understand the phenomena in the latter case. Hence in the present paper we discuss the property of strong-mixing in the theory of covariant representations of C^*-dynamical systems into the set of bounded linear operators on a Hilbert space. Especially we consider C^*-dynamical systems $(C(X), \alpha_\varphi)$ associated with chaotic maps φ such as the tent map, the logistic map on the unit interval and the shift map on the infinite direct product of 2 points. Our result is the following:

(1) A canonical covariant representation π of $(C(X), \alpha_\varphi)$ is implemented by a couple of isometries on the underlying Hilbert space (Theorem 3.2.1).

(2) The property of strong-mixing is extended to the case of $*$-endomorphism of C^*-algebras, and the property follows from the existence of a base on the Hilbert space which has canonical relation with a couple of isometries implementing $*$-endomorphism (Theorem 2.2.3). This base is similar to Walsh series [3], so we call it a base of Walsh type.

(3) The property of strong-mixing of chaotic maps has a large effect on that property of extended $*$-endomorphisms of subalgebras including $\pi(C(X))$ (cf. Sections 3.2, 3.3, 3.4).

Furthermore we note that our study provides some new and interesting examples for the structure theory of crossed-products associated with non-homeomorphic continuous maps. In this paper, we refer to [2], [6] and [5] for theory of topological dynamics and operator algebras respectively; in [5] theory of covariant representations is studied in the case of homeomorphisms on compact spaces.

2. $*$-endomorphisms of C^*-algebras.

2.1. Let A denote a C^*-algebra with unit I on a Hilbert space $\mathfrak{H}$ with inner product $\langle\ ,\ \rangle$, α a $*$-endomorphism of C^*-algebra A with the property $\alpha(I)=I$. First we give three definitions.

DEFINITION 2.1.1. A $*$-endomorphism α of A is said to be implemented by a couple (V_1, V_2) of isometries on $\mathfrak{H}$ if

$$\alpha(a)=V_1aV_1^*+V_2aV_2^*$$

for all a in A. In this case, α is sometimes denoted by α_V.

DEFINITION 2.1.2. A $*$-endomorphism α of A is said to be *strong-mixing* if there exists a unit vector e in $\mathfrak{H}$ such that

$$\lim_{n\to\infty}(\alpha^n(a)\xi, \xi)=(ae, e)$$

for all a in A and ξ in $\mathfrak{H}$ with $\|\xi\|=1$.

The term: strong-mixing has been used in the case of topologically dynamical system (cf. [6: p. 154]). In the definition above, it is the case where A's are abelian C^*-algebras and e's constant functions with value 1.

DEFINITION 2.1.3. Let (V_1, V_2) be a couple of isometries on $\mathfrak{H}$ with the property $V_1V_1^*+V_2V_2^*=I$. A completely orthonormal base $\{e_n\}_{n=1}^{\infty}$ of $\mathfrak{H}$ is said to be of *Walsh type* with respect to (V_1, V_2) if the following relation holds:

$$V_1e_n=e_{2n-1} \quad \text{and} \quad V_2e_n=e_{2n} \qquad \text{for all} \quad n\geq 1 \tag{2.1.3}$$

We note that Walsh series which appears in Example 3.2.3 is a typical example of a base of Walsh type, which is the reason why we use the term: of Walsh type.

2.2. First we give a lemma about a base of Walsh type and then state our theorem. Hereafter, for a positive integer k, $\{1, 2\}^k$ means the set of all k-tuples $\mu=(j_1, \cdots, j_k)$ with j_n in $\{1, 2\}$. Moreover for μ in $\{1, 2\}^k$ we denote by $V(\mu)$ the isometry $V_{j_1}V_{j_2}\cdots V_{j_k}$ on $\mathfrak{H}$ (cf. [1: p. 174]).

LEMMA 2.2.1. *Let $\{e_n\}_{n=1}^{\infty}$ be a base of Walsh type with respect to (V_1, V_2). Then for a fixed positive integer k and an arbitrary positive integer $n\geq k$, there exists a unique k-tuple $(j_1, \cdots, j_k)$ in $\{1, 2\}^k$ such that*

$$V(\mu)^*e_n=\begin{cases} e_1 & \text{if} \quad \mu=(j_1, \cdots, j_k)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where μ is in $\{1, 2\}^k$.

PROOF. First we note that $V_1^*e_1=e_1$ and $V_2^*e_1=0$. For arbitrary $n\leq k$, we have a unique m-tuple $(j_1, \cdots, j_m)$ in $\{1, 2\}^m$ $(m\leq n)$ for which

$$V_{j_m}^*V_{j_{m-1}}^*\cdots V_{j_1}^*e_n=e_1 \quad \text{and} \quad V_{j_{m-1}}^*\cdots V_{j_1}^*e_n\neq e_1\,.$$

Thus, putting $j_{m+1}=\cdots=j_k=1$, we have a unique desired k-tuple $(j_1, \cdots, j_k)$ in $\{1, 2\}^k$.
q.e.d.

REMARK 2.2.2. Let (V_1, V_2) and $\{e_n\}_{n=1}^{\infty}$ be as in Lemma 2.2.1. Besides, let $\mu=(j_1, \cdots, j_n, \cdots)$ be an infinite sequence with j_n in $\{1, 2\}$. We put $\mu(n)=(j_1, \cdots, j_n)$ and $P(\mu(n))=V(\mu(n))V(\mu(n))^*$, $P(\mu)=\text{s-lim}_{n\to\infty}P(\mu(n))$. The C^*-algebra B generated by the two isometries V_1 and V_2 is of course a continuous representation of an abstract simple C^*-algebra O_2 (cf. [1]). We here note that the dimension of $P(\mu)\mathfrak{H}$ is one or zero, which is one of the properties of this representation of O_2. In fact, for each e_n $(n\geq 1)$, there exists a unique m-tuple $(j_1, \cdots, j_m)$ $(m\leq n)$ such that $V_{j_m}^*\cdots V_{j_1}^*e_n=e_1$ and $j_{m-1}=2$ (if $n\geq 2$), $j_m=1$. Put $j_{m+i}=1$ for $i\geq 1$ and $\mu=(j_1, \cdots, j_m, j_{m+1}, \cdots)$. Then $P(\mu)\mathfrak{H}$ is the one-dimensional subspace generated by e_n. Conversely if $\mu=(j_1, \cdots, j_n, \cdots)$ is an infinite sequence such that $j_n=1$ for all $n\geq k$ for some k, there exists a unique vector $e_{n(\mu)}$ such that $P(\mu)\mathfrak{H}$ is the one-dimensional subspace generated by $e_{n(\mu)}$. Otherwise, it is easy to see that the dimension of $P(\mu)\mathfrak{H}$ is zero.

THEOREM 2.2.3. *Let α be a $*$-endomorphism of A implemented by a couple (V_1, V_2) of isometries. If there exists a base $\{e_n\}_{n=1}^{\infty}$ of Walsh type with respect to (V_1, V_2), then α is strong-mixing.*

PROOF. Let a be an arbitrary operator in A with $\|a\| \leq 1$ and ξ an arbitrary unit vector in $\mathfrak{H}$ with Fourier expansion $\xi = \sum_{n=1}^{\infty} c_n e_n$ with respect to $\{e_n\}_{n=1}^{\infty}$. Then for an arbitrary positive number $\varepsilon < 1$, there exists a positive integer k such that $\|\xi - \sum_{n=1}^{k} c_n e_n\| = (1 - \sum_{n=1}^{k} |c_n|^2)^{1/2} < \varepsilon/3$. Put $\xi(k) = \sum_{n=1}^{k} c_n e_n$. Since $\alpha^k(a) = \sum_{\mu \in \{1,2\}^k} V(\mu) a V(\mu)^*$, using Lemma 2.2.1 we have the following:

$$\langle \alpha^k(a)\xi(k), \xi(k)\rangle = \sum_{\mu \in \{1,2\}^k} \sum_{n,m=1}^{k} \langle aV(\mu)^* c_n e_n, V(\mu)^* c_m e_m\rangle$$
$$= \left(\sum_{n=1}^{k} |c_n|^2\right)\langle ae_1, e_1\rangle .$$

Thus it follows that

$$|\langle \alpha^k(a)\xi, \xi\rangle - \langle ae_1, e_1\rangle| \leq |\langle \alpha^k(a)\xi, \xi\rangle - \langle \alpha^k(a)\xi(k), \xi\rangle|$$
$$+ |\langle \alpha^k(a)\xi(k), \xi\rangle - \langle \alpha^k(a)\xi(k), \xi(k)\rangle| + |\langle \alpha^k(a)\xi(k), \xi(k)\rangle - \langle ae_1, e_1\rangle|$$
$$\leq \|\alpha^k(a)\| \cdot \|\xi - \xi(k)\| \cdot \|\xi\| + \|\alpha^k(a)\| \cdot \|\xi(k)\| \cdot \|\xi - \xi(k)\|$$
$$+ |\textstyle\sum_{n=1}^{k} |c_n|^2 - 1| \cdot \|a\| \cdot \|e_1\|^2 \leq \varepsilon/3 + \varepsilon/3 + \varepsilon^2/9 < \varepsilon .$$

q.e.d.

In the remainder of this section, we give two propositions which play an important role in the discussion in Section 3.

PROPOSITION 2.2.4 *Let (V_1, V_2) and (W_1, W_2) be two couples of isometries on $\mathfrak{H}$ with the property:*

$$V_1 V_1^* + V_2 V_2^* = W_1 W_1^* + W_2 W_2^* = I .$$

Then the following conditions are equivalent:

(1) $V_1 a V_1^* + V_2 a V_2^* = W_1 a W_1^* + W_2 a W_2^*$ *for all a in A.*

(2) $W_1 = V_1 h_{11} + V_2 h_{21}$ *and* $W_2 = V_1 h_{12} + V_2 h_{22}$, *where* $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ *is a unitary element in the C^*-tensor product $M_2 \otimes A'$ of the full matrix algebra M_2 and the commutant A' of the C^*-algebra A on the Hilbert space $\mathbf{C}^2 \otimes \mathfrak{H}$.*

PROOF. The implication: (2) $\Rightarrow$ (1) is shown by canonical calculation. On the other hand, putting $h_{ij} = V_i^* W_j$, we can show the converse implication. q.e.d.

PROPOSITON 2.2.5. *Suppose that (V_1, V_2) and (W_1, W_2) satisfy Condition* (1), *equivalently* (2), *of Proposition* 2.2.4 *for $A = \mathfrak{L}(\mathfrak{H})$ (the full operator algebra on $\mathfrak{H}$) and* *there exists a base* $\{e_n\}_{n=1}^{\infty}$ *of Walsh type with respect to* (W_1, W_2). *Then there exists a base* $\{\xi_n\}_{n=1}^{\infty}$ *of Walsh type with respect to* (V_1, V_2) *if and only if* $V_1=W_1$, $V_2=z_2W_2$ *and* $\xi_1=z_1e_1$ *for some complex numbers* z_1 *and* z_2 *with* $|z_1|=|z_2|=1$.

PROOF. We need a proof of only if part. First we note that from the hypothesis, V_1 and V_2 are expressed as follows:

$$V_1=c_{11}W_1+c_{21}W_2\,, \qquad V_2=c_{12}W_1+c_{22}W_2\,,$$

where $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ is in the group of 2×2 unitary matrices $U(2:\mathbf{C})$. Next we assume the existence of a base $\{\xi_n\}_{n=1}^{\infty}$ of Walsh type with respect to (V_1, V_2). Let $\xi_1=\sum_{n=1}^{\infty}c_ne_n$ be the Fourier expansion with respect to $\{e_n\}_{n=1}^{\infty}$. Since $V_1\xi_1=\xi_1$, we have

$$\sum_{n=1}^{\infty}(c_{11}c_ne_{2n-1}+c_{21}c_ne_{2n})=\sum_{n=1}^{\infty}c_ne_n\,.$$

Thus we can see that $c_{11}=1$, $|c_{22}|=1$, $c_{12}=c_{21}=0$ and $|c_1|=1$, $c_n=0$ for all $n\geq 2$.

q.e.d.

3. Covariant representations of topological dynamical systems.

3.1. Let X be a metric space, $C(X)$ the C^*-algebra of all continuous functions on X. Then a continuous map φ from X onto itself induces a $*$-endomorphism α_φ of $C(X)$, which is defined by

$$\alpha_\varphi(f)(x)=f(\varphi(x))\,, \qquad x\in X\,.$$

Hence the topological dynamical system (X, φ) induces a C^*-dynamical system $(C(X), \alpha_\varphi)$.

DEFINITION 3.1.1 (cf. [4: §2]). A map π of $C(X)$ into the full operator algebra $\mathfrak{L}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ is said to be a covariant representation of C^*-dynamical system $(C(X), \alpha)$ of *multiplicity* 2 if π satisfies the following conditions.

(1) π is a continuous homomorphism of the C^*-algebra $C(X)$ into the C^*-algebra $\mathfrak{L}(\mathfrak{H})$.

(2) There exists a couple (V_1, V_2) of isometries on $\mathfrak{H}$ such that

$$\pi(\alpha(f))=V_1\pi(f)V_1^*+V_2\pi(f)V_2^*$$

for all f in $C(X)$.

We here give notations of two kinds of linear operators on the Hilbert space $L^2(X, m)$, where m is a measure on X. We denote by $L^\infty(X, m)$ the set of all complex-valued essentially bounded functions on X. Suppose that f is in $L^\infty(X, m)$ and φ a continuous map of X into itself with the property: the measure $m_{\varphi^{-1}}$ is absolutely continuous with respect to m, where $m_{\varphi^{-1}}(E)=m(\varphi^{-1}(E))$ for a measurable set E in X. Then we can define the multiplication operator M_f and the canonical linear operator T_φ associated with φ, that is, $(M_f\xi)(x)=f(x)\xi(x)$ and $(T_\varphi\xi)(x)=\xi(\varphi(x))$, $(x\in X)$ for ξ in $L^2(X, m)$. When T_φ is not bounded on $L^2(X, m)$, $M_f T_\varphi$ and $T_\varphi M_f$ mean the linear operators defined by $(M_f T_\varphi\xi)(x)=f(x)\xi(\varphi(x))$ and $(T_\varphi M_f\xi)(x)=f(\varphi(x))\xi(\varphi(x))$.

3.2. Let φ be a unimodal map of $[0, 1]$ onto itself in the following sense.

(1) φ is a continuous map of $[0, 1]$ onto $[0, 1]$.

(2) There exists a point c in $(0, 1)$ such that

 (i) $\varphi(0)=\varphi(1)=0$ and $\varphi(c)=1$,

 (ii) φ is strictly monotone increasing on $[0, c]$ and strictly monotone decreasing on $[c, 1]$,

 (iii) φ and the two inverse maps β, γ of φ are absolutely continuous functions on $[0, 1]$, where $\beta([0, 1])=[0, c]$ and $\gamma([0, 1])=[c, 1]$.

We consider that $[0, 1]$ is the unit interval with usual Lebesgue measure dx. Then $\varphi(\beta(x))=\varphi(\gamma(x))=x$ for all x in $[0, 1]$, and by Property (2)-(iii) above, we have $\varphi'(\beta(x))\beta'(x)=\varphi'(\gamma(x))\gamma'(x)=1$ for a.e. x in $[0, 1]$, where $\varphi'=d\varphi/dx$ and so on. Hence we obtain a couple $(V_1, V_2)=(V_1(\varphi), V_2(\varphi))$ of isometries associated with φ by defining as follows:

$$V_1=V_1(\varphi)=M_{\sqrt{\varphi'}}M_{\chi[0,c]}T_\varphi \quad \text{and} \quad V_2=V_2(\varphi)=M_{\sqrt{-\varphi'}}M_{\chi[c,1]}T_\varphi ,$$

where χ_E means the characteristic function of E. In fact we have $V_1^*=M_{\sqrt{\beta'}}T_\beta$, for it follows that

$$\langle V_1\xi, \eta\rangle=\int_0^c \sqrt{\varphi'(x)}\,\xi(\varphi(x))\overline{\eta(x)}dx=\int_0^1 \sqrt{\varphi'(\beta(y))}\,\xi(y)\overline{\eta(\beta(y))}\beta'(y)dy$$

$$=\int_0^1 \xi(y)\frac{\overline{\eta(\beta(y))}}{\sqrt{\varphi'(\beta(y))}}dy=\int_0^1 \xi(y)\overline{\sqrt{\beta'(y)}\eta(\beta(y))}dy .$$

Namely V_1 is an isometry on $L^2(X, m)$ such that $V_1V_1^*=M_{\chi[0,c]}$. Similarly we can see that V_2 is an isometry on $L^2(X, m)$ such that $V_2V_2^*=M_{\chi[c,1]}$ and hence we have $V_1V_1^*+V_2V_2^*=I$. Thus the couple (V_1, V_2) induces an $*$-endomorphism α_V of $\mathfrak{L}(L^2[0, 1])$ and by easy calculation we have

$$(\alpha_V(M_f)\xi)(x)=f(\varphi(x))\xi(x) \qquad (\text{a.e. } x \text{ in } [0, 1])$$

for each f in $L^\infty[0, 1]$. Let $\pi(f)=M_f$ for f in $C[0, 1]$, where $C[0, 1]$ is considered to be embedded into $L^\infty[0, 1]$. Then π is a continuous representation of the C^*-algebra $C[0, 1]$ into $\mathfrak{L}(L^2[0, 1])$ and we have

$$\pi(\alpha_\varphi(f))=\alpha_V(M_f)=V_1\pi(f)V_1^*+V_2\pi(f)V_2^* \qquad \text{for } f \text{ in } C[0, 1] .$$

Therefore we have the following:

THEOREM 3.2.1. *Let φ be a unimodal map on* $[0, 1]$. *Then the representation π of* $C[0, 1]$ *defined by* $\pi(f)=M_f$ *on* $L^2[0, 1]$ *is a covariant representation of multiplicity* 2 *with respect to* φ.

Now suppose that $\alpha_V(=\alpha_{V(\varphi)})=\alpha_W$ on $M_{L^\infty[0,1]}$ for some couple (W_1, W_2) of isometries, with respect to which there exists a base $\{e_n\}_{n=1}^\infty$ of Walsh type. Then by Theorem 2.2.3 and Proposition 2.2.4 we have

$$\lim_{n\to\infty}\int_0^1 f(\varphi^n(x))|\xi(x)|^2dx=\int_0^1 f(x)|e_1(x)|^2dx$$

for all f in $L^\infty[0, 1]$ and ξ in $L^2[0, 1]$ with $\|\xi\|=1$. In this case, putting $f=\chi_{[0,x]}$, we get the distribution function D_φ associated with φ, that is,

$$D_\varphi(x)=\lim_{n\to\infty}\int_0^1 \chi_{[0,x]}(\varphi^n(t))|\xi(t)|^2dt=\int_0^1 \chi_{[0,x]}(t)|e_1(t)|^2dt\,.$$

Of course, D_φ is determined independent of the vector ξ in $L^2[0, 1]$ with $\|\xi\|=1$.

REMARK 3.2.2. The property of strong-mixing is similar to that of being ergodic. The difference of two properties is shown by the example of irrational rotations on the one-dimensional torus, which is not strong-mixing but ergodic.

EXAMPLE 3.2.3. Let τ be the tent map of $[0, 1]$ onto itself, that is, $\tau(x)=1-|1-2x|$. Then $(\tau, [0, 1])$ is a typical chaotic dynamical system and we have

$$V_1=V_1(\tau)=\sqrt{2}\,M_{\chi_{[0,1/2]}}T_\tau \quad\text{and}\quad V_2=V_2(\tau)=\sqrt{2}\,M_{\chi_{[1/2,1]}}T_\tau\,.$$

Now we define another couple (W_1, W_2) of isometries as follows:

$$W_1=W_1(\tau)=\frac{1}{\sqrt{2}}V_1+\frac{1}{\sqrt{2}}V_2(=T_\tau) \quad\text{and}\quad W_2=W_2(\tau)=\frac{1}{\sqrt{2}}V_1-\frac{1}{\sqrt{2}}V_2\,.$$

Then by Proposition 2.2.4 we have

$$\pi(\alpha_\tau(f))=V_1\pi(f)V_1^*+V_2\pi(f)V_2^*=W_1\pi(f)W_1^*+W_2\pi(f)W_2^* \qquad\text{for } f \text{ in } C[0, 1]\,,$$

$$\alpha_V(a)=V_1aV_1^*+V_2aV_2^*=W_1aW_1^*+W_2aW_2^* \qquad\text{for } a \text{ in } \mathfrak{L}(L^2[0, 1])\,.$$

We put $e_1=\chi_{[0,1]}$. Then $W_1e_1=e_1$ and hence we can define e_n for $n\geq 2$ inductively in the manner of (2.1.3), that is, $e_{2n-1}=W_1e_n$ and $e_{2n}=W_2e_n$ for $n=1, 2, \cdots$. Each e_n is a $\{-1, 1\}$-valued function on $[0, 1]$ and the orthonormal system $\{e_n\}_{n=1}^\infty$ is a base of Walsh type with respect to (W_1, W_2), in fact, it is the original Walsh series (cf. [3, p2]). Therefore by Theorem 2.2.3 α_V is strong-mixing on $\mathfrak{L}(L^2[0, 1])$. In particular, for any L^∞-function f on $[0, 1]$ and ξ in $L^2[0, 1]$ with $\|\xi\|=1$, we have

$$\lim_{n\to\infty}\int_0^1 f(\tau^n(x))|\xi(x)|^2dx=\int_0^1 f(x)dx\,,$$

hence we have

$$D_\tau(x)=\lim_{n\to\infty}\int_0^1 \chi_{[0,x]}(\tau^n(t))|\xi(t)|^2dt=\int_0^1 \chi_{[0,x]}(t)dt=x\,.$$

Now we recall that the couple (W_1, W_2) and the base $\{e_n\}_{n=1}^\infty$ of Walsh type are uniquely determined in the sense of Proposition 2.2.5. However when we restrict the definition-domain of $\alpha_{V(\tau)}$ to the C^*-algebra $\pi(C[0, 1])$, such a uniqueness does not valid. We show this fact by giving an example. First we note that the equality $\alpha_{V(\tau)}(=\alpha_{W(\tau)})=\alpha_U$ on $\pi(C[0, 1])$ holds for a couple (U_1, U_2) of isometries if and only if

$$(*)\qquad U_1=V_1M_{h_{11}}+V_2M_{h_{21}} \quad\text{and}\quad U_2=V_1M_{h_{12}}+V_2M_{h_{22}}$$

where $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & _{22}\end{pmatrix}$ is in the unitary group $U(M_2\otimes L^\infty[0, 1])$. Furthermore, a necessary and sufficient condition for a vector ξ in $L^2[0, 1]$ to be fixed by U_1 is the following:

$$(**)\qquad \xi(x)=\begin{cases}\sqrt{2}\,h_{11}(\tau(x))\xi(\tau(x)) & \text{for } 0\le x<1/2\,,\\ \sqrt{2}\,h_{21}(\tau(x))\xi(\tau(x)) & \text{for } 1/2\le x<1\,.\end{cases}$$

We put $I_n=(1/2^{n+1}, 1/2^n]$ for $n=0, 1, 2, \cdots$. Then we have $\tau(I_n)=I_{n-1}$ and $\bigcup_{n=0}^\infty I_n=(0, 1]$. Here we define two functions h and ξ_1 on $[0, 1]$ as follows:

$$h(x)=\begin{cases}(-1)^n & \text{for } x\in I_n\,,\\ 1 & \text{for } x=0\,,\end{cases}\qquad (n=0, 1, \cdots)\,,$$

$$\xi_1(x)=\begin{cases}(-1)^k & \text{for } x\in I_{4n+2k}\cup I_{4n+2k+1}\,,\\ 1 & \text{for } x=0\,,\end{cases}\qquad (n=0, 1, \cdots, \ k=0, 1)\,.$$

Let $h_{11}=h/\sqrt{2}$, $h_{21}=1/(\sqrt{2}\,\xi)$, $h_{12}=h_{21}$ and $h_{22}=-h_{11}$. Then h_{11}, h_{12} and ξ satisfy Condition $(**)$ and $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix}$ belongs to $U(M_2\otimes L^\infty[0, 1])$. Hence, by $(*)$, these h_{ij} induce a couple (U_1, U_2) of isometries which implements $\alpha_{V(\tau)}$ on $\pi(C[0, 1])$ and ξ_1 generates a base $\{\xi_n\}_{n=1}^\infty$ of Walsh type with respect to (U_1, U_2). Namely we have $\lim_{n\to\infty}\langle\alpha_U^n(a)\xi, \xi\rangle=\langle a\xi_1, \xi_1\rangle$ for each $a\in\mathfrak{L}(L^2[0, 1])$ and ξ in $L^2[0, 1]$ with $\|\xi\|=1$. Especially we have

$$\begin{aligned}\lim_{n\to\infty}\langle\alpha_U^n(M_f)\xi, \xi\rangle&=\lim_{n\to\infty}\langle\alpha_{V(\tau)}^n(M_f)\xi, \xi\rangle\\ &=\langle M_f\xi_1, \xi_1\rangle=\int_0^1 f(x)dx=\langle M_fe_1, e_1\rangle\end{aligned}$$

for f in $L^\infty[0, 1]$.

Now we suppose that a unimodal map φ is topologically conjugate to the tent map, that is, $\varphi=h\circ\tau\circ h^{-1}$ for some homeomorphism h of $[0, 1]$ onto itself. In our case, the maps h and h^{-1} are assumed to be absolutely continuous functions on [0, 1].

Let $U(h)$ be the unitary operator on $L^2[0, 1]$ defined by $(U(h)\xi)(x)=\sqrt{h'(x)}\xi(h(x))$ for ξ in $L^2[0, 1]$. Then $(U(h)^*\xi)(x)=\sqrt{(h^{-1})'(x)}\xi(h^{-1}(x))$ and we put

$$V_i(\varphi)=U(h)^*V_i(\tau)U(h)\,,\qquad W_i(\varphi)=U(h)^*W_i(\tau)U(h)$$

for $i=1, 2$. Moreover let $\xi_n=U(h)^*e_n$ for $n=1, 2, \cdots$, where $\{e_n\}_{n=1}^{\infty}$ is the Walsh series. Then we have

$$\pi(\alpha_{\varphi}(f))=V_1(\varphi)\pi(f)V_1(\varphi)^*+V_2(\varphi)\pi(f)V_2(\varphi)^* \qquad \text{for} \quad f \text{ in } C[0, 1]\,,$$

$$\alpha_{V(\varphi)}(a)=\alpha_{W(\varphi)}(a) \qquad \text{for} \quad a \text{ in } \mathfrak{L}(L^2[0, 1])\,,$$

$$W_1(\varphi)=M_{\sqrt{|\varphi'|/2}}T_{\varphi}$$

and $\{\xi_n\}_{n=1}^{\infty}$ is a base of Walsh type with respect to $(W_1(\varphi), W_2(\varphi))$. Thus by Theorem 2.2.3 we have the following:

THEOREM 3.2.4. *Let φ be topologically conjugate to the tent map τ with cojugacy h. Then we have*

$$\lim_{n\to\infty}\langle\alpha^n_{V(\varphi)}(a)\xi, \xi\rangle=\langle a\xi_1, \xi_1\rangle$$

for each a in $\mathfrak{L}(L^2[0, 1])$, where $\xi_1(x)=\sqrt{(h^{-1})'(x)}$.

In the theorem above, if $a=M_f$ for f in $L^{\infty}[0, 1]$ it follows that

$$\lim_{n\to\infty}\int_0^1 f(\varphi^n(x))|\xi(x)|^2dx=\langle M_f\xi_1, \xi_1\rangle=\int_0^1 f(x)(h^{-1})'(x)dx\,.$$

Thus we have $D_{\varphi}(x)=h^{-1}(x)$.

EXAMPLE 3.2.5. Let λ be the logistic map of [0, 1] onto itself, that is, $\lambda(x)=4x(1-x)$. Then λ is topologically conjugate to the tent map τ. Namely $\lambda=h\circ\tau\circ h^{-1}$, where $h(x)=\sin^2(\pi x/2)$. Hence we have

$$\pi(\alpha_{\lambda}(f))=V_1(\lambda)\pi(f)V_1(\lambda)^*+V_2(\lambda)\pi(f)V_2(\lambda)^* \qquad \text{for} \quad f \text{ in } C[0, 1]\,,$$

$$\alpha_{V(\lambda)}(a)=\alpha_{W(\lambda)}(a) \qquad \text{for} \quad a \text{ in } \mathfrak{L}(L^2[0, 1])\,,$$

$$W_1(\lambda)=M_{\sqrt{|2-4x|}}T_{\lambda}$$

and there exists a base $\{\xi_n\}_{n=1}^{\infty}$ of Walsh type with respect to $(W_1(\tau), W_2(\tau))$ with $\xi_1(x)=1/(\pi(x(1-x))^{1/2})^{1/2}$. Moreover we have

$$D_{\lambda}(x)=\frac{2}{x}\arcsin\sqrt{x}\,.$$

EXAMPLE 3.2.6. Let $h(x)=x^n$ and $\varphi=h\circ\tau\circ h^{-1}$. Then we have

$$\varphi(x)=\begin{cases} 2^n x & \text{for } 0\leq x\leq 1/2^n, \\ (2-2\sqrt[n]{x})^n & \text{for } 1/2^n\leq x\leq 1. \end{cases}$$

As in the example above, we have $\xi_1(x)=(1/\sqrt{n})x^{(1-n)/2n}$ and $D_\varphi(x)=\sqrt[n]{x}$.

We have seen those maps which are topologically conjugate to the tent map τ. Of course there are a lot of unimodal maps which are not topologically conjugate to τ. In addition, there are many cases where $*$-endomorphisms on $\pi(C[0, 1])$ associated with maps on $[0, 1]$ are not strong-mixing. Hence, in this subsection, we leave two questions concerning relationship between general unimodal maps and strong-mixing maps.

(1) Does there exist a unimodal map φ such that $\alpha_{V(\varphi)}$ is strong-mixing on $\pi(C[0, 1])$ but not on $\mathfrak{L}(L^2[0, 1])$?

(2) Is a unimodal map φ topologically conjugate to the tent map τ if $\alpha_{V(\varphi)}$ is strong-mixing on $\mathfrak{L}(L^2[0, 1])$ or $\pi(C[0, 1])$?

3.3. Here we study covariant representations of chaotic dynamical systems on Cantor set. Let X be the compact infinite direct product $\prod_{n=1}^{\infty}\{1, 2\}$. Moreover we denote by σ and β, γ the unilateral shift on X and the two inverse maps of σ, that is,

$$\sigma((x_1, x_2, x_3, \cdots))=(x_2, x_3, x_4, \cdots),$$

$$\beta((x_1, x_2, x_3, \cdots))=(1, x_1, x_2, \cdots),$$

$$\gamma((x_1, x_2, x_3, \cdots))=(2, x_1, x_2, \cdots),$$

where $(x_n)_{n=1}^{\infty}$ is in X. In addition, for $\mu=(j_1, \cdots, j_k)$ in $\{1, 2\}^k=\prod_{n=1}^{k}\{1, 2\}$, we denote by $C(\mu)$ or $C(j_1, \cdots, j_k)$ the cylinder set $\{x=(x_n)_{n=1}^{\infty}\in X: x_i=j_i \text{ for } i=1, 2, \cdots, k\}$. For a vector $P=\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $Q=\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, the associated measure $m=m(P, Q)$ is determined by $m(C(j_1, \cdots, j_k))=p_{j_1}p_{j_1j_2}\cdots p_{j_{k-1}j_k}$. Let $P=\begin{pmatrix} \lambda/(\lambda+\mu) \\ \mu/(\lambda+\mu) \end{pmatrix}$ and $Q=\begin{pmatrix} 1-\mu & \mu \\ \lambda & 1-\lambda \end{pmatrix}$, where $0<\lambda, \mu<1$. Then $m=m(\lambda, \mu)=m(P, Q)$ is a faithful σ-invariant measure on X and the three operators T_σ, T_β and T_γ are bounded on $L^2(X, m)$. We put

$$V_1=M_{\chi_{c(1)}}T_\sigma\left(\frac{1}{\sqrt{1-\mu}}M_{\chi_{c(1)}}+\frac{1}{\sqrt{\lambda}}M_{\chi_{c(2)}}\right),$$

$$V_2=M_{\chi_{c(2)}}T_\sigma\left(\frac{1}{\sqrt{\mu}}M_{\chi_{c(1)}}+\frac{1}{\sqrt{1-\lambda}}M_{\chi_{c(2)}}\right).$$

Let $\pi(f)=M_f$ on $L^2(X, m)$ for f in $C(X)$. Then we have $\pi(\alpha_\sigma(f))=V_1\pi(f)V_1^*+V_2\pi(f)V_2^*$ for f in $C(X)$. Moreover we put

$$W_1=V_1(\sqrt{1-\mu}M_{\chi_{c(1)}}+\sqrt{\lambda}M_{\chi_{c(2)}})+V_2(\sqrt{\mu}M_{\chi_{c(1)}}+\sqrt{1-\lambda}M_{\chi_{c(2)}}),$$
$$W_2=V_1(\sqrt{\mu}M_{\chi_{c(1)}}+\sqrt{1-\lambda}M_{\chi_{c(2)}})-V_2(\sqrt{1-\mu}M_{\chi_{c(1)}}+\sqrt{\lambda}M_{\chi_{c(2)}}).$$

Then we have $W_1=T_\sigma$ and $\alpha_V=\alpha_W$ on $\mathfrak{L}(L^2(X, m))$ (if $\lambda+\mu=1$) or the commutant $B(1, 2)$ of the C^*-agebra $A(1, 2)$ generated by $\{M_{\chi_{c(1)}}, M_{\chi_{c(2)}}\}$ (if $\lambda+\mu\neq 1$). For $\xi_1(x)=1$ on X and (W_1, W_2), let $\{\xi_n\}_{n=1}^{\infty}$ be the orthonormal system defined by (2.1.3). Since σ is strong-mixing with respect to the measure $m=m(P, Q)$ (cf. [6: Theorem 1.31]), $\{\xi_n\}_{n=1}^{\infty}$ is complete. Hence it is a base of Walsh type with respect to (W_1, W_2) and α_V is strong-mixing on $\mathfrak{L}(L^2(X, m))$ or the C^*-algebra $B(1, 2)$.

In the case where $\lambda+\mu=1$, we have

$$V_1=\frac{1}{\sqrt{\lambda}}M_{\chi_{c(1)}}T_\sigma, \quad V_2=\frac{1}{\sqrt{\mu}}M_{\chi_{c(2)}}T_\sigma,$$

$$W_1=\sqrt{\lambda}V_1+\sqrt{\mu}V_2=T_\sigma, \quad W_2=\sqrt{\mu}V_1-\sqrt{\lambda}V_2=\sqrt{\frac{\mu}{\lambda}}M_{\chi_{c(1)}}T_\sigma+\sqrt{\frac{\lambda}{\mu}}M_{\chi_{c(2)}}T_\sigma.$$

Therefore we obtained the following:

THEOREM 3.3.1. *The representation π of $C(X)$ into $\mathfrak{L}(L^2(X, m(\lambda, \mu))$ defined by $\pi(f)=M_f$ is a covariant representation of multiplicity* 2, *and the associated $*$-endomorphism α_V is strong-mixing on $\mathfrak{L}(L^2(X, m(\lambda, \mu)))$ (if $\lambda+\mu=1$) or the C^*-algebra $B(1, 2)$ (if $\lambda+\mu\neq 1$).*

REMARK 3.3.2. Let us recall the family of logistic maps $\lambda_c(x)=cx(1-x)$, $c>0$. Suppose $c>2+\sqrt{5}$ and

$$\Lambda=\{x\in[0, 1] : \lambda_c^n(x) \text{ is in } [0, 1] \text{ for each positive integer } n\}.$$

Then the topological dynamical system (Λ, λ_c) is topologically conjugate to $(\prod_{n=1}^{\infty}\{1, 2\}, \sigma)$ (cf. [2: §1.7. Theorem 7.3]). Hence the example discussed above is regarded as a covariant representation of (Λ, λ_c).

REMARK 3.3.3. Let $X=\prod_{n=1}^{\infty}\{1, 2\}$ and $A=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$. We denote by X_A and σ_A, the set $\{x=(x_n)_{n=1}^{\infty}\in X : x_n=1 \text{ can be followed by } x_{n+1}=2\}$ and the restriction of σ to X_A. Moreover we put

$$p=\begin{pmatrix}\lambda/(1+\lambda)\\ 1/(1+\lambda)\end{pmatrix} \quad\text{and}\quad Q=\begin{pmatrix}0 & 1\\ \lambda & 1-\lambda\end{pmatrix}.$$

Then the associated measure $m=m(P, Q)$ is a faithful σ_A-invariant measure on X_A (cf. [6: Theorem 1.31]). Let

$$V_1 = M_{\chi_{C(2,1)}} T_{\sigma_A}, \quad V_2 = \frac{1}{\sqrt{\lambda}} M_{\chi_{C(1)}} T_{\sigma_A}, \quad V_3 = \frac{1}{\sqrt{1-\lambda}} M_{\chi_{C(2,2)}} T_{\sigma_A}.$$

Then (V_1, V_2, V_3) are three isometries on $L^2(X, m)$ such that $V_1V_1^* + V_2V_2^* + V_3V_3^* = I$ and if $\pi(f) = M_f$ on $L^2(X, m)$ it follows that

$$\pi(\alpha_{\sigma_A}(f)) = V_1\pi(f)V_1^* + V_2\pi(f)V_2^* + V_3\pi(f)V_3^*.$$

Furthermore we note that the topological dynamical system (X_A, σ_A) is topologically conjugate to (Λ, λ_c), where $\lambda_c(x) = cx(1-x)$ and c is approximately equal to 3.839 $(>1+\sqrt{8})$ and

$$\Lambda = \{x \in [0, 1] : \{\lambda_c^{3n}(x)\}_{n=1}^{\infty} \text{ does not converge}\}$$

(cf. [2: §1.3, Theorem 13.7].) The covariant representation of this dynamical systems is studied in author's subsequent paper related to representations of O_3.

3.4. Let φ be the map on the set of positive integers $\mathbf{N}$ defined by $\varphi(2n-1) = n = \varphi(2n)$ for n in $\mathbf{N}$. We denote by $\{e_n\}_{n=1}^{\infty}$ the canonical base of $l^2(\mathbf{N})$, that is, $e_n(i) = \delta_{n,i}$. Moreover let π_ψ be the canonical representation of $l^\infty(\mathbf{N})$ into $\mathfrak{L}(l^2(\mathbf{N}))$, that is, $\pi_\psi(g)e_n = g(n)e_n$, $(g \in l^\infty(\mathbf{N}), n \in \mathbf{N})$. Furthermore let $W_1(\psi)$ and $W_2(\psi)$ be two isometries on $l^2(\mathbf{N})$ determined by

$$W_1(\psi)e_n = e_{2n-1}, \quad W_2(\psi)e_n = e_{2n} \qquad (n = 1, 2, \cdots).$$

Then we have $\pi_\psi(\alpha_\psi(g)) = W_1(\psi)\pi_\psi(g)W_1(\psi)^* + W_2(\psi)\pi_\psi(g)W_2(\psi)^*$. Since $\{e_n\}_{n=1}^{\infty}$ is a base of Walsh type with respect to $(W_1(\psi), W_2(\psi))$, $*$-endomorphism $\alpha_{W(\psi)}$ is strong-mixing on $\mathfrak{L}(l^2(\mathbf{N}))$.

Now suppose that a topological dynamical system (X, φ) has a strong-mixing covariant representation π of multiplicity 2 into $\mathfrak{L}(L^2(X, m))$, that is,

$$\pi(\alpha_\varphi(f)) = W_1(\varphi)\pi(f)W_1(\varphi)^* + W_2(\varphi)\pi(f)W_2(\varphi)^*$$

for a couple $(W_1(\varphi), W_2(\varphi))$ of isometries with respect to which there exists a base $\{e_n\}_{n=1}^{\infty}$ of Walsh type. By identifying the Hilbert space $L^2(X, m)$ with $l^2(\mathbf{N})$, we can recognize the property of strong-mixing as follows:

$$\pi(\alpha_\varphi(f)) = \alpha_{W(\varphi)}(\pi(f)) = \alpha_{W(\psi)}(\pi(f)) \qquad \text{for } f \text{ in } C(X),$$

$$\pi_\psi(\alpha_\psi(g)) = \alpha_{W(\varphi)}(\pi_\psi(g)) = \alpha_{W(\varphi)}(\pi_\psi(g)) \qquad \text{for } g \text{ in } l^\infty(\mathbf{N}).$$

Namely every strong-mixing representation π is prolonged to π_ψ by $\alpha_{W(\varphi)} = \alpha_{W(\psi)}$.

ACKNOWLEDGEMENT. The author wishes to express his thanks to the members of the seminar on operator algebras and operator theory in Sendai for useful discussion with them, and to the refree for his/her adequate advice.

References

[1] J. Cuntz, Simple C^*-algebras generated by isometries, Comm. Math. Phys. **57** (1977), 173–185.

[2] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd Ed., Addison-Wesley (1989).

[3] F. Schipp, W. R. Wade and P. Simon, *Walsh Series*, Adam Hilger (1990).

[4] P. J. Stacey, Crossed product of C^*-algebras by $*$-endomorphisms, J. Austral. Math. Soc. **54** (1993), 204–212.

[5] J. Tomiyama, *Invitation to C^*-Algebras and Topological Dynamics*, World Scientific (1987).

[6] P. Walter, *An Introduction to Ergodic Theory*, GTM **79**, Springer (1982).

Present Address:
Department of Mathematical Sciences, Faculty of Science, Yamagata University,
Yamagata, 990 Japan.