TOKYO J. MATH.
VOL. 20, NO. 1, 1997

Hydrodynamic Limit for an Infinite Spin System on Z

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(Communicated by M. Maejima)

0. Introduction.

In this paper we study the hydrodynamic limit for a Markov process of a certain spin system on $\mathbf{Z}$. In the process a $[0, \infty)$-valued spin is attached to each site of $\mathbf{Z}$. In this paper the word "energy" is often used instead of "spin" since the value is non-negative. At random times the values of energy on each pair of adjacent sites evolve simultaneously according to a certain law which conserves their sum. The jump rate is the sum of the values of energy. Since the jump rate is unbounded, the construction of the process is not easy. A Markov process describing this infinite system is constructed in [5]. The process is reversible with respect to an infinite product measure and of gradient type.

We show that, under an appropriate scaling of lattice spacing and time, the macroscopic energy distribution converges to a deterministic limit which is characterized by a non-linear diffusion equation on the one-dimensional real line: $(\partial/\partial t)\rho(t, w) = 2^{-1}(\partial^2/\partial w^2)P(\rho(t, w))$, $(t, w) \in (0, \infty) \times \mathbf{R}$. A sufficient condition for the uniqueness of non-negative weak solutions of this equation was given in terms of the non-linear function $P(\rho)$, $\rho \geq 0$, in [1], [2]. We show that, under some assumptions for our system, $P(\rho)$, $\rho \geq 0$, satisfies this sufficient condition. To do this we make use of the Laplace method.

The hydrodynamic limit for the corresponding model on a periodic lattice is studied in Suzuki-Uchiyama ([6], [7]), and a non-linear diffusion equation on a torus is obtained.

In Guo-Papanicolaou-Varadhan ([4]) the hydrodynamic limit for a finite system of interacting diffusions is studied by using certain estimates deduced from an assumed bound for entropy of initial distributions. In Fritz ([3]) the hydrodynamic limit for

Received September 6, 1995
Revised October 20, 1995
This research was supported by the Japan Society for the Promotion of Science for Japanese Junior Scientists (April, 1993–March, 1995) and by Grant-in-Aid for Scientific Research (No. 015-10002667), Ministry of Education, Science and Culture.

infinite systems of interacting diffusions is studied by extending the entropy arguments of [4] to infinite systems. Namely, in [3], a priori bounds for the flow of entropy in finite volumes are derived for infinite systems. Since the method of [3] for obtaining the bounds depends heavily on the dynamics of the systems, it does not seem to be applicable to our system. Our method is different from that of [3], namely, we make use of the approximation of our infinite system by certain finite systems.

1. The model and results.

Before describing our infinite system on $\mathbf{Z}$, we introduce a system on two sites. Let X and Y be independent positive random variables with a common law having a positive continuous density $p(x)$, $x>0$. We assume

$$\text{(A.1)} \qquad \int_0^\infty x^2 p(x)dx < \infty .$$

Let $\gamma(\cdot\,;r)$ be the conditional distribution of X given $X+Y=r$, $r>0$, namely,

$$\gamma(du\,;r)=\frac{p(u)p(r-u)}{\int_0^r p(v)p(r-v)dv}\mathbf{1}_{(0,r)}(u)du .$$

Here $\mathbf{1}_A$ stands for the indicator function of a set A. Let $\mathbf{R}_+=[0,\infty)$ and $\mathbf{C}_b(\mathbf{R}_+^2)$ be the set of bounded continuous functions on $\mathbf{R}_+^2=\mathbf{R}_+\times\mathbf{R}_+$. For $\phi\in\mathbf{C}_b(\mathbf{R}_+^2)$ define

$$\begin{aligned}\Gamma\phi(x,y)&=E[\phi(X,Y)\mid X+Y=x+y]\\ &=\int\phi(u,x+y-u)\gamma(du\,;x+y), \qquad \text{if } x+y>0,\end{aligned}$$

and $\Gamma\phi(0,0)=\phi(0,0)$. The generator of the Markov process on $\mathbf{R}_+^2$ describing the dynamics of a system on two sites is given by L:

$$L\phi(x,y)=(x+y)(\Gamma-I)\phi(x,y), \qquad \phi\in\mathbf{C}_b(\mathbf{R}_+^2).$$

Here I is the identity operator. The sum $x+y$ is conserved by the dynamics. Observing $E[\psi(X,Y)\Gamma\phi(X,Y)]=E[E[\psi(X,Y)\mid X+Y]E[\phi(X,Y)\mid X+Y]]$, we see that Γ is symmetric relative to the product measure $p(x)p(y)dxdy$. This product measure is a reversible measure for the Markov process with generator L.

Now we describe our infinite system on $\mathbf{Z}$. Let

$$\mathfrak{X}=\mathbf{R}_+^{\mathbf{Z}}=\{\mathbf{x}=(x_i\,;i\in\mathbf{Z}):x_i\in\mathbf{R}_+ \text{ for } i\in\mathbf{Z}\}.$$

It is a Polish space endowed with the product topology. Here x_i is viewed as a value of energy on a site i and $\mathbf{x}$ as a configuration of values of energy on $\mathbf{Z}$. The process we are going to study is a Markov process on $\mathfrak{X}$, in which the values of energy x_i and

x_{i+1} on each pair of adjacent sites evolve simultaneously according to the law determined by Γ at rate x_i+x_{i+1}. Roughly speaking, it is a Markov process with generator

$$\mathscr{L}\phi(\mathbf{x})=\sum_{i\in\mathbf{Z}} L_i\phi(\mathbf{x}), \qquad L_i\phi(\mathbf{x})=L\phi_{\mathbf{x},i}(x_i, x_{i+1}),$$

where $\phi_{\mathbf{x},i}(x, y)=\phi(\mathbf{x})|_{x_i=x, x_{i+1}=y}$ is a function of (x, y) with $(x_k)_{k\neq i,i+1}$ fixed. The product measure $\Phi(d\mathbf{x})=\prod p(x_i)dx_i$, where i runs over $\mathbf{Z}$, is supposed to be a reversible measure of the process. According to [5], we have an equilibrium Markov process on $\mathfrak{X}$ associated with $\mathscr{L}$ and with initial (therefore marginal) distribution Φ under the assumption (A.1), and a non-equilibrium Markov process on $\mathfrak{X}$ associated with $\mathscr{L}$ under some additional assumptions for $p(x)$, $x>0$, and an initial distribution.

For $\varepsilon>0$ let

$$\hat{\mathbf{X}}^\varepsilon(t)=(\hat{X}_i^\varepsilon(t)\ ;\ i\in\mathbf{Z}), \qquad t\geq 0, \tag{1.1}$$

be a Markov process on $\mathfrak{X}$ associated with $\varepsilon^{-2}\mathscr{L}$. The factor ε^{-2} corresponds to speeding up of time. (Precisely speaking, we consider the process in (1.8) below.) We are interested in the limiting behavior, as $\varepsilon\downarrow 0$, of the empirical energy distribution α_t^ε:

$$\alpha_t^\varepsilon(dw)=\varepsilon\sum_{i\in\mathbf{Z}}\hat{X}_i^\varepsilon(t)\delta_{\varepsilon i}(dw), \qquad w\in\mathbf{R}. \tag{1.2}$$

Here δ_w is the δ-measure at w. We regard α_t^ε, $t\geq 0$, as a random process taking values in $\mathfrak{M}(\mathbf{R})$, the set of measures on $\mathbf{R}$ with the topology of vague convergence. We prove under suitable conditions that, as $\varepsilon\downarrow 0$, α_t^ε converges in probability to a deterministic limit which is characterized by a certain non-linear diffusion equation.

First of all we state the result in [5] concerning the construction of a non-equilibrium Markov process in our context. Let $\mathscr{P}(\mathfrak{X})$ denote the set of probability measures on $\mathfrak{X}$ and for any non-negative integer m define a probability measure Φ_m on $\mathbf{R}_+^{\{-m,\cdots,m\}}$ by

$$\Phi_m(d\mathbf{x}_m)=\sum_{i=-m}^{m} p(x_i)dx_i.$$

We introduce the relative entropy of $\mu\in\mathscr{P}(\mathfrak{X})$ in $\{-m, \cdots, m\}$ with respect to Φ_m defined by

$$H_m(\mu)=\begin{cases}\displaystyle\int\log\{(d\mu_m)/(d\Phi_m)\}d\mu_m, & \text{if}\quad \mu_m\ll\Phi_m,\\ \infty, & \text{otherwise},\end{cases}$$

where μ_m denotes the projection of μ to $\mathbf{R}_+^{\{-m,\cdots,m\}}$. Let $M=\{-M^-, M^+\}$ be a pair of integers $-M^-$ and M^+ such that $M^-\geq 1$ and $M^+\geq 1$ and let

$$\mathbf{X}^M(t)=(X_i^M(t)\ ;\ i\in\mathbf{Z}), \qquad t\geq 0, \tag{1.3}$$

be the Markov process on $\mathfrak{X}$ generated by $\mathscr{L}^M=\sum L_i$, where the summation is taken over the integers i's such that $-M^- \le i < M^+$. In this process, only the values of energy on each pair of adjacent sites in $[-M^-, M^+]$ evolve simultaneously and the values of energy on sites outside $[-M^-, M^+]$ remain unchanged with time. Our process is obtained as a limit of $\mathbf{X}^M(t)$, $t\ge 0$, as $M^-\to\infty$, $M^+\to\infty$. Let us construct $\mathbf{X}^M(t)$, $t\ge 0$, on a common probability space for all $M^-\ge 1$ and $M^+\ge 1$ by solving certain stochastic differential equations (abbreviated to SDE's) associated with common Poisson random measures following [5]. Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space equipped with a suitable filtration $\{\mathfrak{F}_t\}_{t\ge 0}$. Let $\{N_i(dsd\eta d\xi), i\in\mathbf{Z}\}$ be an independent family of $\mathfrak{F}_t$-adapted Poisson random measures on $(0,\infty)\times(0,\infty)\times(0,1)$ with intensity measure $dsd\eta d\xi$, and let $\mathbf{X}=(X_i\,;\, i\in\mathbf{Z})$ be an $\mathfrak{X}$-valued random variable which is distributed according to $\mu^0\in\mathscr{P}(\mathfrak{X})$ and is $\mathfrak{F}_0$-measurable (so is independent of $\{N_i(\cdot), i\in\mathbf{Z}\}$). Moreover, for $r>0$ let $a(\xi\,;r)$, $\xi\in(0,1)$, be the inverse function of $A(\zeta\,;r)\equiv\int_0^\zeta \gamma(du\,;r)$, $\zeta\in(0,r)$. For fixed r, $a(\xi\,;r)$ is a $(0,r)$-valued random variable on the probability space $((0,1), d\xi)$ having probability distribution $\gamma(\cdot\,;r)$. For each $M^-\ge 1$ and $M^+\ge 1$, the process $\mathbf{X}^M(t)$, $t\ge 0$, with initial distribution μ^0 can be constructed on $(\Omega, \mathfrak{F}, \mathbf{P})$ as the unique solution of the SDE

$$
\begin{aligned}
(1.4)\qquad X_i^M(t)=X_i&+\mathbf{1}_{[-M^-,M^+)}(i)\int_0^t\int_0^{X_i^M(s-)+X_{i+1}^M(s-)}\int_0^1\{a(\xi\,;X_i^M(s-)\\
&+X_{i+1}^M(s-))-X_i^M(s-)\}N_i(dsd\eta d\xi)\\
&+\mathbf{1}_{[-M^-,M^+)}(i-1)\int_0^t\int_0^{X_{i-1}^M(s-)+X_i^M(s-)}\int_0^1\{X_{i-1}^M(s-)\\
&-a(\xi\,;X_{i-1}^M(s-)+X_i^M(s-))\}N_{i-1}(dsd\eta d\xi)\,,\qquad i\in\mathbf{Z},\quad t\ge 0\,.
\end{aligned}
$$

Now we state a theorem concerning the construction of a non-equilibrium Markov process which is slightly modified from that of [5]. Let $g(x)$, $x>0$, be a positive increasing function satisfying

$$
\text{(A.2)}\qquad \limsup_{x\to\infty} g(cx)g(x)^{-1}<\infty \qquad \text{for all}\quad c>0\,;
$$

for example, $g(x)=x^\alpha$, $\alpha\ge 0$.

THEOREM A ([5]). *Suppose that $p(x)$, $x>0$, and $\mu^0\in\mathscr{P}(\mathfrak{X})$ satisfy the following conditions* (A.3)–(A.5):

(A.3) $\int_0^\infty e^{\gamma x^\beta}p(x)dx<\infty$ *for some* $\beta\ge 2$ *and* $\gamma>0$,

(A.4) *there exists* $C>0$ *such that* $H_m(\mu^0)\le Cg(m)$ *for all sufficiently large* m, *where* g *is some positive increasing function satisfying* (A.2),

(A.5) $\lim_{x\to\infty} g(x)x^{-(\beta^2-\beta)/(2\beta-1)}=0$ *for* g *in* (A.4) *and* $\beta\ge 2$ *in* (A.3).

Let $M_k=\{-M_k^-, M_k^+\}$, $k\ge 1$, *satisfy* $1\le M_1^-\le M_2^-\le\cdots$, $1\le M_1^+\le M_2^+\le\cdots$ *and*

(A.6)

$$\begin{cases} \text{(i)} & \textit{there exists } C>1 \textit{ such that } 1/C \le M_k^+/M_k^- \le C \textit{ and } M_{k+1}^+ \le CM_k^+ \textit{ for all sufficiently large } k; \\ \text{(ii)} & \sum_{k=1}^{\infty} g(M_k^+)^{2/\beta - 1/\beta^2}(M_k^+)^{-1+1/\beta} < \infty \textit{ for } g \textit{ in } (\mathrm{A.4}) \textit{ and } \beta \ge 2 \textit{ in } (\mathrm{A.3}) \\ & (\lim_{x\to\infty} g(x)^{2/\beta - 1/\beta^2} x^{-1+1/\beta} = 0 \textit{ by virtue of } (\mathrm{A.5})). \end{cases}$$

Then for any $T>0$ *we have*

$$\mathbf{P}\{\textit{for any } i \in \mathbf{Z} \textit{ there exists } X_i(t) \textit{ such that } X_i(t) = X_i^{M_k}(t) \quad \textit{for all } t \in [0, T] \textit{ and for all sufficiently large } k\} = 1 \, .$$

Moreover $\mathbf{X}(t) = (X_i(t) \, ; i \in \mathbf{Z})$, $t \ge 0$, *depends neither on* $\{M_k, k \ge 1\}$ *satisfying* (A.6) *nor on* T, *and is a Markov process on* $\mathfrak{X}$ *whose initial distribution is* μ^0 *and satisfies the SDE*

(1.5)

$$\begin{aligned} X_i(t) = X_i + & \int_0^t \int_0^{X_i(s-)+X_{i+1}(s-)} \int_0^1 \{a(\xi \, ; X_i(s-)+X_{i+1}(s-)) - X_i(s-)\} N_i(ds d\eta d\xi) \\ & + \int_0^t \int_0^{X_{i-1}(s-)+X_i(s-)} \int_0^1 \{X_{i-1}(s-) \\ & \qquad - a(\xi \, ; X_{i-1}(s-)+X_i(s-))\} N_{i-1}(ds d\eta d\xi) \, , \qquad i \in \mathbf{Z} \, , \quad t \ge 0 \, . \end{aligned}$$

Before studying the limiting behavior of (1.2), we study the hydrodynamic limit for finite systems. For $\varepsilon>0$ and $M=\{-M^-, M^+\}$, a pair of integers $-M^-$ and M^+ satisfying M^-, $M^+ \ge 1$, denote by

$$\mathbf{X}^{\varepsilon,M}(t) = (X_i^{\varepsilon,M}(t) \, ; i \in \mathbf{Z}) \, , \qquad t \ge 0 \, ,$$

the Markov process on $\mathfrak{X}$ with initial distribution $\mu_\varepsilon^0 \in \mathscr{P}(\mathfrak{X})$ generated by $\mathscr{L}^M$. For each $\varepsilon>0$ we construct these processes on $(\Omega, \mathfrak{F}, \mathbf{P})$ from the same initial value for all $M^- \ge 1$ and $M^+ \ge 1$ by solving SDE's associated with the common Poisson random measures $N_i(\cdot)$, $i \in \mathbf{Z}$ (see (1.4)). Define a process

$$\hat{\mathbf{X}}^{\varepsilon,M}(t) = (\hat{X}_i^{\varepsilon,M}(t) \, ; i \in \mathbf{Z}) \, , \qquad t \ge 0 \, ,$$

by $\hat{X}_i^{\varepsilon,M}(t) = X_i^{\varepsilon,M}(t/\varepsilon^2)$. This is a Markov process on $\mathfrak{X}$ speeded up, namely, with generator $\mathscr{L}^{\varepsilon,M} \equiv \varepsilon^{-2}\mathscr{L}^M$ and with initial distribution μ_ε^0. For $\varepsilon>0$ let $N(\varepsilon)=\{-N(\varepsilon)^-, N(\varepsilon)^+\}$ be a pair of integers $-N(\varepsilon)^-$ and $N(\varepsilon)^+$ such that $N(\varepsilon)^- \ge 1$ and $N(\varepsilon)^+ \ge 1$. We assume $\varepsilon N(\varepsilon)^+ \to \infty$ and $\varepsilon N(\varepsilon)^- \to \infty$ as $\varepsilon \downarrow 0$. We consider the finite system $\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(t)$, $t \ge 0$, and study the limiting behavior, as $\varepsilon \downarrow 0$, of the empirical energy distribution $\alpha_t^{\varepsilon,N(\varepsilon)}$:

$$\alpha_t^{\varepsilon,N(\varepsilon)}(dw) = \varepsilon \sum_{i \in \mathbf{Z}} \hat{X}_i^{\varepsilon,N(\varepsilon)}(t) \delta_{\varepsilon i}(dw) \, , \qquad w \in \mathbf{R} \, .$$

To do this, we make the following assumptions (A.7)–(A.10) for $p(x)$ and μ_ε^0 ($\in \mathscr{P}(\mathfrak{X})$):

(A.7) $\lim_{x\to\infty} e^{\lambda x} p(x) = 0$ for all $\lambda \ge 0$.

(A.8) $\int_0^\infty e^{\gamma x^2} p(x) dx < \infty$ for all $\gamma > 0$.

(A.9) There exists $C>0$ such that $H_{n(\varepsilon)}(\mu_\varepsilon^0)\leq C/\varepsilon$ for all $\varepsilon>0$, where $n(\varepsilon)=N(\varepsilon)^- \vee N(\varepsilon)^+$.

(A.10) There exists $C>0$ such that $\int x_i^2 d\mu_\varepsilon^0 \leq C$ for all $i\notin[-n(\varepsilon), n(\varepsilon)]$ and $\varepsilon>0$.

To describe the limiting non-linear diffusion equation, we introduce some notation. Put for $\lambda\in\mathbf{R}$

$$M(\lambda)=\int_0^\infty e^{\lambda x}p(x)dx\,, \qquad \omega(\lambda)=\log M(\lambda)\,,$$

and for $y\geq 0$

$$h(y)=\sup_{\lambda\in\mathbf{R}}\{\lambda y-\omega(\lambda)\}\,.$$

Then $h(\cdot)$ and $\omega(\cdot)$ are a pair of conjugate convex functions and

$$\lambda=h'(y) \text{ if and only if } y=\omega'(\lambda)\,.$$

We also know that $h'(\cdot)$ and $\omega'(\cdot)$ are smooth strictly increasing functions. For $\lambda\in\mathbf{R}$ define a probability density p_λ on $\mathbf{R}_+$ by

$$p_\lambda(x)=M(\lambda)^{-1}e^{\lambda x}p(x)\,.$$

Then we have

$$\int_0^\infty x p_\lambda(x)dx=\omega'(\lambda)\,.$$

We put for $\rho>0$

$$P(\rho)=\int_0^\infty x^2 p_{h'(\rho)}(x)dx\,,$$

and $P(0)=0$. The function $P(\cdot)$ is smooth and $P'(\rho)$ is positive for all $\rho>0$. We make the following assumptions (A.11) and (A.12) for $P(\rho)$, $\rho\geq 0$ (these are assumptions for $p(x)$, $x>0$):

(A.11) $\limsup_{\rho\to\infty} P(\rho)\rho^{-2}<\infty$.

(A.12) There exists $C>2$ such that $1+1/C\leq \rho P'(\rho)/P(\rho)\leq C$ for almost all $\rho\geq 0$.

REMARK 1.1. A sufficient condition for $p(x)$, $x>0$, that guarantees (A.11) is obtained in [6]. Suppose that $p(x)$ is of the form $p(x)=e^{-c(x)}b(x)$, $x>0$, where $b(x)$, $x>0$, is bounded away from zero and infinity. If $c(\cdot)$ is convex on some semi-infinite interval (n,∞), then (A.11) is satisfied (see [6]). The condition (A.12) is a sufficient condition for the uniqueness of non-negative weak solutions of the non-linear diffusion equation (1.6) appearing in the next theorem ([1], [2]). Some sufficient conditions in terms of $p(x)$, $x>0$, which ensure (A.12) will be given in Section 6. Under each of the conditions on $p(x)$, $x>0$, assumed in Proposition 6.2 and Proposition 6.3, we have

$\lim_{\rho\to\infty} P(\rho)\rho^{-2}=1$, which clearly implies (A.11).

The result concerning the hydrodynamic limit for the finite system is as follows.

THEOREM 1.1. *Suppose that all the hypotheses* (A.7)–(A.12) *are satisfied. Suppose further that as* $\varepsilon\downarrow 0$ *the random measure* $\alpha_0^{\varepsilon,N(\varepsilon)}$ *converges in probability to a non-random measure* $\rho_0(w)dw$ *in* $\mathbf{R}$. *Then for every* $t>0$, $\alpha_t^{\varepsilon,N(\varepsilon)}$ *also converges in probability to a non-random limit* $\rho(t,w)dw$ *as* $\varepsilon\downarrow 0$ *and* $\rho(t,w)$ *is a unique non-negative weak solution of the non-linear diffusion equation*

$$\text{(1.6)}\qquad \begin{cases} \dfrac{\partial}{\partial t}\rho(t,w)=\dfrac{1}{2}\dfrac{\partial^2}{\partial w^2}P(\rho(t,w))\,, \\ \rho(t,w)\big|_{t=0}=\rho_0(w)\,. \end{cases}$$

By *a weak solution* to the Cauchy problem (1.6) it is meant that both $\rho(t,w)$ and $P(\rho(t,w))$ are locally integrable on $(0,\infty)\times\mathbf{R}$ and

$$\langle\rho(t,\cdot),J\rangle=\langle\rho_0,J\rangle+\frac{1}{2}\int_0^t\langle P(\rho(s,\cdot)),J''\rangle ds\,,\qquad t>0\,,$$

for all $J\in\mathbf{C}_0^\infty(\mathbf{R})$, the space of smooth functions on $\mathbf{R}$ with compact support. Here $\langle v,J\rangle$ denotes the integral of a function J by $v\in\mathfrak{M}(\mathbf{R})$; when $v(dw)=\rho(w)dw$, the integral $\langle v,J\rangle$ is denoted also by $\langle\rho,J\rangle$.

REMARK 1.2. An example of a pair of $\rho_0(\cdot)$ and μ_ε^0 which satisfies the conditions in Theorem 1.1 is given below. For $R_0>0$, let $\rho_0(\cdot)$ be a positive continuous function on $\mathbf{R}$ satisfying

$$\rho_0(w)=\int_0^\infty xp(x)dx\,,\qquad w\notin[-R_0,R_0]\,,$$

and let

$$\mu_\varepsilon^0(dx)=\prod_{|i|\le R_0/\varepsilon}p_{h'(\rho_0(\varepsilon i))}(x_i)dx_i\prod_{|i|>R_0/\varepsilon}p(x_i)dx_i\,.$$

Then this pair of $\rho_0(\cdot)$ and μ_ε^0 satisfies the conditions in Theorem 1.1.

In order to study the hydrodynamic limit for our infinite system, we approximate our infinite system by certain finite systems. Let $F(x)$, $x>0$, be a positive strictly increasing function; the case $F(x)=x^\alpha$, $\alpha>0$, is of our interest. Let G be the inverse function of F. For the approximation by finite systems we make the following assumptions (A.13)–(A.16) for $p(x)$, μ_ε^0 and $F(x)$:

(A.13) $\int_0^\infty e^{\gamma x^\beta}p(x)dx<\infty$ for some $\beta\ge 2$ and $\gamma>0$.

(A.14) For any $c>0$ there exists $c'>0$ such that $cF(x)\le F(c'x)$ for all sufficiently large x.

(A.15) $\lim_{x\to\infty}F(x)x^{-(4\beta^2-1)/(\beta^2-\beta)}=\infty$ for $\beta\ge 2$ in (A.13).

(A.16) There exists $C>0$ such that $H_m(\mu_\varepsilon^0)\le CG(m)$ for all $\varepsilon>0$ and $m\ge[F(1/\varepsilon)]+1$, where $[u]$ denotes the integral part of $u \in \mathbf{R}$.

REMARK 1.3. If $F(x)$ is of the form $F(x)=x^\alpha$, $\alpha>0$, then (A.15) implies $\alpha>(4\beta^2-1)/(\beta^2-\beta)>4$ for $\beta\geq 2$ in (A.13). In this case, by (A.16), the relative entropy of initial distributions is assumed to satisfy $H_m(\mu_\varepsilon^0)\leq Cm^{1/\alpha}$ for all $\varepsilon>0$ and $m\geq[\varepsilon^{-\alpha}]+1$.

By (A.14) we have (A.2) for $g=G$. Moreover since

$$\lim_{x\to\infty} G(x)x^{-(\beta^2-\beta)/(4\beta^2-1)}=0 \tag{1.7}$$

by virtue of (A.15), (A.5) is satisfied by $g=G$. Therefore, by Theorem A, under the assumptions (A.13)–(A.16) we obtain for each $\varepsilon>0$ a Markov process

$$\mathbf{X}^\varepsilon(t)=(X_i^\varepsilon(t)\ ;\ i\in\mathbf{Z}), \qquad t\geq 0,$$

on $\mathfrak{X}$ associated with $\mathscr{L}$ whose initial distribution is μ_ε^0 as a limit of $\mathbf{X}^{\varepsilon,M_k}(t)$, $t\geq 0$, as $k\to\infty$, where $M_k=\{-M_k^-, M_k^+\}$, $k\geq 1$, satisfy $1\leq M_1^-\leq M_2^-\leq\cdots$, $1\leq M_1^+\leq M_2^+\leq\cdots$ and (A.6) for $g=G$. For $\varepsilon>0$ let $K(\varepsilon)=\{-K(\varepsilon)^+, K(\varepsilon)^+\}$ be a pair of integers defined by

$$K(\varepsilon)^+=[F(1/\varepsilon)]+1\ .$$

Then, by (A.15) $\varepsilon K(\varepsilon)^+\to\infty$ as $\varepsilon\downarrow 0$, and by (A.16) there exists $C>0$ such that $H_{K(\varepsilon)^+}(\mu_\varepsilon^0)\leq C/\varepsilon$ for all $\varepsilon>0$.

The result concerning the approximation of our infinite system by finite systems is as follows.

THEOREM 1.2. *Suppose that all the hypotheses* (A.13)–(A.16) *are satisfied. Then for any* $T>0$ *and* $R>0$

$$\lim_{\varepsilon\downarrow 0}\mathbf{P}\{X_i^\varepsilon(t)=X_i^{\varepsilon,K(\varepsilon)}(t) \textit{ for all } i\in[-[R/\varepsilon],[R/\varepsilon]] \textit{ and } t\in[0, T/\varepsilon^2]\}=1\ .$$

For $\varepsilon>0$ define a process

$$\hat{\mathbf{X}}^\varepsilon(t)=(\hat{X}_i^\varepsilon(t)\ ;\ i\in\mathbf{Z}), \qquad t\geq 0, \tag{1.8}$$

by $\hat{X}_i^\varepsilon(t)=X_i^\varepsilon(t/\varepsilon^2)$. Then this is a Markov process on $\mathfrak{X}$ associated with $\varepsilon^{-2}\mathscr{L}$ whose initial distribution is μ_ε^0. We study the limiting behavior of (1.2) for $\hat{\mathbf{X}}^\varepsilon(t)$, $t\geq 0$, in (1.8). Let $J\in C_0^\infty(\mathbf{R})$ and $suppJ\subset[-R, R]$ for some $R>0$. We are going to examine the limiting behavior, as $\varepsilon\downarrow 0$, of

$$\langle\alpha_t^\varepsilon, J\rangle=\varepsilon\sum_{|i|\leq R/\varepsilon} J(\varepsilon i)\hat{X}_i^\varepsilon(t), \qquad 0\leq t\leq T, \tag{1.9}$$

where T is an arbitrary positive constant. To do this, we combine Theorem 1.1 and Theorem 1.2. By Theorem 1.2 we have for any $T>0$ and $R>0$

(1.10)

$$\lim_{\varepsilon\downarrow 0} \mathbf{P}\{\hat{X}_i^\varepsilon(t)=\hat{X}_i^{\varepsilon,K(\varepsilon)}(t) \text{ for all } i\in[-[R/\varepsilon],[R/\varepsilon]] \text{ and } t\in[0,T]\}=1 .$$

Under the conditions in Theorem 1.2 we can take $N(\varepsilon)=K(\varepsilon)$ in Theorem 1.1, and then we have the limiting behavior, as $\varepsilon\downarrow 0$, of

$$\langle\alpha_t^{\varepsilon,K(\varepsilon)}, J\rangle=\varepsilon \sum_{|i|\le R/\varepsilon} J(\varepsilon i)\hat{X}_i^{\varepsilon,K(\varepsilon)}(t)\,, \qquad 0\le t\le T. \tag{1.11}$$

Combining (1.10) and the limiting behavior of (1.11), we obtain the limiting behavior of (1.9).

Our main result in this paper concerning the hydrodynamic limit for the infinite system is as follows.

THEOREM 1.3. *Suppose that all the hypotheses* (A.7), (A.11), (A.12), (A.14), (A.16) *and the following* (A.17)–(A.19) *are satisfied*:

(A.17) $\int_0^\infty e^{\gamma x^\beta} p(x)dx<\infty \begin{cases} \text{for some } \beta>2 \text{ and some } \gamma>0\,, \\ \text{for } \beta=2 \text{ and all } \gamma>0\,, \end{cases}$

(A.18) $\lim_{x\to\infty} F(x)x^{-(4\beta^2-1)/(\beta^2-\beta)}=\infty$ *for* $\beta\ge 2$ *in* (A.17),

(A.19) *there exists* $C>0$ *such that* $\int x_i^\beta d\mu_\varepsilon^0\le C$ *for all* $i\notin[-[F(1/\varepsilon)]-1, [F(1/\varepsilon)]+1]$ *and* $\varepsilon>0$, *where* $\beta\ge 2$ *is a constant in* (A.17).

Suppose further that as $\varepsilon\downarrow 0$ *the random measure* α_0^ε *converges in probability to a non-random measure* $\rho_0(w)dw$ *in* $\mathbf{R}$. *Then for every* $t>0$, α_t^ε *also converges in probability to a non-random limit* $\rho(t,w)dw$ *as* $\varepsilon\downarrow 0$ *and* $\rho(t,w)$ *is a unique non-negative weak solution of the non-linear diffusion equation* (1.6).

REMARK 1.4. By (A.16), (1.7) and noting $(\beta^2-\beta)/(4\beta^2-1)<1/4$ for $\beta\ge 2$, we see that our condition (A.16) for the relative entropy of initial distributions is stronger than that of [3], where it is assumed to satisfy $H_m(\mu_\varepsilon^0)=O(m)$ as $m\to\infty$ for all $\varepsilon>0$. The pair of $\rho_0(\cdot)$ and μ_ε^0 given in Remark 1.2 also gives an example which satisfies the conditions in Theorem 1.3.

In Sections 2, 3 and 4 we prove Theorem 1.1, and in Section 5 we prove Theorem 1.2. In Section 6 we discuss some sufficient conditions for $p(x)$, $x>0$, which guarantee the condition (A.12).

2. Relative entropy and I-function for the finite system.

In this section we make some estimates of the relative entropy and the I-function associated with $\mathscr{L}^{N(\varepsilon)}$ which will be used for proving Theorem 1.1. To examine the limiting behavior of $\alpha_t^{\varepsilon,N(\varepsilon)}$, $0\le t\le T$, we may assume $T=1$ without loss of generality. For each $\varepsilon>0$ and $0\le t\le 1$, denote by $\mu_\varepsilon^t\in\mathscr{P}(\mathfrak{X})$ the distribution of $\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(t)$. Recall

$n(\varepsilon)=N(\varepsilon)^- \vee N(\varepsilon)^+$. The evolution of the process gives us a density $f_\varepsilon^t=\{d(\mu_\varepsilon^t)_{n(\varepsilon)}\}/\{d\Phi_{n(\varepsilon)}\}$ at time t which is characterized by the forward equation

$$\frac{\partial f_\varepsilon^t}{\partial t}=\mathscr{L}^{\varepsilon,N(\varepsilon)} f_\varepsilon^t .$$

Let $\bar{\mu}_\varepsilon$ be the time average of μ_ε^t, namely,

$$\bar{\mu}_\varepsilon=\int_0^1 \mu_\varepsilon^t dt ,$$

and put $\bar{f}_\varepsilon=\{d(\bar{\mu}_\varepsilon)_{n(\varepsilon)}\}/\{d\Phi_{n(\varepsilon)}\}=\int_0^1 f_\varepsilon^t dt$.

Let $M=\{-M^-, M^+\}$ be a pair of integers $-M^-$ and M^+ such that $M^-\geq 1$ and $M^+\geq 1$, and put $m=M^- \vee M^+$. For a probability density $f(\mathbf{x}_m)$, $\mathbf{x}_m\in\mathbf{R}_+^{\{-m,\dots,m\}}$, with respect to Φ_m, define the I-function $I_M(f)$ associated with $\mathscr{L}^M$ by

$$I_M(f)=\sum_{i=-M^-}^{M^+-1} I_i^{(m)}(f) ,$$

$$I_i^{(m)}(f)=\frac{1}{2}\int(x_i+x_{i+1})d\Phi_m\int\{\sqrt{f}_{\mathbf{x}_m,i}(u,\, x_i+x_{i+1}-u)-\sqrt{f}(\mathbf{x}_m)\}^2\gamma(du\,;\, x_i+x_{i+1}) ,$$

where $f_{\mathbf{x}_m,i}(x, y)=f(\mathbf{x}_m)|_{x_i=x,x_{i+1}=y}$. If $\int x_i f d\Phi_m<\infty$ for all $-M^-\leq i\leq M^+$, then $I_M(f)<\infty$ and we can write

$$I_M(f)=-\int\sqrt{f}\mathscr{L}^M\sqrt{f}d\Phi_m .$$

The functional $I_M(\,\cdot\,)$ is non-negative and convex.

Let us estimate $H_{n(\varepsilon)}(\bar{\mu}_\varepsilon)$ and $I_{N(\varepsilon)}(\bar{f}_\varepsilon)$ under the assumption (A.9) according to [4]. For $0\leq t\leq 1$ we have

$$\frac{d}{dt}H_{n(\varepsilon)}(\mu_\varepsilon^t)\leq-\frac{4}{\varepsilon^2}I_{N(\varepsilon)}(f_\varepsilon^t)$$

in the same way as in [6]. Therefore $H_{n(\varepsilon)}(\mu_\varepsilon^t)$ is non-increasing in t. Moreover by integrating both sides of the above inequality with respect to t, we obtain the following lemma as in [6] and [7].

LEMMA 2.1. *Suppose that the hypothesis* (A.9) *is satisfied. Then there exists* $C>0$ *such that for any* $\varepsilon>0$

$$H_{n(\varepsilon)}(\bar{\mu}_\varepsilon)\leq C/\varepsilon , \tag{2.1}$$

$$I_{N(\varepsilon)}(\bar{f}_\varepsilon)\leq C\varepsilon . \tag{2.2}$$

3. Derivation of the non-linear diffusion equation from the finite systems.

In this section we outline the proof of Theorem 1.1. Let $J \in \mathbf{C}_0^\infty(\mathbf{R})$ and $suppJ \subset [-R, R]$ for some $R>0$. We are going to study the limiting behavior, as $\varepsilon \downarrow 0$, of

$$\langle \alpha_t^{\varepsilon,N(\varepsilon)}, J\rangle = \varepsilon \sum_{|i| \le R/\varepsilon} J(\varepsilon i)\hat{X}_i^{\varepsilon,N(\varepsilon)}(t), \qquad 0 \le t \le 1 .$$

Define $\hat{M}^\varepsilon(t)$, $0 \le t \le 1$, by the relation

$$\langle \alpha_t^{\varepsilon,N(\varepsilon)}, J\rangle = \langle \alpha_0^{\varepsilon,N(\varepsilon)}, J\rangle + \int_0^t b_\varepsilon(\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(s))ds + \hat{M}^\varepsilon(t), \tag{3.1}$$

where

$$b_\varepsilon(\mathbf{x}) = \mathscr{L}^{\varepsilon,N(\varepsilon)}\left\{\varepsilon \sum_{|i| \le R/\varepsilon} J(\varepsilon i)x_i\right\}.$$

Then $\hat{M}^\varepsilon(t)$ is a martingale and its quadratic variational process is written as

$$\langle \hat{M}^\varepsilon\rangle(t) = \int_0^t c_\varepsilon(\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(s))ds , \tag{3.2}$$

$$c_\varepsilon(\mathbf{x}) = \sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]} \{J(\varepsilon(i+1)) - J(\varepsilon i)\}^2(x_i + x_{i+1}) \left(\int u^2\gamma(du\,;\, x_i + x_{i+1}) - x_i x_{i+1}\right). \tag{3.3}$$

The following lemma is due to K. Uchiyama.

LEMMA 3.1. *For any $t>0$, $\hat{M}^\varepsilon(t)$ converges to 0 in probability as $\varepsilon \downarrow 0$.*

PROOF. By (3.2) and (3.3), we have only to show that for any $R>0$

$$\varepsilon^2 \int_0^1 \sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1} \{\hat{X}_i^{\varepsilon,N(\varepsilon)}(t)\}^3 dt \to 0 \qquad \text{as } \varepsilon \downarrow 0 \text{ in probability} .$$

Considering the decomposition

$$1 = \mathbf{1}(\hat{X}_i^{\varepsilon,N(\varepsilon)}(t) \le (\delta/\varepsilon)^{2/3}) + \mathbf{1}(\hat{X}_i^{\varepsilon,N(\varepsilon)}(t) > (\delta/\varepsilon)^{2/3})$$

for $\delta>0$, we have for any $a>0$

$$\begin{aligned} &\mathbf{P}\left\{\varepsilon^2 \int_0^1 \sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1} \{\hat{X}_i^{\varepsilon,N(\varepsilon)}(t)\}^3 dt > a\right\} \\ &\le \mathbf{P}\left\{\varepsilon\delta \int_0^1 \sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1} \{\hat{X}_i^{\varepsilon,N(\varepsilon)}(t)\}^{3/2} dt > a/2\right\} \end{aligned} \tag{3.4}$$

$$+\mathbf{P}\left\{\varepsilon^2\int_0^1\sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1}\{\hat{X}_i^{\varepsilon,N(\varepsilon)}(t)\}^3\mathbf{1}(\hat{X}_i^{\varepsilon,N(\varepsilon)}(t)>(\delta/\varepsilon)^{2/3})dt>a/2\right\}.$$

Denote by $\mathbf{E}$ the expectation with respect to $\mathbf{P}$. First let us estimate

$$\mathbf{E}\left[\int_0^1\sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1}\{\hat{X}_i^{\varepsilon,N(\varepsilon)}(t)\}^{3/2}dt\right]=\int\sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1}x_i^{3/2}d\bar{\mu}_\varepsilon, \tag{3.5}$$

by making use of the entropy inequality

$$\int fd\mu\le\log\int e^f d\Phi_n+H_n(\mu), \tag{3.6}$$

where f is a measurable function on $\mathbf{R}_+^{\{-n,\cdots,n\}}$ and $\mu\in\mathscr{P}(\mathfrak{X})$. By taking $n=n(\varepsilon)=N(\varepsilon)^-\vee N(\varepsilon)^+$ and $f(\mathbf{x}_n)=\sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1}x_i^{3/2}$ in (3.6) and applying (A.8) and (2.1), we see that the right-hand side of (3.5) is dominated by const.ε^{-1}. Therefore, by Chebyshev's inequality, the first term of the right-hand side of (3.4) is dominated by const.δ/a, which can be made arbitrarily small independent of ε by choosing δ small enough.

Next we estimate the second term of the right-hand side of (3.4). Let $\mathbf{D}([0, 1]\,;\mathfrak{X})$ be the set of functions on $[0, 1]$ with values in $\mathfrak{X}$ which are right continuous and have left limits. Let $P_{\mu_\varepsilon^0}$ be the distribution of $\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(t)$, $0\le t\le 1$, with initial distribution μ_ε^0. This is a probability measure on $\mathbf{D}([0, 1]\,;\mathfrak{X})$. Since for the process $\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(t)$, $0\le t\le 1$, the values of energy on sites outside $[-n(\varepsilon), n(\varepsilon)]$ remain unchanged with time, this process can be regarded as a process on $\mathbf{R}_+^{\{-n(\varepsilon),\cdots,n(\varepsilon)\}}$. Let $\tilde{P}_{\Phi_{n(\varepsilon)}}$ be the distribution of the process on $\mathbf{R}_+^{\{-n(\varepsilon),\cdots,n(\varepsilon)\}}$ with initial distribution $\Phi_{n(\varepsilon)}$. This is a probability measure on $\mathbf{D}([0, 1]\,;\mathbf{R}_+^{\{-n(\varepsilon),\cdots,n(\varepsilon)\}})$. For $\mathbf{x}_{n(\varepsilon)}(\cdot)\in\mathbf{D}([0, 1]\,;\mathbf{R}_+^{\{-n(\varepsilon),\cdots,n(\varepsilon)\}})$ put

$$Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\cdot))=\mathbf{1}\left(\varepsilon^2\int_0^1\sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1}\{x_i(t)\}^3\mathbf{1}(x_i(t)>(\delta/\varepsilon)^{2/3})dt>\frac{a}{2}\right).$$

Then the second term of the right-hand side of (3.4) is equal to $\int Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\cdot))dP_{\mu_\varepsilon^0}$. By making use of the entropy inequality for $P_{\mu_\varepsilon^0}\in\mathscr{P}(\mathbf{D}([0, 1]\,;\mathfrak{X}))$, we get for any $b>0$

$$\int Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\cdot))dP_{\mu_\varepsilon^0}\le\frac{\varepsilon}{b}\left\{\log\int e^{(b/\varepsilon)Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\cdot))}d\tilde{P}_{\Phi_{n(\varepsilon)}}+H_{n(\varepsilon)}(\mu_\varepsilon^0)\right\}. \tag{3.7}$$

Since $\{\mathbf{x}_{n(\varepsilon)}(t), 0\le t\le 1, \tilde{P}_{\Phi_{n(\varepsilon)}}\}$ is an equilibrium process, we have by Chebyshev's inequality

$$\begin{aligned}\tilde{P}_{\Phi_{n(\varepsilon)}}\{Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\cdot))=1\}&\le\frac{2}{a}\varepsilon^2\int\sum_{i=-[R/\varepsilon+1]}^{[R/\varepsilon]+1}x_i^3\mathbf{1}(x_i>(\delta/\varepsilon)^{2/3})d\Phi_{n(\varepsilon)}\\&\le\text{const.}a^{-1}\varepsilon\int_{(\delta/\varepsilon)^{2/3}}^{\infty}x^3p(x)dx.\end{aligned}$$

Moreover, by the Schwarz inequality, we have for any $\lambda>0$

$$\text{(3.8)} \qquad \int_{(\delta/\varepsilon)^{2/3}}^{\infty} x^3 p(x)dx \le \left(\int_{(\delta/\varepsilon)^{2/3}}^{\infty} x^6 e^{\lambda x^{3/2}} p(x)^2 dx\right)^{1/2} \left(\int_{(\delta/\varepsilon)^{2/3}}^{\infty} e^{-\lambda x^{3/2}} dx\right)^{1/2} .$$

The first integral in the right-hand side of (3.8) is finite by virtue of (A.7) and (A.8), and the second integral is dominated by const.$\lambda^{-1} e^{-(\lambda\delta)/\varepsilon}$. Therefore, by taking suitable λ, we have

$$\text{(3.9)} \qquad \int e^{(b/\varepsilon)Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\cdot))} d\tilde{P}_{\Phi_{n(\varepsilon)}} \le C$$

for some $C>0$ independent of $\varepsilon>0$. By (3.7), (3.9) and (A.9), we get

$$\limsup_{\varepsilon\downarrow 0} \int Y_\varepsilon(\mathbf{x}_{n(\varepsilon)}(\,\cdot\,))dP_{\mu_\varepsilon^0} \le \frac{C}{b} ,$$

which can be made arbitrarily small by choosing b large enough. Hence we obtain the desired result.

Let us examine the limiting behavior, as $\varepsilon\downarrow 0$, of $\int_0^t b_\varepsilon(\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(s))ds$, the second term of the right-hand side of (3.1). Since

$$b_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon} \sum_{i=-N(\varepsilon)^-}^{N(\varepsilon)^+-1} (x_i + x_{i+1})(J(\varepsilon(i+1)) - J(\varepsilon i))\left(x_i - \int u\gamma(du\,;\, x_i + x_{i+1})\right)$$

and $\int u\gamma(du\,;\,x_i+x_{i+1}) = (x_i+x_{i+1})/2$, we have

$$b_\varepsilon(\mathbf{x}) = \frac{1}{2\varepsilon} \sum_{i=-N(\varepsilon)^-}^{N(\varepsilon)^+-1} x_i^2 \{J(\varepsilon(i+1)) - 2J(\varepsilon i) + J(\varepsilon(i-1))\} .$$

By exploiting the smoothness of J and using the entropy inequality (3.6) and then (A.8) and (2.1) as in [4], we have

$$\text{(3.10)} \qquad \lim_{n\to\infty} \limsup_{l\to\infty} \limsup_{\varepsilon\downarrow 0} \mathbf{E}\left[\int_0^t \left| b_\varepsilon(\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(s)) - \frac{\varepsilon}{2} \sum_{|i|\le R/\varepsilon} J''(\varepsilon i) \frac{1}{2l+1} \sum_{j=i-l}^{i+l} \phi_n(\hat{X}_j^{\varepsilon,N(\varepsilon)}(s)) \right| ds\right] = 0 ,$$

where $\phi_n(x) = x^2 \wedge n$, $x \ge 0$, $n>0$.

Before examining the limiting behavior further, we introduce for $\rho\in[0,\infty)$ a probability measure ν_ρ on $\mathfrak{X}$ defined by

$$\nu_\rho(d\mathbf{x}) = \prod_{i\in\mathbf{Z}} p_{h'(\rho)}(x_i)dx_i \quad \text{if } \rho>0 , \qquad \nu_0 = \delta_{\mathbf{0}} \quad \text{if } \rho=0 .$$

This ν_ρ is a reversible measure for the Markov process on $\mathfrak{X}$ associated with $\mathscr{L}$ whose average value of energy per site is ρ, namely, $\int x_i \nu_\rho(d\mathbf{x}) = \rho$, $i\in\mathbf{Z}$.

Let $\mathbf{D}[0, 1]=\mathbf{D}([0, 1]; \mathfrak{M}(\mathbf{R}))$ be the set of functions on $[0, 1]$ with values in $\mathfrak{M}(\mathbf{R})$ which are right continuous and have left limits. We equip $\mathbf{D}[0, 1]$ with the Skorohod topology. The sample path $\alpha_t^{\varepsilon,N(\varepsilon)}$, $0\leq t\leq 1$, defines a random variable in $\mathbf{D}[0, 1]$. Denote by Q_ε the probability law on $\mathbf{D}[0, 1]$ induced by $\alpha_t^{\varepsilon,N(\varepsilon)}$, $0\leq t\leq 1$, from $\mathbf{P}$.

LEMMA 3.2. *The family $\{Q_\varepsilon, \varepsilon>0\}$ is relatively compact. Moreover if Q is a limit point of it, then sample paths α_t, $0\leq t\leq 1$, are continuous almost surely with respect to Q,*

$$Q\{\alpha. : \text{for a.a. } t,\ \alpha_t(dw)=\rho(t, w)dw \text{ for some } \rho(t, w)\}=1, \tag{3.11}$$

and there exists $C>0$ such that for any $a<b$

$$E^Q\left[\int_0^1 dt \int_a^b \rho^2(t, w)dw\right]\leq C(b-a+1), \tag{3.12}$$

where E^Q denotes the expectation with respect to Q.

PROOF. To prove that $\{Q_\varepsilon\}$ is relatively compact, we have only to show for any $R>0$

$$\lim_{K\to\infty} \limsup_{\varepsilon\downarrow 0} \mathbf{P}\left\{\sup_{0\leq t\leq 1} \varepsilon \sum_{|i|\leq R/\varepsilon} \hat{X}_i^{\varepsilon,N(\varepsilon)}(t)\geq K\right\}=0$$

and for any $a>0$ and $J\in \mathbf{C}_0^\infty(\mathbf{R})$

$$\lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} Q_\varepsilon\left\{\alpha. : \sup_{\substack{0\leq s,t\leq 1\\ |t-s|\leq\delta}} |\langle\alpha_t, J\rangle-\langle\alpha_s, J\rangle|>a\right\}=0.$$

These can be shown in the same way as in [4] by (2.1) and Lemma 3.1. We also obtain (3.11) and (3.12) in the same way as in [6] using (2.1).

We are going to prove that for every limit point Q the density function $\rho(t)=\rho(t, \cdot)$ is a weak solution of (1.6), namely, for all $J\in \mathbf{C}_0^\infty(\mathbf{R})$

$$Q\left\{\langle\rho(t), J\rangle-\langle\rho_0, J\rangle=\frac{1}{2}\int_0^t \langle P(\rho(s)), J''\rangle ds\right\}=1. \tag{3.13}$$

It is expected that in the limit the system would be locally in the equilibrium state. Namely if we look at our process $\hat{\mathbf{X}}^{\varepsilon,N(\varepsilon)}(\cdot)$ in a microscopic neighborhood of $w\in\mathbf{R}$ at time t, it would be almost in an equilibrium state ν_ρ where the parameter ρ would be identified with a macroscopic density $\rho(t, w)$. For our evolution the local equilibrium is built up in the sense expressed in the following two theorems, which will be proved in the next section.

Let us introduce a shift transformation Θ on $\mathfrak{X}$. For $\mathbf{x}=(x_i; i\in\mathbf{Z})\in\mathfrak{X}$, $\Theta\mathbf{x}\in\mathfrak{X}$ is defined by

$$(\Theta\mathbf{x})_i=x_{i+1}, \qquad i\in\mathbf{Z}.$$

A function ϕ of $\mathbf{x} \in \mathfrak{X}$ is called local if it is actually a function of $(x_i\,;\,|i|<K)$ for some constant $K<\infty$.

THEOREM 3.3. *For any $R>0$ and local function ϕ on $\mathfrak{X}$ which is bounded and continuous*

$$\lim_{l\to\infty}\limsup_{\varepsilon\downarrow 0}\int \varepsilon\sum_{|i|\le R/\varepsilon}\left|\frac{1}{2l+1}\sum_{j=i-l}^{i+l}\phi(\Theta^j\mathbf{x})-\hat{\phi}\left(\frac{1}{2l+1}\sum_{j=i-l}^{i+l}x_j\right)\right|d\bar{\mu}_\varepsilon=0\,,$$

where $\hat{\phi}(\rho)=\int\phi(\mathbf{x})\nu_\rho(d\mathbf{x})$.

THEOREM 3.4. *For any $R>0$*

$$\lim_{\delta\downarrow 0}\limsup_{l\to\infty}\limsup_{\varepsilon\downarrow 0}\int \varepsilon\sum_{|i|\le R/\varepsilon}\left|\frac{1}{2l+1}\sum_{j=i-l}^{i+l}x_j-\frac{1}{2[\delta/\varepsilon]+1}\sum_{j=i-[\delta/\varepsilon]}^{i+[\delta/\varepsilon]}x_j\right|d\bar{\mu}_\varepsilon=0\,.$$

Once we prove Theorem 3.3 and Theorem 3.4, we obtain for any $a>0$

$$\lim_{n\to\infty}\lim_{\delta\downarrow 0}\limsup_{\varepsilon\downarrow 0}Q_\varepsilon\{|F^{\varepsilon,t}_{n,\delta}(\alpha.(dw))|>a\}=0\,, \tag{3.14}$$

where

$$F^{\varepsilon,t}_{n,\delta}(\alpha.(dw))=\langle\alpha_t,J\rangle-\langle\alpha_0,J\rangle$$
$$-\frac{\varepsilon}{2}\int_0^t\sum_{|i|\le R/\varepsilon}J''(\varepsilon i)\hat{\phi}_n\left(\frac{1}{\varepsilon(2[\delta/\varepsilon]+1)}\alpha_s([\varepsilon(i-[\delta/\varepsilon]),\varepsilon(i+[\delta/\varepsilon])])\right)ds\,,$$
$$\alpha.(dw)\in\mathbf{D}[0,1]\,,$$

by (3.1), Lemma 3.1 and (3.10) in the same way as in [4]. The functionals $F^{\varepsilon,t}_{n,\delta}(\alpha.(dw))$ converge, as $\varepsilon\downarrow 0$, to $F^{t}_{n,\delta}(\alpha.(dw))$ defined by

$$F^{t}_{n,\delta}(\alpha.(dw))=\langle\alpha_t,J\rangle-\langle\alpha_0,J\rangle-\frac{1}{2}\int_0^t\int_{-\infty}^{\infty}J''(w)\hat{\phi}_n\left(\frac{1}{2\delta}\alpha_s([w-\delta,w+\delta])\right)dsdw\,,$$

and the convergence is uniform on compact subsets of $\mathbf{D}[0,1]$. Therefore for any $a>0$

$$\limsup_{\varepsilon\downarrow 0}Q_\varepsilon\{|F^{\varepsilon,t}_{n,\delta}(\alpha.(dw))|>a\}\ge Q\{|F^{t}_{n,\delta}(\alpha.(dw))|>a\}\,. \tag{3.15}$$

By (3.14), (3.15) and Fatou's lemma, we obtain (3.13). Since under the assumption (A.12) a non-negative weak solution of the non-linear diffusion equation (1.6) is unique ([1], [2]), the proof of Theorem 1.1 is complete.

4. Proof of Theorem 3.3 and Theorem 3.4.

In this section we prove Theorem 3.3 and Theorem 3.4 under the assumptions (A.7), (A.9), (A.11), (A.13) and the following (A.10′) in order to apply the proof for proving Theorem 1.3:

(A.10′) There exists $C>0$ such that $\int x_i^\beta d\mu_\varepsilon^0 \le C$ for all $i \notin [-n(\varepsilon), n(\varepsilon)]$ and $\varepsilon>0$, where $n(\varepsilon)=N(\varepsilon)^- \vee N(\varepsilon)^+$ and $\beta \ge 2$ is a constant in (A.13).

Let $\beta \ge 2$ be a constant appearing in (A.13) throughout this section. For $\varepsilon>0$, $k\in\mathbf{Z}$ and $i\in\mathbf{Z}$ let $\mu_{\varepsilon,k,i}^{(2)}$ be the probability measure on $\mathfrak{X}\times\mathfrak{X}$ induced by $(\Theta^i\mathbf{x}, \Theta^{i+k}\mathbf{x})$ from $\bar{\mu}_\varepsilon$. Let $R>0$ be fixed throughout this section. We define a probability measure $\mu_{\varepsilon,k}^{(2)}$ on $\mathfrak{X}\times\mathfrak{X}$ by

$$\mu_{\varepsilon,k}^{(2)}=\frac{1}{2[R/\varepsilon]+1}\sum_{|i|\le R/\varepsilon}\mu_{\varepsilon,k,i}^{(2)} .$$

LEMMA 4.1. *Suppose that the hypotheses* (A.9), (A.10′) *and* (A.13) *are satisfied. Then the family* $\{\mu_{\varepsilon,k}^{(2)}, \varepsilon>0, k=0, \pm1, \pm2, \cdots\}$ *is a relatively compact subset of* $\mathscr{P}(\mathfrak{X}\times\mathfrak{X})$, *the set of probability measures on* $\mathfrak{X}\times\mathfrak{X}$ *equipped with the topology of weak convergence.*

PROOF. Let us show that there exists $C>0$ such that

$$\int y_j^\beta \mu_{\varepsilon,k}^{(2)}(d\mathbf{y}d\mathbf{z})\le C\,, \qquad \int z_j^\beta \mu_{\varepsilon,k}^{(2)}(d\mathbf{y}d\mathbf{z})\le C \tag{4.1}$$

for all $j\in\mathbf{Z}$, $\varepsilon>0$, and $k\in\mathbf{Z}$. Note that

$$\int y_j^\beta \mu_{\varepsilon,k}^{(2)}(d\mathbf{y}d\mathbf{z})=\frac{1}{2[R/\varepsilon]+1}\int\sum_{i=j-[R/\varepsilon]}^{j+[R/\varepsilon]} x_i^\beta d\bar{\mu}_\varepsilon . \tag{4.2}$$

For the integers j's such that $|j|\le n(\varepsilon)-[R/\varepsilon]$, by making use of the entropy inequality (3.6), we see that the right-hand side of (4.2) is dominated by

$$\frac{1}{(2[R/\varepsilon]+1)\gamma}\left\{\log\int e^{\gamma\sum_{i=j-[R/\varepsilon]}^{j+[R/\varepsilon]}x_i^\beta}d\Phi_{n(\varepsilon)}+H_{n(\varepsilon)}(\bar{\mu}_\varepsilon)\right\},$$

where γ is a positive constant. Therefore by (A.13) and (2.1), we obtain the first estimate in (4.1) for these j's. Since for $t>0$ the projection of μ_ε^t to $\mathbf{R}_+^{\mathbf{Z}-\{-n(\varepsilon),\cdots,n(\varepsilon)\}}$ equals that of μ_ε^0, there exists $C>0$ such that

$$\int x_i^\beta d\bar{\mu}_\varepsilon=\int x_i^\beta d\mu_\varepsilon^0\le C \tag{4.3}$$

for all $i\notin[-n(\varepsilon), n(\varepsilon)]$ and $\varepsilon>0$ by virtue of (A.10′). Hence we also obtain the first estimate in (4.1) for the other j's by using (4.3) and the entropy inequality. The second estimate in (4.1) can be shown in the same way. Hence the required result follows.

LEMMA 4.2. *Suppose that the same hypotheses as in Lemma* 4.1 *and also* (A.7) *are satisfied. Then every limit point* $\mu^{(2)}$ *of* $\{\mu_{\varepsilon,k}^{(2)}\}$, *as* $\varepsilon\downarrow 0$ *and* $|k|\to\infty$ *under the restriction that* $|k|<1/\varepsilon$, *is a convex combination of* $\nu_{\rho_1}\otimes\nu_{\rho_2}$, *namely,*

$$\mu^{(2)}(d\mathbf{y}d\mathbf{z})=\int_{\mathbf{R}_+^2}\nu_{\rho_1}(d\mathbf{y})\nu_{\rho_2}(d\mathbf{z})\hat{\tau}(d\rho_1 d\rho_2) \tag{4.4}$$

for some probability measure $\hat{\tau}$ *on* $\mathbf{R}_+^2$. *Moreover for some* $C>0$

$$\int_{\mathbf{R}_+^2}(\rho_1^\beta+\rho_2^\beta)\hat{\tau}(d\rho_1 d\rho_2)\le C. \tag{4.5}$$

Theorem 3.3 follows from Lemma 4.2 by considering the first marginal in the expression (4.4) and then applying the law of large numbers for independent random variables.

PROOF OF LEMMA 4.2. We prove this lemma by following the argument in [4] (see also [7]). Put for any integer $l\ge 1$

$$\mathfrak{X}^{(l)}=\mathbf{R}_+^{\{-l,\cdots,l\}}=\{\mathbf{x}_l=(x_{-l},\cdots,x_l):x_i\in\mathbf{R}_+ \text{ for } -l\le i\le l\}.$$

For $-l\le i\le l-1$ define operators $L_i^{Y,(l)}$ and $L_i^{Z,(l)}$ on $\mathbf{C}_b(\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)})$ by

$$L_i^{Y,(l)}\phi(\mathbf{y}_l,\mathbf{z}_l)=(y_i+y_{i+1})\{\Gamma\phi^Y_{\mathbf{y}_l,\mathbf{z}_l,i}(y_i,y_{i+1})-\phi(\mathbf{y}_l,\mathbf{z}_l)\},$$

$$L_i^{Z,(l)}\phi(\mathbf{y}_l,\mathbf{z}_l)=(z_i+z_{i+1})\{\Gamma\phi^Z_{\mathbf{y}_l,\mathbf{z}_l,i}(z_i,z_{i+1})-\phi(\mathbf{y}_l,\mathbf{z}_l)\},$$

where $\phi^Y_{\mathbf{y}_l,\mathbf{z}_l,i}(u,v)=\phi(\mathbf{y}_l,\mathbf{z}_l)|_{y_i=u,y_{i+1}=v}$ and $\phi^Z_{\mathbf{y}_l,\mathbf{z}_l,i}(u,v)=\phi(\mathbf{y}_l,\mathbf{z}_l)|_{z_i=u,z_{i+1}=v}$. For $\mu\in\mathscr{P}(\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)})$ define the I-function associated with $L_i^{Y,(l)}$ by

$$I_i^{Y,(l)}(\mu)=\sup\left\{\int_{\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)}}\frac{L_i^{Y,(l)}\phi}{\phi}d\mu:\phi\in\mathbf{C}_b(\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)}),\ \inf_{\mathbf{y}_l,\mathbf{z}_l}\phi(\mathbf{y}_l,\mathbf{z}_l)>0\right\}.$$

The I-function associated with $L_i^{Z,(l)}$, denoted by $I_i^{Z,(l)}(\mu)$, $\mu\in\mathscr{P}(\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)})$, is defined in the same way.

Let $\mu^{(2)}$ be a limit point of $\{\mu^{(2)}_{\varepsilon,k}\}$ as $\varepsilon\downarrow 0$ and $|k|\to\infty$ under the restriction that $|k|<1/\varepsilon$, and $\mu_l^{(2)}$ be the $\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)}$-marginal of $\mu^{(2)}$. Then for each $l\ge 1$ $\mu_l^{(2)}$ is a limit point of $(\mu^{(2)}_{\varepsilon,k})_l$, the $\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)}$-marginal of $\mu^{(2)}_{\varepsilon,k}$. If $2l+1\le|k|<1/\varepsilon$, then we have by convexity of $I_i^{Y,(l)}(\cdot)$

$$\sum_{i=-l}^{l-1}I_i^{Y,(l)}((\mu^{(2)}_{\varepsilon,k})_l)\le\frac{1}{2[R/\varepsilon]+1}\sum_{i=-l}^{l-1}\sum_{|r|\le R/\varepsilon}I_{r+i}^{(n(\varepsilon))}(\bar{f}_\varepsilon)\le\frac{2l}{2[R/\varepsilon]+1}I_{N(\varepsilon)}(\bar{f}_\varepsilon).$$

Therefore we obtain $\sum_{i=-l}^{l-1}I_i^{Y,(l)}(\mu_l^{(2)})=0$ by (2.2) and Fatou's lemma. In the same way as above, we also obtain $\sum_{i=-l}^{l-1}I_i^{Z,(l)}(\mu_l^{(2)})=0$. These imply that $\mu_l^{(2)}$ is an invariant measure for the Markov process on $\mathfrak{X}^{(l)}\times\mathfrak{X}^{(l)}$ generated by

$$\mathscr{L}^{Y,Z,(l)}\phi(\mathbf{y}_l,\mathbf{z}_l)\equiv\left(\sum_{i=-l}^{l-1}L_i^{Y,(l)}\phi(\cdot,\mathbf{z}_l)\right)(\mathbf{y}_l)+\left(\sum_{i=-l}^{l-1}L_i^{Z,(l)}\phi(\mathbf{y}_l,\cdot)\right)(\mathbf{z}_l).$$

Therefore we see that $\mu_l^{(2)}$ is a convex combination of $\Phi^{(l)}_{\rho_1}\otimes\Phi^{(l)}_{\rho_2}$, where $\Phi^{(l)}_\rho$ is the conditional distribution of Φ_l conditioned on $(2l+1)^{-1}\sum_{i=-l}^{l} x_i=\rho$. Namely, we have

$$\mu_l^{(2)}(d\mathbf{y}_l d\mathbf{z}_l)=\int_{\mathbf{R}_+^2} \Phi_{\rho_1}^{(l)}(d\mathbf{y}_l)\Phi_{\rho_2}^{(l)}(d\mathbf{z}_l)\hat{\tau}_l(d\rho_1 d\rho_2)\,,$$

where $\hat{\tau}_l$ is the distribution of $(\rho_1, \rho_2)=((2l+1)^{-1}\sum_{|j|\le l} y_j,\ (2l+1)^{-1}\sum_{|j|\le l} z_j)$ on $\mathbf{R}_+^2$ under $\mu_l^{(2)}$. Moreover, by the proof of Lemma 4.1, we have for some $C>0$

$$\int y_j^\beta \mu^{(2)}(d\mathbf{y}d\mathbf{z})\le C\,,\quad \int z_j^\beta \mu^{(2)}(d\mathbf{y}d\mathbf{z})\le C\,,\qquad j\in\mathbf{Z}\,,$$

and then

$$\int_{\mathbf{R}_+^2} (\rho_1^\beta+\rho_2^\beta)\hat{\tau}_l(d\rho_1 d\rho_2)\le 2C\,,\qquad l\ge 1\,.$$

Hence we obtain the lemma by using Theorem I.3 in [6] in the same way as in [6].

We turn to proving Theorem 3.4.

PROOF OF THEOREM 3.4. We prove this theorem by following the proof of Theorem 7 of [7]. We have only to prove that there exists $C>0$ such that for any $\delta\in(0, 1)$

$$\limsup_{l\to\infty}\ \limsup_{\substack{\varepsilon\downarrow 0\\ K\to\infty}}\ \max_{K\le|k|\le\delta/\varepsilon}\int \varepsilon\sum_{|i|\le R/\varepsilon}\left|\frac{1}{2l+1}\sum_{j=i-l}^{i+l} x_j - \frac{1}{2l+1}\sum_{j=i+k-l}^{i+k+l} x_j\right|^{(2\beta)/(\beta+1)} d\bar{\mu}_\varepsilon\le C\delta^{(2\beta)/(\beta+1)}\,, \tag{4.6}$$

since an average over a long block of length $2[\delta/\varepsilon]+1$ is approximated by an average of averages over short blocks of length $2l+1$ (see [4]). Let $K\ge 2l+1$, and for any integer k satisfying $K\le|k|\le\delta/\varepsilon$ put

$$Y_l^i(k)=\frac{1}{2l+1}\sum_{j=i-l}^{i+l} x_j-\frac{1}{2l+1}\sum_{j=i+k-l}^{i+k+l} x_j\,,$$

$$h_l^i(k)=(\operatorname{sgn} Y_l^i(k))|\,Y_l^i(k)\,|^{(\beta-1)/(\beta+1)}\,.$$

It is easily seen that

$$|\,Y_l^i(k)\,|^{(2\beta)/(\beta+1)}=h_l^i(k)\frac{1}{2l+1}\sum_{|j|\le l}(x_{i+j}-x_{i+k+j})\,.$$

We only discuss the case where $k>0$. Considering for $-l\le j\le l$ the decomposition

$$\begin{aligned} x_{i+j}-x_{i+k+j}&=(x_{i+j}-x_{i+l+1})+(x_{i+l+1}-x_{i+l+2})\\ &\quad+\cdots+(x_{i+k-l-2}-x_{i+k-l-1})+(x_{i+k-l-1}-x_{i+k+j})\,, \end{aligned}$$

we have

$$\text{(4.7)}\qquad \int \varepsilon \sum_{|i|\le R/\varepsilon} |Y_l^i(k)|^{(2\beta)/(\beta+1)} d\bar{\mu}_\varepsilon$$
$$= \int \varepsilon \sum_{|i|\le R/\varepsilon} W_{k,l,i} d\bar{\mu}_\varepsilon + \int \varepsilon \sum_{|i|\le R/\varepsilon} h_l^i(k) \sum_{j=l+1}^{k-l-2} (x_{i+j} - x_{i+j+1}) d\bar{\mu}_\varepsilon ,$$

where

$$W_{k,l,i} = h_l^i(k) \frac{1}{2l+1} \sum_{|j|\le l} \{(x_{i+j} - x_{i+l+1}) + (x_{i+k-l-1} - x_{i+k+j})\} .$$

By the same argument as in [7], the second term of the right-hand side of (4.7) is dominated by

$$2^{3/2}\left[\int \varepsilon \sum_{|i|\le R/\varepsilon} \left\{\sum_{j=l+1}^{k-l-2} (x_{i+j} + x_{i+j+1})\right\}^\beta \bar{f}_\varepsilon d\Phi_{n(\varepsilon)}\right]^{1/(2\beta)}$$
$$\times \left[\int \varepsilon \sum_{|i|\le R/\varepsilon} |Y_l^i(k)|^{(2\beta)/(\beta+1)} \bar{f}_\varepsilon d\Phi_{n(\varepsilon)}\right]^{(\beta-1)/(2\beta)} \left[\varepsilon \sum_{|i|\le R/\varepsilon} \sum_{j=l+1}^{k-l-2} I_{i+j}^{(n(\varepsilon))}(\bar{f}_\varepsilon)\right]^{1/2}$$
$$\equiv 2^{3/2} A_1^{1/(2\beta)} A_2^{(\beta-1)/(2\beta)} A_3^{1/2} .$$

Now we estimate A_1 and A_3 in the above. By using Hölder's inequality, noticing $R/\varepsilon + k \le (R+1)/\varepsilon$ and then applying the entropy inequality, we have

$$\text{(4.8)}\qquad A_1 \le 2^\beta \varepsilon (k-2l-2)^\beta \int \sum_{|i|\le (R+1)/\varepsilon} x_i^\beta \bar{f}_\varepsilon d\Phi_{n(\varepsilon)} \le \text{const.}(\delta/\varepsilon)^\beta .$$

On the other hand, we have

$$\text{(4.9)}\qquad A_3 \le \varepsilon(k-2l-2) I_{N(\varepsilon)}(\bar{f}_\varepsilon) \le \text{const.}\varepsilon\delta .$$

Therefore by (4.7), (4.8) and (4.9), we obtain for some $C_0 > 0$

$$\text{(4.10)}\qquad \int \varepsilon \sum_{|i|\le R/\varepsilon} |Y_l^i(k)|^{(2\beta)/(\beta+1)} d\bar{\mu}_\varepsilon$$
$$\le \int \varepsilon \sum_{|i|\le R/\varepsilon} W_{k,l,i} d\bar{\mu}_\varepsilon + C_0 \delta \left\{ \int \varepsilon \sum_{|i|\le R/\varepsilon} |Y_l^i(k)|^{(2\beta)/(\beta+1)} d\bar{\mu}_\varepsilon \right\}^{(\beta-1)/(2\beta)} .$$

Next we show that the first term of the right-hand side of (4.10) converges to 0 uniformly in $k \in [K, \delta/\varepsilon]$ as we let $\varepsilon \downarrow 0$, $K \to \infty$ and $l \to \infty$ in this order. By Lemma 4.2, we have

$$\text{(4.11)}\qquad \limsup_{\substack{\varepsilon\downarrow 0\\ K\to\infty}} \max_{K\le|k|\le\delta/\varepsilon} \int \varepsilon \sum_{|i|\le R/\varepsilon} W_{k,l,i} d\bar{\mu}_\varepsilon \le \text{const.} \sup_{\hat{\tau}\in\mathscr{A}_\beta(C)} \int_{\mathbf{R}_+^2} f_l(\rho_1, \rho_2) \hat{\tau}(d\rho_1 d\rho_2) ,$$

where

$$f_l(\rho_1, \rho_2) = \int_{\mathfrak{X}\times\mathfrak{X}} h_l(\mathbf{y}, \mathbf{z}) \frac{1}{2l+1} \sum_{|j|\le l} \{(y_j - y_{l+1}) + (z_{-l-1} - z_j)\} \nu_{\rho_1}(d\mathbf{y}) \nu_{\rho_2}(d\mathbf{z}) ,$$

$$h_l(\mathbf{y}, \mathbf{z}) = (\operatorname{sgn} Y_l(\mathbf{y}, \mathbf{z})) | Y_l(\mathbf{y}, \mathbf{z}) |^{(\beta-1)/(\beta+1)} ,$$

$$Y_l(\mathbf{y}, \mathbf{z}) = \frac{1}{2l+1} \sum_{i=-l}^{l} y_i - \frac{1}{2l+1} \sum_{i=-l}^{l} z_i ,$$

and $\mathscr{A}_\beta(C)$ is the set of probability measures $\hat{\tau}$ on $\mathbf{R}_+^2$ satisfying (4.5). We see that $f_l(\rho_1, \rho_2)$ converges to 0 locally uniformly in (ρ_1, ρ_2) as $l \to \infty$, and for some $C_1 > 0$

$$(4.12) \qquad | f_l(\rho_1, \rho_2) |^{(\beta+1)/\beta} \le C_1 \left(\int y_0^2 \nu_{\rho_1}(d\mathbf{y}) + \int z_0^2 \nu_{\rho_2}(d\mathbf{z}) \right), \qquad l \ge 1 .$$

Since, by (A.11), the right-hand side of (4.12) is dominated by const.$(\rho_1^2 + \rho_2^2)$ for all sufficiently large ρ_1 and ρ_2, we have for some $C' > 0$

$$\sup_{\hat{\tau} \in \mathscr{A}_\beta(C)} \int_{\mathbf{R}_+^2} | f_l(\rho_1, \rho_2) |^{(\beta+1)/\beta} \hat{\tau}(d\rho_1 d\rho_2) \le C' , \qquad l \ge 1 .$$

Therefore the right-hand side and then the left-hand side of (4.11) converges to 0 as $l \to \infty$. Hence, by (4.10), we obtain (4.6).

5. Proof of Theorem 1.2.

In this section we prove Theorem 1.2. Let $M = \{-M^-, M^+\}$ be a pair of integers $-M^-$ and M^+ such that $M^- \ge 1$ and $M^+ \ge 1$. Recall that for each $\varepsilon > 0$ the processes $\mathbf{X}^{\varepsilon, M}(t)$, $t \ge 0$, are constructed on $(\Omega, \mathfrak{F}, \mathbf{P})$ from the same initial value for all $M^- \ge 1$ and $M^+ \ge 1$ by solving SDE's associated with the common Poisson random measures $N_i(\cdot)$, $i \in \mathbf{Z}$. Let us introduce the first interaction time between the i-th and the $(i+1)$-th energy of the process $\mathbf{X}^{\varepsilon, M}(t)$, $0 \le t \le T/\varepsilon^2$. For any integer $i \in [-M^-, M^+)$ put

$$\tau_i^{\varepsilon, M} = \min \left\{ 0 < t \le T/\varepsilon^2 : \int_0^t \int_0^{X_i^{\varepsilon, M}(s-) + X_{i+1}^{\varepsilon, M}(s-)} \int_0^1 N_i(ds d\eta d\xi) > \int_0^{t-} \int_0^{X_i^{\varepsilon, M}(s-) + X_{i+1}^{\varepsilon, M}(s-)} \int_0^1 N_i(ds d\eta d\xi) \right\} ,$$

where we adopt the convention that the minimum over the empty set is ∞.

LEMMA 5.1 (Lemma 3.3 (2) of [5]). *Suppose that the hypotheses* (A.13) *and* (A.16) *are satisfied. Then there exists $C > 0$ such that*

$$\mathbf{P}\{\tau_i^{\varepsilon, M} \le t, \tau_j^{\varepsilon, M} \le t\} \le CG(M^- \vee M^+)^{2/\beta} t^2 + CG(M^- \vee M^+)^{2/\beta - 1/\beta^2} t^{2 - 1/\beta}$$

for all $\varepsilon>0$, $M=\{-M^-, M^+\}$ *satisfying* $M^-\vee M^+\geq[F(1/\varepsilon)]+1$, $-M^-\leq i<j<M^+$ *and* $t>0$, *where* $\beta\geq 2$ *is a constant in* (A.13).

PROOF OF THEOREM 1.2. Let $M_k=\{-M_k^-, M_k^+\}$, $k\geq 1$, satisfy $1\leq M_1^-\leq M_2^-\leq \cdots$, $1\leq M_1^+\leq M_2^+\leq\cdots$ and the following conditions (5.1) and (5.2):

(5.1) There exists $C>1$ such that $1/C\leq M_k^+/M_k^-\leq C$ and $M_{k+1}^+\leq CM_k^+$ for all sufficiently large k.

$$\sum_{k=1}^{\infty} G(M_k^+)^{4-1/\beta^2}(M_k^+)^{-1+1/\beta}<\infty \qquad \text{for} \quad \beta\geq 2 \text{ in (A.13)}. \tag{5.2}$$

(Note that $\lim_{x\to\infty} G(x)^{4-1/\beta^2}x^{-1+1/\beta}=0$ by virtue of (1.7).) Let $R>0$ be fixed. Since $\{M_k, k\geq 1\}$ satisfies (A.6) for $g=G$, we have for each $\varepsilon>0$, by Theorem A,

$$\begin{aligned}\mathbf{P}\{X_i^\varepsilon(t)=X_i^{\varepsilon,M_k}(t) \text{ for all } i\in[-[R/\varepsilon],[R/\varepsilon]],\\ t\in[0,T/\varepsilon^2] \text{ and all sufficiently large } k\}=1.\end{aligned} \tag{5.3}$$

For $\varepsilon>0$ put

$$k(\varepsilon)=\min\{k\geq 1 : M_k^-, M_k^+\geq K(\varepsilon)^+\}.$$

Then we have

(5.4)

$$\begin{aligned}&\mathbf{P}\{X_i^\varepsilon(t)\neq X_i^{\varepsilon,K(\varepsilon)}(t) \text{ for some } i\in[-[R/\varepsilon],[R/\varepsilon]] \text{ and some } t\in[0,T/\varepsilon^2]\}\\ &\leq \mathbf{P}\{X_i^\varepsilon(t)\neq X_i^{\varepsilon,M_{k(\varepsilon)}}(t) \text{ for some } i\in[-[R/\varepsilon],[R/\varepsilon]] \text{ and some } t\in[0,T/\varepsilon^2]\}\\ &\quad+\mathbf{P}\{X_i^{\varepsilon,M_{k(\varepsilon)}}(t)\neq X_i^{\varepsilon,K(\varepsilon)}(t) \text{ for some } i\in[-[R/\varepsilon],[R/\varepsilon]] \text{ and some } t\in[0,T/\varepsilon^2]\}.\end{aligned}$$

By (5.3), the first term of the right-hand side of (5.4) is dominated by

$$\begin{aligned}\sum_{l=k(\varepsilon)}^{\infty}\mathbf{P}\{X_i^{\varepsilon,M_l}(t)\neq X_i^{\varepsilon,M_{l+1}}(t) \text{ for some } i\in[-[R/\varepsilon],[R/\varepsilon]]\\ \text{and some } t\in[0,T/\varepsilon^2]\}.\end{aligned} \tag{5.5}$$

To estimate this, we follow the argument of the proof of Theorem A (Theorem 1.2 of [5]). Divide the time interval $[0, T/\varepsilon^2]$ into short pieces of length $\delta_l\equiv(T/\varepsilon^2)/[(M_l^-\wedge M_l^+-[R/\varepsilon])/3]$, where $l\geq k(\varepsilon)$. We apply Lemma 5.1 both for $M=M_l$, $t=\delta_l$ and for $M=M_{l+1}$, $t=\delta_l$. Note that $\varepsilon^{-1}\leq G(K(\varepsilon)^+)\leq G(M_{k(\varepsilon)}^-\vee M_{k(\varepsilon)}^+)$. Since G satisfies (A.2), we see by (5.1) and (1.7) that there exists $C>0$ such that

$$\begin{aligned}\mathbf{P}\{\tau_i^{\varepsilon,M_l}\leq\delta_l, \tau_j^{\varepsilon,M_l}\leq\delta_l\}\leq CG(M_l^+)^{4-1/\beta^2}(M_l^+)^{-2+1/\beta}, &\qquad -M_l^-\leq i<j<M_l^+,\\ \mathbf{P}\{\tau_i^{\varepsilon,M_{l+1}}\leq\delta_l, \tau_j^{\varepsilon,M_{l+1}}\leq\delta_l\}\leq CG(M_l^+)^{4-1/\beta^2}(M_l^+)^{-2+1/\beta}, &\qquad -M_{l+1}^-\leq i<j<M_{l+1}^+,\end{aligned}$$

for all $\varepsilon>0$ and $l\geq k(\varepsilon)$. Therefore, by the same argument as in [5], we get for $l\geq k(\varepsilon)$

$$\begin{aligned}\mathbf{P}\{X_i^{\varepsilon,M_l}(t)\neq X_i^{\varepsilon,M_{l+1}}(t) \text{ for some } i\in[-[R/\varepsilon],[R/\varepsilon]] \text{ and some } t\in[0,T/\varepsilon^2]\}\\ \leq CG(M_l^+)^{4-1/\beta^2}(M_l^+)^{-1+1/\beta},\end{aligned} \tag{5.6}$$

where C is a positive constant independent of ε and l. By (5.6) and (5.2), we see that (5.5) converges to 0 as $\varepsilon\downarrow 0$.

To estimate the second term of the right-hand side of (5.4), we divide the time interval $[0, T/\varepsilon^2]$ into short pieces of length $\delta'_\varepsilon \equiv (T/\varepsilon^2)/[(K(\varepsilon)^+ - [R/\varepsilon])/3]$. By applying Lemma 5.1 both for $M=K(\varepsilon)$, $t=\delta'_\varepsilon$ and for $M=M_{k(\varepsilon)}$, $t=\delta'_\varepsilon$, we obtain, in the same way as above,

$$(5.7)\quad \mathbf{P}\{X_i^{\varepsilon,M_{k(\varepsilon)}}(t) \neq X_i^{\varepsilon,K(\varepsilon)}(t) \text{ for some } i\in[-[R/\varepsilon],[R/\varepsilon]] \text{ and some } t\in[0,T/\varepsilon^2]\}$$
$$\le CG(M_{k(\varepsilon)}^+)^{4-1/\beta^2}(M_{k(\varepsilon)}^+)^{-1+1/\beta}\,.$$

The right-hand side of (5.7) converges to 0 as $\varepsilon\downarrow 0$ by (1.7). The proof of the theorem is finished.

6. Uniqueness.

Under the condition (A.12), a non-negative weak solution of the non-linear diffusion equation (1.6) is unique ([1], [2]). In this section we discuss some sufficient conditions for $p(x)$, $x>0$, which guarantee the condition (A.12). Throughout this section we assume that $p(x)$, $x>0$, is positive and continuous and satisfies $\int_0^\infty e^{\lambda x}p(x)dx<\infty$ for all $\lambda>0$. We notice that for $\rho>0$

$$(6.1)\qquad \rho=\frac{M'(\lambda)}{M(\lambda)}\,,\quad P(\rho)=\frac{M''(\lambda)}{M(\lambda)}\,,\qquad \lambda=h'(\rho)\,,$$

and $\lambda=h'(\rho)$ ranges from $-\infty$ to ∞ as ρ varies from 0 to ∞. Put for non-negative integer n

$$M_n(\lambda)=\int_0^\infty x^n e^{\lambda x}p(x)dx\,,\qquad \lambda\in\boldsymbol{R}\,.$$

Using the expressions in (6.1) and noticing that for $n\ge 0$ $M^{(n)}(\lambda)$, the n-th derivative of $M(\lambda)$, is equal to $M_n(\lambda)$, we have

$$(6.2)\qquad \frac{\rho P'(\rho)}{P(\rho)}=\frac{M_0(\lambda)M_1(\lambda)M_3(\lambda)-M_1(\lambda)^2M_2(\lambda)}{M_0(\lambda)M_2(\lambda)^2-M_1(\lambda)^2M_2(\lambda)}\,,\qquad \lambda=h'(\rho)\,,\quad \rho>0\,.$$

By (6.2) and the Schwarz inequality we obtain

$$(6.3)\qquad \frac{\rho P'(\rho)}{P(\rho)}>1\,,\qquad \rho>0\,.$$

We prove that under some conditions for $p(x)$, $x>0$, the ratio $\{\rho P'(\rho)\}/P(\rho)$ converges to 2 as $\rho\downarrow 0$ (Proposition 6.1) and as $\rho\to\infty$ (Proposition 6.2 or Proposition 6.3). Combining (6.3) and these propositions, we see that under both of the conditions in Proposition 6.1 and Proposition 6.2 (or in Proposition 6.1 and Proposition 6.3) (A.12) is satisfied and hence a non-negative weak solution of (1.6) is unique.

We put

$$\psi(x) = -\log p(x)\,, \qquad x>0\,;$$

namely, $p(x)=e^{-\psi(x)}$. Suppose $|\psi(0+)|<\infty$ throughout this section. To examine the limiting behaviors of $\{\rho P'(\rho)\}/P(\rho)$ both as $\rho\downarrow 0$ and as $\rho\to\infty$, we may assume $\psi(0+)=0$ because of the expression in (6.2).

PROPOSITION 6.1. *Suppose that there exists $C>-\infty$ such that*

(H.1) $$\psi(x)\geq C\,, \qquad x>0\,.$$

Then

$$\lim_{\rho\downarrow 0}\frac{\rho P'(\rho)}{P(\rho)}=2\,.$$

PROOF. We have only to examine the limiting behavior of the right-hand side of (6.2) as $\lambda\to-\infty$. For $\lambda<0$ we have

(6.4) $$M_n(\lambda)=\frac{1}{(-\lambda)^{n+1}}\int_0^\infty x^n e^{-x-\psi(-x/\lambda)}dx\,.$$

To examine the limiting behavior, as $\lambda\to-\infty$, of the integral in the right-hand side of (6.4), we can apply the dominated convergence theorem by virtue of (H.1). Therefore we have

(6.5) $$\lim_{\lambda\to-\infty}\int_0^\infty x^n e^{-x-\psi(-x/\lambda)}dx=n!\,.$$

By (6.2), (6.4) and (6.5), we obtain the required result.

PROPOSITION 6.2. *Suppose that ψ is five times continuously differentiable and that*

(H.2) $$\liminf_{x\to\infty}\psi^{(2)}(x)x^{-(A-2)}>0\,,$$

(H.3) $$\limsup_{x\to\infty}|\psi^{(k)}(x)|x^{-(B-k)}<\infty\,, \qquad 2\leq k\leq 5\,,$$

hold ($\psi^{(k)}$ denotes the k-th derivative of ψ) for some constants A and B satisfying one of the three conditions (i) $2\vee(2B/3)<A\leq B$, (ii) $4B/5<A\leq B$; $1<A\leq 2$ *and* (iii) $A=1$, $1\leq B<10/9$. *Then*

$$\lim_{\rho\to\infty}\frac{\rho P'(\rho)}{P(\rho)}=2\,.$$

REMARK 6.1. Examples of ψ satisfying the conditions in Proposition 6.2 are given below. Let $\psi(x)=cx^\gamma$, $x>0$, where $c>0$ and $\gamma>1$. Then this ψ satisfies the conditions in Proposition 6.2 for $A=B=\gamma>1$. Next let $\psi(x)=cx(\log x)^{\gamma}$, $x>0$, where $c>0$ and $\gamma\geq 1$. Then this ψ satisfies the conditions for $A=1$ and $B>1$ if $\gamma>1$, and for $A=B=1$ if $\gamma=1$. In general, let $m\geq 1$ be fixed and let

$$\psi(x)=cx^{\gamma_0}(\log x)^{\gamma_1}(\log\log x)^{\gamma_2}\cdots(\log\log\cdots\log x)^{\gamma_m}, \qquad x>0, \tag{6.6}$$

where $c>0$. If $\gamma_0>1$ and $\gamma_k\in\mathbf{R}$, $1\leq k\leq m$, then ψ in (6.6) satisfies the conditions for suitable constants $A>1$ and $B>1$. If $\gamma_0=1$, $\gamma_1>1$ and $\gamma_k\in\mathbf{R}$, $2\leq k\leq m$, then ψ in (6.6) satisfies the conditions for $A=1$ and $B>1$.

Let $l(x)$, $x>0$, be a strictly increasing and differentiable function satisfying

$$\lim_{x\to\infty} l(x)(\log x)^{-1}=\infty, \tag{H.4}$$

$$\limsup_{x\to\infty} l'(x)l(x)^{-1}<\infty; \tag{H.5}$$

for example

$$l(x)=x^{\gamma}, \qquad \gamma>0, \tag{6.7}$$

and

$$l(x)=e^{ax^{\gamma}}, \qquad a>0, \quad 0<\gamma\leq 1. \tag{6.8}$$

PROPOSITION 6.3. *Suppose that ψ is five times continuously differentiable and that*

$$\liminf_{x\to\infty} \psi^{(2)}(x)e^{-Al(x)}>0, \tag{H.6}$$

$$\limsup_{x\to\infty} |\psi^{(5)}(x)|e^{-Bl(x)}<\infty, \tag{H.7}$$

hold for some constants $A>0$ and $B>0$ satisfying $2/3<A/B\leq 1$ and for some strictly increasing and differentiable function $l(x)$, $x>0$, satisfying (H.4) *and* (H.5). *Then*

$$\lim_{\rho\to\infty}\frac{\rho P'(\rho)}{P(\rho)}=2.$$

REMARK 6.2. Examples of ψ satisfying the conditions in Proposition 6.3 are given below. Let

$$\psi(x)=ce^{bl(x)}, \qquad x>0, \tag{6.9}$$

where $b, c>0$ and $l(x)$, $x>0$, is a function defined by (6.7) or (6.8). Then ψ in (6.9) satisfies the conditions in Proposition 6.3 for $l(x)$, $x>0$, appearing in the right-hand side of (6.9).

To prove Propositions 6.2 and 6.3, we are going to give the asymptotic expansions of $M_n(\lambda)$, $0\leq n\leq 3$, as $\lambda\to\infty$ by making use of the Laplace method. From the expansions we will obtain the limiting behavior of the right-hand side of (6.2) as $\lambda \to \infty$. If we put for $\lambda > 0$

$$\psi_\lambda(x) = \lambda x - \psi(x), \qquad x > 0,$$

then we have

$$M_n(\lambda) = \int_0^\infty x^n e^{\psi_\lambda(x)} dx. \tag{6.10}$$

Since $\psi^{(2)}(x) > 0$ for all sufficiently large x under each of the conditions in Proposition 6.2 and Proposition 6.3, $\psi'(x)$ is strictly increasing in x. Therefore for large $\lambda > 0$ there exists a unique $x_\lambda > 0$ such that $\psi'(x_\lambda) = \lambda$. We see that x_λ is the unique maximizing point of ψ_λ and $\psi_\lambda^{(2)}(x_\lambda) < 0$. By the assumption $\psi(0+) = 0$, we have $\psi_\lambda(x_\lambda) > 0$. We note that $x_\lambda \to \infty$ as $\lambda \to \infty$. By the Laplace method we will see that $M_n(\lambda)$ is expanded asymptotically as follows if we only consider the main term of the expansion:

$$M_n(\lambda) \sim \sqrt{\pi}\, C_\lambda x_\lambda^n e^{\psi_\lambda(x_\lambda)} \qquad \text{as} \quad \lambda \to \infty. \tag{6.11}$$

Here $C_\lambda = \sqrt{2/\psi^{(2)}(x_\lambda)}$. If we examine the asymptotics of $M_0(\lambda)M_1(\lambda)M_3(\lambda)$, $M_1(\lambda)^2 M_2(\lambda)$ and $M_0(\lambda)M_2(\lambda)^2$ appearing in the right-hand side of (6.2) by using (6.11), then we have the same asymptotics for all of them. So we cannot get the asymptotic behavior of the right-hand side of (6.2) as $\lambda \to \infty$. Therefore we need to obtain more precise estimates on the asymptotics of $M_n(\lambda)$, $0 \le n \le 3$, as $\lambda \to \infty$.

For $0 \le n \le 3$ put

$$L_n(\lambda) = \int_0^\infty x^n e^{\psi_\lambda(x) - \psi_\lambda(x_\lambda)} dx, \qquad \lambda > 0.$$

Then, by (6.10), we have

$$M_n(\lambda) = e^{\psi_\lambda(x_\lambda)} L_n(\lambda). \tag{6.12}$$

To examine the asymptotic behavior of $L_n(\lambda)$ as $\lambda \to \infty$, we apply Taylor's theorem to $\psi_\lambda(x)$. Since $\psi_\lambda'(x_\lambda) = 0$, we have

$$\psi_\lambda(x) - \psi_\lambda(x_\lambda) = -C_\lambda^{-2}(x - x_\lambda)^2 - a_\lambda(3)(x - x_\lambda)^3 - a_\lambda(4)(x - x_\lambda)^4 - W_\lambda(x), \tag{6.13}$$

where

$$a_\lambda(k) = \psi^{(k)}(x_\lambda)/k!, \quad k = 3, 4, \qquad W_\lambda(x) = \int_{x_\lambda}^x \frac{(x-y)^4}{4!} \psi^{(5)}(y) dy.$$

We remark that the condition (H.2) in Proposition 6.2 implies

$$\begin{cases} \liminf_{x\to\infty} \psi'(x) x^{-(A-1)} > 0, & \text{if} \quad A > 1, \\ \liminf_{x\to\infty} \psi'(x)(\log x)^{-1} > 0, & \text{if} \quad A = 1, \end{cases} \tag{6.14}$$

$$(6.15)\qquad \begin{cases} \liminf_{x\to\infty} \psi(x)x^{-A}>0\,, & \text{if } A>1\,,\\ \liminf_{x\to\infty} \psi(x)(x\log x)^{-1}>0\,, & \text{if } A=1\,,\end{cases}$$

and the condition (H.3) implies

$$(6.16)\qquad \begin{cases} \limsup_{x\to\infty} |\psi'(x)|x^{-(B-1)}<\infty\,, & \text{if } B>1\,,\\ \limsup_{x\to\infty} |\psi'(x)|(\log x)^{-1}<\infty\,, & \text{if } B=1\,.\end{cases}$$

By (6.14) we have

$$(6.17)\qquad \begin{cases} \limsup_{\lambda\to\infty} x_\lambda \lambda^{-1/(A-1)}<\infty\,, & \text{if } A>1\,,\\ \limsup_{\lambda\to\infty} (\log x_\lambda)\lambda^{-1}<\infty\,, & \text{if } A=1\,.\end{cases}$$

On the other hand, under the conditions in Proposition 6.3 there exists $C>0$ such that $\psi^{(2)}(x)\geq Ce^{Al(x)}l'(x)l(x)^{-1}\geq Ce^{(A/2)l(x)}l'(x)$ for sufficiently large x by virtue of (H.6) and (H.5). Thus we have, in this case,

$$(6.18)\qquad \liminf_{x\to\infty} \psi'(x)e^{-(A/2)l(x)}>0\,,$$

and in the same way

$$(6.19)\qquad \liminf_{x\to\infty} \psi(x)e^{-(A/3)l(x)}>0\,.$$

Moreover the condition (H.7) implies

$$(6.20)\qquad \limsup_{x\to\infty} |\psi^{(k)}(x)|x^{-(5-k)}e^{-Bl(x)}<\infty\,, \qquad 1\leq k\leq 4\,.$$

Under the conditions in Proposition 6.3 since

$$(6.21)\qquad \lim_{x\to\infty} \psi(x)x^{-2}=\infty\,,$$

we have

$$(6.22)\qquad \lim_{\lambda\to\infty} x_\lambda\lambda^{-1}=0\,.$$

Furthermore we note that under each of the conditions in Proposition 6.2 and Proposition 6.3 the following (6.23) and (6.24) hold:

$$(6.23)\qquad a_\lambda(3)C_\lambda^3\to 0\,,\quad a_\lambda(4)C_\lambda^4\to 0 \qquad \text{as} \quad \lambda\to\infty\,,$$

$$(6.24)\qquad C_\lambda x_\lambda^{-1}\to 0 \qquad \text{as} \quad \lambda\to\infty\,.$$

The following lemma concerns with the asymptotic expansions of $L_n(\lambda)$, $0\leq n\leq 3$, as $\lambda\to\infty$.

LEMMA 6.4. *Under each of the conditions in Proposition* 6.2 *and Proposition* 6.3, *the following assertions* (1)–(4) *hold.*

(1) $L_0(\lambda)=\sqrt{\pi}\,C_\lambda+g_0(\lambda)+o(C_\lambda^3x_\lambda^{-2})$ *as* $\lambda\to\infty$.

(2) $L_1(\lambda)=\sqrt{\pi}\,C_\lambda x_\lambda+g_0(\lambda)x_\lambda+g_1(\lambda)+o(C_\lambda^3x_\lambda^{-1})$ *as* $\lambda\to\infty$.

(3) $L_2(\lambda)=\sqrt{\pi}\,C_\lambda x_\lambda^2+(\sqrt{\pi}/2)C_\lambda^3+g_0(\lambda)x_\lambda^2+2g_1(\lambda)x_\lambda+g_2(\lambda)+o(C_\lambda^3)$ *as* $\lambda\to\infty$.

(4) $L_3(\lambda)=\sqrt{\pi}\,C_\lambda x_\lambda^3+(3\sqrt{\pi}/2)C_\lambda^3x_\lambda+g_0(\lambda)x_\lambda^3+3g_1(\lambda)x_\lambda^2+3g_2(\lambda)x_\lambda+g_3(\lambda)$ $+o(C_\lambda^3x_\lambda)$ *as* $\lambda\to\infty$.

Here

$$g_n(\lambda)=\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m=0}^{k}\binom{k}{m}a_\lambda(3)^m a_\lambda(4)^{k-m}C_\lambda^{4k+1+n-m}\int_{-\infty}^{\infty}y^{4k+n-m}e^{-y^2}dy$$

for $0\le n\le 3$, *and* K *is a constant determined according to each of the conditions.*

PROOF. When we prove this lemma under the conditions in Proposition 6.2, we define $r_\lambda>0$ for sufficiently large λ by

$$r_\lambda=\begin{cases}\psi^{(2)}(x_\lambda)^{-\gamma_1}, & \text{if } A>2,\\ \log x_\lambda, & \text{if } A=2,\\ x_\lambda^{\gamma_2}, & \text{if } 1<A<2,\\ x_\lambda^{\gamma_3}, & \text{if } A=1,\end{cases}\tag{6.25}$$

where γ_1, γ_2 and γ_3 are positive constants such that $1/5+\{(B-3)\vee 0\}/\{5(A-2)\}<\gamma_1<1/2$, $1-A/2<\gamma_2<1-2B/5$ and $B/2<\gamma_3<1-2B/5$. On the other hand, when we want to prove this lemma under the conditions in Proposition 6.3, we must define $r_\lambda>0$ for sufficiently large λ by

$$r_\lambda=\psi^{(2)}(x_\lambda)^{-\gamma_4},\tag{6.26}$$

where γ_4 is a positive constant satisfying $1/5+B/(5A)<\gamma_4<1/2$. Then the following (6.27)–(6.31) hold as $\lambda\to\infty$:

(6.27) $x_\lambda^{-1}r_\lambda\to 0$,

(6.28) $C_\lambda^{-1}r_\lambda\to\infty$,

(6.29) $a_\lambda(3)r_\lambda^3\to 0$, $\quad a_\lambda(4)r_\lambda^4\to 0$,

(6.30) $C_\lambda^{-2}x_\lambda^{B-3}r_\lambda^5\to 0$ under the conditions in Proposition 6.2,

(6.31) $C_\lambda^{-2}x_\lambda^2e^{(1+a)Bl(x_\lambda)}r_\lambda^5\to 0$ for sufficiently small $a>0$ under the conditions in Proposition 6.3.

Under the conditions in Proposition 6.2 there exists $\kappa>0$ such that

$$\psi(x)\geq\begin{cases}\kappa x^{A}, & \text{if } A>1,\\ \kappa x\log x, & \text{if } A=1,\end{cases} \tag{6.32}$$

for all sufficiently large x by (6.15). When we prove this lemma under the conditions in Proposition 6.2, we define $m_\lambda>0$ for $\lambda>0$ by

$$m_\lambda=\begin{cases}\{(1+b)/\kappa\}^{1/(A-1)}\lambda^{1/(A-1)}, & \text{if } A>1,\\ e^{\{(1+b)/\kappa\}\lambda}, & \text{if } A=1,\end{cases} \tag{6.33}$$

where κ is a positive constant appearing in (6.32) and b is a positive constant such that

$$x_\lambda<m_\lambda \quad \text{for all sufficiently large } \lambda. \tag{6.34}$$

(See (6.17).) By (6.32) and (6.33) we have, for sufficiently large λ,

$$\psi(x)\geq(1+b)\lambda x \quad \text{if } x>m_\lambda. \tag{6.35}$$

On the other hand, when we want to prove this lemma under the conditions in Proposition 6.3, we define $m_\lambda>0$ for $\lambda>0$ by

$$m_\lambda=\lambda. \tag{6.36}$$

In this case, by (6.21) and (6.22), we have (6.34) and (6.35) for any $b>0$.

By (6.27) and (6.34), for sufficiently large λ we can divide $L_n(\lambda)$ into the following four parts:

$$L_n(\lambda)=R_n^{(1)}(\lambda)+\tilde{L}_n(\lambda)+R_n^{(2)}(\lambda)+R_n^{(3)}(\lambda), \tag{6.37}$$

$$R_n^{(1)}(\lambda)=\int_0^{x_\lambda-r_\lambda}x^n e^{\psi_\lambda(x)-\psi_\lambda(x_\lambda)}dx, \qquad \tilde{L}_n(\lambda)=\int_{x_\lambda-r_\lambda}^{x_\lambda+r_\lambda}x^n e^{\psi_\lambda(x)-\psi_\lambda(x_\lambda)}dx,$$

$$R_n^{(2)}(\lambda)=\int_{x_\lambda+r_\lambda}^{m_\lambda}x^n e^{\psi_\lambda(x)-\psi_\lambda(x_\lambda)}dx, \qquad R_n^{(3)}(\lambda)=\int_{m_\lambda}^{\infty}x^n e^{\psi_\lambda(x)-\psi_\lambda(x_\lambda)}dx.$$

First we examine the limiting behavior of $\tilde{L}_n(\lambda)$ as $\lambda\to\infty$. By (6.13), we have

$$\tilde{L}_n(\lambda)=\tilde{L}_n^{(1)}(\lambda)+\tilde{L}_n^{(2)}(\lambda), \tag{6.38}$$

$$\tilde{L}_n^{(1)}(\lambda)=\int_{x_\lambda-r_\lambda}^{x_\lambda+r_\lambda}x^n e^{-C_\lambda^{-2}(x-x_\lambda)^2-a_\lambda(3)(x-x_\lambda)^3-a_\lambda(4)(x-x_\lambda)^4}dx,$$

$$\tilde{L}_n^{(2)}(\lambda)=\int_{x_\lambda-r_\lambda}^{x_\lambda+r_\lambda}x^n e^{-C_\lambda^{-2}(x-x_\lambda)^2-a_\lambda(3)(x-x_\lambda)^3-a_\lambda(4)(x-x_\lambda)^4}(e^{-W_\lambda(x)}-1)dx.$$

Let us estimate $\tilde{L}_n^{(1)}(\lambda)$. By changing the variable of integration according to $y=C_\lambda^{-1}(x-x_\lambda)$ and putting $y_\lambda=C_\lambda^{-1}r_\lambda$, we have

$$(6.39)\qquad \tilde{L}_n^{(1)}(\lambda)=C_\lambda\int_{-y_\lambda}^{y_\lambda}(x_\lambda+C_\lambda y)^n e^{-y^2}dy$$
$$+C_\lambda\int_{-y_\lambda}^{y_\lambda}(x_\lambda+C_\lambda y)^n e^{-y^2}(e^{-a_\lambda(3)C_\lambda^3y^3-a_\lambda(4)C_\lambda^4y^4}-1)dy\,.$$

By (6.28), we note that $y_\lambda\to\infty$ as $\lambda\to\infty$. Let us estimate the second term of the right-hand side of (6.39). If $|y|\le y_\lambda$, then we observe

$$(6.40)\qquad |a_\lambda(3)C_\lambda^3y^3|+|a_\lambda(4)C_\lambda^4y^4|\le|a_\lambda(3)|r_\lambda^3+|a_\lambda(4)|r_\lambda^4\,.$$

Therefore, by (6.29), we get for any $K\ge 1$

$$(6.41)\qquad e^{-a_\lambda(3)C_\lambda^3y^3-a_\lambda(4)C_\lambda^4y^4}-1=\sum_{k=1}^{K}\frac{1}{k!}(-a_\lambda(3)C_\lambda^3y^3$$
$$-a_\lambda(4)C_\lambda^4y^4)^k+O((|a_\lambda(3)|r_\lambda^3+|a_\lambda(4)|r_\lambda^4)^{K+1})\qquad\text{as}\quad\lambda\to\infty\,.$$

In addition, we note that $|C_\lambda y|\le r_\lambda$ for $|y|\le y_\lambda$. Therefore, by (6.27) and (6.41), we see that the second term of the right-hand side of (6.39) equals

$$(6.42)\qquad C_\lambda\int_{-y_\lambda}^{y_\lambda}(x_\lambda+C_\lambda y)^n e^{-y^2}\sum_{k=1}^{K}\frac{1}{k!}(-a_\lambda(3)C_\lambda^3y^3-a_\lambda(4)C_\lambda^4y^4)^k dy$$
$$+O(C_\lambda x_\lambda^n(|a_\lambda(3)|r_\lambda^3+|a_\lambda(4)|r_\lambda^4)^{K+1})\qquad\text{as}\quad\lambda\to\infty\,.$$

By (6.39) and (6.42), we obtain for any $K\ge 1$

$$(6.43)\quad \tilde{L}_n^{(1)}(\lambda)=C_\lambda\int(x_\lambda+C_\lambda y)^n e^{-y^2}dy-C_\lambda\int_{|y|>y_\lambda}(x_\lambda+C_\lambda y)^n e^{-y^2}dy$$
$$+\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m=0}^{k}\binom{k}{m}a_\lambda(3)^m a_\lambda(4)^{k-m}C_\lambda^{4k+1-m}\int_{-\infty}^{\infty}(x_\lambda+C_\lambda y)^n y^{4k-m}e^{-y^2}dy$$
$$-\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m=0}^{k}\binom{k}{m}a_\lambda(3)^m a_\lambda(4)^{k-m}C_\lambda^{4k+1-m}\int_{|y|>y_\lambda}(x_\lambda+C_\lambda y)^n y^{4k-m}e^{-y^2}dy$$
$$+O(C_\lambda x_\lambda^n(|a_\lambda(3)|r_\lambda^3+|a_\lambda(4)|r_\lambda^4)^{K+1})\qquad\text{as}\quad\lambda\to\infty\,.$$

Note that

$$\int_{|y|>y_\lambda}y^k e^{-y^2}dy=\begin{cases}0\,, & \text{if}\quad k\ge 0\text{ is odd}\,,\\ O(y_\lambda^{k-1}e^{-y_\lambda^2})\quad\text{as}\quad\lambda\to\infty\,, & \text{if}\quad k\ge 0\text{ is even}\,.\end{cases}$$

Therefore both the second and the fourth terms of the right-hand side of (6.43) are equal to $o(C_\lambda^3x_\lambda^{n-2})$ as $\lambda\to\infty$ for $0\le n\le 3$ under each of the conditions. Let us show that the fifth term of the right-hand side of (6.43) is also equal to $o(C_\lambda^3x_\lambda^{n-2})$ as $\lambda\to\infty$ for $0\le n\le 3$ if we choose K large enough. By the definition of C_λ and $a_\lambda(k)$, $k=3, 4$, we have

$$C_\lambda^{-3}x_\lambda^{-n+2}\{C_\lambda x_\lambda^n(|a_\lambda(3)|r_\lambda^3+|a_\lambda(4)|r_\lambda^4)^{K+1}\} \tag{6.44}$$
$$\leq\psi^{(2)}(x_\lambda)x_\lambda^2(|\psi^{(3)}(x_\lambda)|r_\lambda^3+|\psi^{(4)}(x_\lambda)|r_\lambda^4)^{K+1}.$$

Under the conditions in Proposition 6.2 we see that there exists $C>0$ such that the right-hand side of (6.44) is dominated by

$$\begin{cases} Cx_\lambda^{A+\{B-3-3\gamma_1(A-2)\}(K+1)}, & \text{if } A>2, \\ Cx_\lambda^{B+(B-3)(K+1)}(\log x_\lambda)^{3(K+1)}, & \text{if } A=2, \\ Cx_\lambda^{B+(B-3+3\gamma_2)(K+1)}, & \text{if } 1<A<2, \\ Cx_\lambda^{B+(B-3+3\gamma_3)(K+1)}, & \text{if } A=1, \end{cases} \tag{6.45}$$

for all sufficiently large λ. We note that in the case $A>2$, $B-3-3\gamma_1(A-2)<0$, in the case $A=2$, $B<5/2$, in the case $1<A<2$, $B-3+3\gamma_2<0$, and in the case $A=1$, $B-3+3\gamma_3<0$. Therefore in each case, by choosing K large enough, (6.45) converges to 0 as $\lambda\to\infty$. On the other hand, under the conditions in Proposition 6.3 we see, by using (H.6) and (6.20), that there exists $C>0$ such that the right-hand side of (6.44) is dominated by

$$Cx_\lambda^{2+2(K+1)}e^{\{A+(B-3\gamma_4A)(K+1)\}l(x_\lambda)} \tag{6.46}$$

for all sufficiently large λ. Since $B-3\gamma_4A<0$, (6.46) converges to 0 as $\lambda\to\infty$ for sufficiently large K. Therefore in each case the following (6.47) holds as $\lambda\to\infty$:

(6.47)

$$\begin{cases} \tilde{L}_0^{(1)}(\lambda)=\sqrt{\pi}\,C_\lambda+g_0(\lambda)+o(C_\lambda^3x_\lambda^{-2}), \\ \tilde{L}_1^{(1)}(\lambda)=\sqrt{\pi}\,C_\lambda x_\lambda+g_0(\lambda)x_\lambda+g_1(\lambda)+o(C_\lambda^3x_\lambda^{-1}), \\ \tilde{L}_2^{(1)}(\lambda)=\sqrt{\pi}\,C_\lambda x_\lambda^2+(\sqrt{\pi}/2)C_\lambda^3+g_0(\lambda)x_\lambda^2+2g_1(\lambda)x_\lambda+g_2(\lambda)+o(C_\lambda^3), \\ \tilde{L}_3^{(1)}(\lambda)=\sqrt{\pi}\,C_\lambda x_\lambda^3+(3\sqrt{\pi}/2)C_\lambda^3x_\lambda+g_0(\lambda)x_\lambda^3+3g_1(\lambda)x_\lambda^2+3g_2(\lambda)x_\lambda+g_3(\lambda) \\ \qquad +o(C_\lambda^3x_\lambda). \end{cases}$$

Next we estimate $\tilde{L}_n^{(2)}(\lambda)$. Under the condition (H.3) in Proposition 6.2 there exists $C>0$ such that

$$|W_\lambda(x)|\leq\begin{cases} Cr_\lambda^5(x_\lambda+r_\lambda)^{B-5}/4!, & \text{if } B\geq5, \\ Cr_\lambda^5(x_\lambda-r_\lambda)^{B-5}/4!, & \text{if } B<5, \end{cases} \tag{6.48}$$

for all sufficiently large λ and $x\in[x_\lambda-r_\lambda, x_\lambda+r_\lambda]$. By (6,27), (6.30) and (6.24), the right-hand side of (6.48) tends to 0 as $\lambda\to\infty$ in each case. Let us estimate $|W_\lambda(x)|$ under the conditions in Proposition 6.3. By (H.7), there exists $C>0$ such that

$$|W_\lambda(x)|\leq Cr_\lambda^5e^{Bl(x_\lambda+r_\lambda)}/4! \tag{6.49}$$

for all sufficiently large λ and $x\in[x_\lambda-r_\lambda, x_\lambda+r_\lambda]$. By (H.5), we have

(6.50) $$\limsup_{\delta\downarrow 0}\limsup_{x\to\infty} l(x+\delta)l(x)^{-1}=1 .$$

Indeed, by (H.5) there exists $C'>0$ such that

$$l(x+\delta)=l(x)+\int_x^{x+\delta} l'(y)dy\le l(x)+C'\int_x^{x+\delta} l(y)dy$$

for sufficiently large x and $\delta>0$. Since $l(x)$, $x>0$, is strictly increasing in x, we have

$$l(x+\delta)\le l(x)+C'\delta l(x+\delta) .$$

Therefore (6.50) follows. Since $r_\lambda\to 0$ as $\lambda\to\infty$, we have

(6.51) $$|W_\lambda(x)|\le Cr_\lambda^5 e^{(1+a)Bl(x_\lambda)}/4!$$

for all sufficiently large λ and $a>0$ by (6.49) and (6.50). By (6.31) and (6.24), the right-hand side of (6.51) tends to 0 as $\lambda\to\infty$ for sufficiently small $a>0$. Thus in each case we have

(6.52) $$|e^{-W_\lambda(x)}-1|=O(|W_\lambda(x)|) \qquad \text{as} \quad \lambda\to\infty .$$

By (6.48) and (6.30) or by (6.51) and (6.31), the right-hand side of (6.52) is equal to $o(C_\lambda^2 x_\lambda^{-2})$ uniformly in $x\in[x_\lambda-r_\lambda, x_\lambda+r_\lambda]$ as $\lambda\to\infty$. Therefore we get

(6.53) $$\tilde{L}_n^{(2)}(\lambda)=\tilde{L}_n^{(1)}(\lambda)\cdot o(C_\lambda^2 x_\lambda^{-2}) \qquad \text{as} \quad \lambda\to\infty .$$

Once we prove

(6.54) $$\tilde{L}_n^{(1)}(\lambda)=O(C_\lambda x_\lambda^n) \quad \text{as } \lambda\to\infty , \qquad 0\le n\le 3 ,$$

we will have, by combining this with (6.53),

(6.55) $$\tilde{L}_n^{(2)}(\lambda)=o(C_\lambda^3 x_\lambda^{n-2}) \quad \text{as } \lambda\to\infty , \qquad 0\le n\le 3 .$$

To show (6.54), let us examine the asymptotic behaviors, as $\lambda\to\infty$, of $g_n(\lambda)$, $0\le n\le 3$, appearing in (6.47). Since

$$Z_k\equiv\int_{-\infty}^{\infty} y^k e^{-y^2}dy=0 \qquad \text{if} \quad k\ge 1 \text{ is odd} ,$$

we can express $g_n(\lambda)$, $0\le n\le 3$, as follows:

$$g_0(\lambda)=\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m'=0}^{[k/2]}\binom{k}{2m'}a_\lambda(3)^{2m'}a_\lambda(4)^{k-2m'}C_\lambda^{4k-2m'+1}Z_{4k-2m'} ,$$

$$g_1(\lambda)=\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m'=0}^{[(k-1)/2]}\binom{k}{2m'+1}a_\lambda(3)^{2m'+1}a_\lambda(4)^{k-2m'-1}C_\lambda^{4k-2m'+1}Z_{4k-2m'} ,$$

$$g_2(\lambda)=\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m'=0}^{[k/2]}\binom{k}{2m'}a_\lambda(3)^{2m'}a_\lambda(4)^{k-2m'}C_\lambda^{4k-2m'+3}Z_{4k-2m'+2}\,,$$

$$g_3(\lambda)=\sum_{k=1}^{K}\frac{(-1)^k}{k!}\sum_{m'=0}^{[(k-1)/2]}\binom{k}{2m'+1}a_\lambda(3)^{2m'+1}a_\lambda(4)^{k-2m'-1}C_\lambda^{4k-2m'+3}Z_{4k-2m'+2}\,.$$

Therefore, by (6.23), the following (6.56) holds as $\lambda\to\infty$:

$$\text{(6.56)}\qquad \begin{cases} g_0(\lambda)=O(a_\lambda(3)^2C_\lambda^7)+O(a_\lambda(4)C_\lambda^5)\,, & g_1(\lambda)=O(a_\lambda(3)C_\lambda^5)\,,\\ g_2(\lambda)=O(a_\lambda(3)^2C_\lambda^9)+O(a_\lambda(4)C_\lambda^7)\,, & g_3(\lambda)=O(a_\lambda(3)C_\lambda^7)\,.\end{cases}$$

By (6.47), (6.56), (6.23) and (6.24), we obtain (6.54), and then (6.55).

Next we estimate $R_n^{(k)}(\lambda)$, $k=1, 2, 3$. Since $\psi_\lambda(x)$ is strictly increasing in $x\in(0, x_\lambda)$, we have

$$R_n^{(1)}(\lambda)\le x_\lambda^{n+1}e^{\psi_\lambda(x_\lambda-r_\lambda)-\psi_\lambda(x_\lambda)}\,.$$

We use (6.13) for $x=x_\lambda-r_\lambda$. Then, by using (6.29) and noting $W_\lambda(x_\lambda-r_\lambda)\to 0$ as $\lambda\to\infty$, we have

$$R_n^{(1)}(\lambda)\le Cx_\lambda^{n+1}e^{-\psi^{(2)}(x_\lambda)r_\lambda^2/2}\,,$$

when C is a positive constant independent of λ. Let us show

$$\text{(6.57)}\qquad R_n^{(1)}(\lambda)=o(C_\lambda^3x_\lambda^{n-2})\quad\text{as }\lambda\to\infty\,,\qquad 0\le n\le 3\,.$$

Indeed we have

$$\text{(6.58)}\qquad R_n^{(1)}(\lambda)C_\lambda^{-3}x_\lambda^{-n+2}\le Cx_\lambda^3\psi^{(2)}(x_\lambda)^{3/2}e^{-\psi^{(2)}(x_\lambda)r_\lambda^2/2}\,.$$

Under the conditions in Proposition 6.2 the right-hand side of (6.58) is dominated by

$$\text{(6.59)}\qquad \begin{cases} C'x_\lambda^{3+(3/2)(B-2)}\exp\{-C''x_\lambda^{(A-2)(1-2\gamma_1)}\}\,, & \text{if } A>2\,,\\ C'x_\lambda^{3+(3/2)(B-2)}\exp\{-C''(\log x_\lambda)^2\}\,, & \text{if } A=2,\\ C'x_\lambda^{3+(3/2)(B-2)}\exp\{-C''x_\lambda^{A-2+2\gamma_2}\}\,, & \text{if } 1<A<2\,,\\ C'x_\lambda^{3+(3/2)(B-2)}\exp\{-C''x_\lambda^{-1+2\gamma_3}\}\,, & \text{if } A=1\,,\end{cases}$$

for all sufficiently large λ, where C' and C'' are positive constants independent of λ. Since (6.59) converges to 0 as $\lambda\to\infty$ in each case, we get (6.57) under the conditions in Proposition 6.2. On the other hand, under the conditions in Proposition 6.3 the right-hand side of (6.58) is dominated by

$$\text{(6.60)}\qquad C'x_\lambda^3(x_\lambda^3e^{Bl(x_\lambda)})^{3/2}\exp\{-C''e^{(1-2\gamma_4)Al(x_\lambda)}\}$$

for all sufficiently large λ, where C' and C'' are positive constants independent of λ. Since (6.60) converges to 0 as $\lambda\to\infty$, we get (6.57) in this case, too.

Now we examine $R_n^{(2)}(\lambda)$. Since $\psi_\lambda(x)$ is strictly decreasing in $x\in(x_\lambda, \infty)$, we have

$$R_n^{(2)}(\lambda) \le m_\lambda^{n+1} e^{\psi_\lambda(x_\lambda + r_\lambda) - \psi_\lambda(x_\lambda)} \le C m_\lambda^{n+1} e^{-\psi^{(2)}(x_\lambda) r_\lambda^2/2}$$

for some $C>0$ independent of λ. Recall that m_λ is defined by (6.33) or (6.36) and $\lambda = \psi'(x_\lambda)$. Therefore, by using (6.16) or (6.20), we obtain

$$R_n^{(2)}(\lambda) = o(C_\lambda^3 x_\lambda^{n-2}) \qquad \text{as } \lambda \to \infty, \quad 0 \le n \le 3, \tag{6.61}$$

in the same way as we obtained (6.57).

Finally we estimate $R_n^{(3)}(\lambda)$. Since $\psi_\lambda(x_\lambda) > 0$, we have

$$R_n^{(3)}(\lambda) \le \int_{m_\lambda}^{\infty} x^n e^{\lambda x - \psi(x)} dx .$$

By (6.35) we get, for sufficiently large λ,

$$R_n^{(3)}(\lambda) \le \int_{m_\lambda}^{\infty} x^n e^{-b\lambda x} dx \le C m_\lambda^n e^{-b\lambda m_\lambda},$$

where C is a positive constant independent of λ. By using (6.14) or (6.18), we obtain

$$R_n^{(3)}(\lambda) = o(C_\lambda^3 x_\lambda^{n-2}) \qquad \text{as } \lambda \to \infty, \quad 0 \le n \le 3, \tag{6.62}$$

in the same way as we obtained (6.57). Hence, by (6.37), (6.38), (6.47), (6.55), (6.57), (6.61) and (6.62), we obtain the required results.

Proof of Proposition 6.2 and Proposition 6.3. By Lemma 6.4, (6.56), (6.23) and (6.24), we have

$$L_0(\lambda)L_1(\lambda)L_3(\lambda) - L_1(\lambda)^2 L_2(\lambda) = \pi\sqrt{\pi}\, C_\lambda^5 x_\lambda^2 + o(C_\lambda^5 x_\lambda^2) \qquad \text{as } \lambda \to \infty,$$

$$L_0(\lambda)L_2(\lambda)^2 - L_1(\lambda)^2 L_2(\lambda) = (\pi\sqrt{\pi}/2) C_\lambda^5 x_\lambda^2 + o(C_\lambda^5 x_\lambda^2) \qquad \text{as } \lambda \to \infty .$$

Hence, by (6.2) and (6.12), we obtain the required result in each of the propositions.

Acknowledgment. I would like to express may thanks to Professor H. Tanaka and Professor Y. Tamura for valuable suggestions and constant encouragement. I thank Professor K. Uchiyama; I owe the proof of Lemma 3.1 to him. I also thank the referee for suggesting some improvement in Proposition 6.2.

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