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On the Generalized Thomas-Fermi Differential Equations and Applicability of Saito's Transformation

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Dedicated to Professor Junji Kato on his sixtieth birthday

1. Introduction.

Let us consider the generalized Thomas-Fermi differential equation

$$x''=P(t)x^{1+\alpha}, \qquad '=d/dt, \quad x\geqq 0 \tag{1.1}$$

where α is a nonzero real constant and $x^{1+\alpha}$ denotes a nonnegative-valued branch.

In the papers [5], [6] Saito succeeded in investigating the asymptotic behavior of solutions of (1.1) where $P(t)=t^{\alpha\lambda-2}$ (λ is a positive constant) with the aid of a transformation

$$y=\psi(t)^{-\alpha}\phi(t)^{\alpha}, \qquad z=ty' \tag{1.2}$$

which transforms (1.1) to a first order algebraic differential equation

$$\frac{dz}{dy}=\frac{-\lambda(\lambda+1)\alpha^2y^2+(2\lambda+1)\alpha yz-(1-\alpha)z^2+\lambda(\lambda+1)\alpha^2y^3}{\alpha yz}. \tag{1.3}$$

In (1.2), $\psi(t)=[\lambda(\lambda+1)]^{1/\alpha}t^{-\lambda}$ is a particular solution of (1.1) and $\phi(t)$ is an arbitrary solution of (1.1). Moreover in [8], [9] we considered the case $P(t)=\pm e^{\alpha\lambda t}$ where λ is a real constant, using a transformation in a form similar to (1.2) such as

$$y=\psi(t)^{-\alpha}\phi(t)^{\alpha}, \qquad z=y'$$

where $\psi(t)=\pm\lambda^{2/\alpha}e^{-\lambda t}$. This transforms (1.1) to a first order algebraic differential equation also.

Since the coefficients of y' in the two transformations above differ, we consider a more general transformation

$$y=\psi(t)^{-\alpha}\phi(t)^{\alpha}, \qquad z=\theta(t)y' \tag{1.4}$$

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where $\psi(t)$, $\theta(t)$ are sufficiently smooth functions. We call this Saito's transformation. The purpose of this paper is to determine $P(t)$ of (1.1) such that we can transform (1.1) to a first order algebraic differential equation. In determining this, $\psi(t)$, $\theta(t)$ will be suitably chosen. This purpose will be achieved in §2. From the conclusion of that section, we shall find that the four differential equations

$$x'' = \pm t^{\beta} x^{1+\alpha}, \qquad x'' = \pm e^{\alpha\lambda t} x^{1+\alpha}$$

whose importance is stated in [1] have the form of $P(t)$ specified in section 2 so as to satisfy our purpose. Therefore $x'' = -t^{\beta} x^{1+\alpha}$ can be treated in the same way as in the papers [5], [6] and so on, while we have already dealt with the other three differential equations. So, following the form of the differential equation given in [5], let us consider in section 3 this equation in the form

$$x'' = -t^{\alpha\lambda - 2} x^{1+\alpha} \tag{1.5}$$

where α, λ are positive constants.

Recently there appeared many papers (cf. [4] and its references) where the positive radial solutions of the partial differential equation

$$\Delta u + K(|x|)u^{p} = 0, \qquad x \in \mathbf{R}^{n}, \quad p > 1 \tag{1.6}$$

are considered. Such solutions satisfy

$$(r^{n-1}u_{r})_{r} + r^{n-1}K(r)u^{p} = 0$$

where $r = |x|$. Let us make a change of letters $t = r$, $x = u$, $p = 1 + \alpha$ taking account of (1.1). Then from a simple calculation we get

$$x'' + ((n-1)/t)x' + Kx^{1+\alpha} = 0\,. \tag{1.7}$$

Put $c = n - 1$. Then from (1.7) we get

$$x'' + (c/t)x' + Kx^{1+\alpha} = 0 \qquad (t \geqq 0)\,. \tag{1.8}$$

Applying the determination of $P(t)$ in (1.1) to (1.8), we shall determine the function K of (1.6) in section 2 so that (1.8) can be transformed to a first order algebraic differential equation by (1.4). For K so determined, the arguments of [5] through [10] would make the asymptotic behavior of the positive radial solutions of (1.5) easier to investigate.

2. The determination of the generalized Thomas-Fermi differential equations by Saito's transformation.

First we suppose

$$\theta(t)>0\,. \tag{2.1}$$

For brevity we omit the variable t. Now we transform (1.1) by (1.4). From (1.4) we have $\phi=\psi y^{1/\alpha}$. Differentiating this, we obtain

$$\phi'=\psi' y^{1/\alpha}+(1/\alpha)\psi y^{(1/\alpha)-1}y'\,,$$

$$\phi''=\psi'' y^{1/\alpha}+(2/\alpha)\psi' y^{(1/\alpha)-1}y'+(1/\alpha)((1/\alpha)-1)\psi y^{(1/\alpha)-2}(y')^2+(1/\alpha)\psi y^{(1/\alpha)-1}y''\,.$$

However since ϕ satisfies (1.1), $\phi''=P\psi^{1+\alpha}y^{(1/\alpha)+1}$. Therefore we get

$$y''=\alpha P\psi^{\alpha}y^2-\alpha\psi^{-1}\psi'' y-2\psi^{-1}\psi' y'-((1/\alpha)-1)y^{-1}(y')^2\,.$$

Substituting $y'=\theta^{-1}z$, we obtain

$$y''=\alpha P\psi^{\alpha}y^2-\alpha\psi^{-1}\psi'' y-2\psi^{-1}\psi'\theta^{-1}z-((1/\alpha)-1)y^{-1}\theta^{-2}z^2\,.$$

Hence we get

$$\begin{aligned} z'=\theta' y'+\theta y''=\theta^{-1}\theta' z+\alpha\theta P\psi^{\alpha}y^2-\alpha\theta\psi^{-1}\psi'' y\\ -2\psi^{-1}\psi' z-((1/\alpha)-1)\theta^{-1}y^{-1}z^2\,. \end{aligned}$$

From $y'=\theta^{-1}z$ and this fact, we conclude that

$$\begin{aligned} dz/dy=z'/y'=\{(\alpha-1)z^2+\alpha(\theta'-2\theta\psi^{-1}\psi')yz\\ +\alpha^2\theta^2P\psi^{\alpha}y^3-\alpha^2\theta^2\psi^{-1}\psi'' y^2\}/\alpha yz\,. \end{aligned}$$

If the coefficients of the polynomial of y, z between the braces are constants, then there exist constants c_1, c_2, c_3 such that

$$\theta'-2\theta\psi^{-1}\psi'=c_1\,, \tag{2.2}$$

$$\theta^2P\psi^{\alpha}=c_2\,, \tag{2.3}$$

$$\theta^2\psi^{-1}\psi''=c_3\,. \tag{2.4}$$

From (2.2)

$$\psi'=((\theta'-c_1)/2\theta)\psi\,.$$

Solve this. Then noticing (2.1), we get

$$\psi=c_4\theta^{1/2}J(\theta) \tag{2.5}$$

where c_4 is a constant and

$$J(\theta)=\exp(-c_1/2)\int(1/\theta)dt\,.$$

However $J'(\theta)=(-c_1/2\theta)J(\theta)$. Consequently

$$\psi' = (c_4/2)\theta^{-1/2}(\theta' - c_1)J(\theta),$$
$$\psi'' = (c_4/4)\theta^{-3/2}\{-\theta'(\theta' - c_1) + 2\theta\theta'' - c_1(\theta' - c_1)\}J(\theta).$$

Thus we get

$$\theta^2\psi^{-1}\psi'' = \{-(\theta')^2 + 2\theta\theta'' + c_1^2\}/4. \tag{2.6}$$

Substitute this into (2.4). Then $-(\theta')^2 + 2\theta\theta'' = 4c_3 - c_1^2$. Differentiating both sides, we obtain $\theta''' = 0$. Namely

$$\theta = pt^2 + qt + r \tag{2.7}$$

where p, q, r are real constants.

It follows from (2.5) and (2.7) that

$$\psi = c_4(pt^2 + qt + r)^{1/2} \exp(-c_1/2) \int (1/(pt^2 + qt + r))dt. \tag{2.8}$$

Furthermore from (2.3), (2.7) and (2.8)

$$\begin{aligned} P &= c_2\theta^{-2}\psi^{-\alpha} \\ &= c_2 c_4^{-\alpha}(pt^2 + qt + r)^{-2-(\alpha/2)} \exp(\alpha c_1/2) \int (1/(pt^2 + qt + r))dt. \end{aligned} \tag{2.9}$$

Conversely if θ, ψ, P are given as (2.7), (2.8), (2.9) respectively, then (2.2) and (2.3) are evidently valid. Moreover we have (2.6). Therefore from (2.4) and (2.7) it follows that

$$c_3 = (-q^2 + 4pr + c_1^2)/4. \tag{2.10}$$

Thus (1.1) with (2.9) can be transformed by (1.4) with (2.8) into a first order algebraic differential equation

$$\frac{dz}{dy} = \frac{(\alpha-1)z^2 + \alpha c_1 yz + \alpha^2 c_2 y^3 - \alpha^2 c_3 y^2}{\alpha yz}. \tag{2.11}$$

Summarizing these, we have

THEOREM A. *If* (1.1) *can be transformed by* (1.4) *into a first order algebraic differential equation, then* $P(t)$ *has the form* (2.9). *Furthermore* $\theta(t)$, $\psi(t)$ *are determined as* (2.7), (2.8) *respectively. In this case,* (1.1) *is transformed into* (2.11) *by* (1.4).

If $\theta < 0$, then the same conclusion follows. Actually it suffices to put $\tilde{\theta} = -\theta$, $\tilde{c}_1 = -c_1$.

Rewriting (2.9) by using θ, we get

$$P = c_2 c_4^{-\alpha}\theta^{-2-(\alpha/2)} \exp(\alpha c_1/2) \int (1/\theta)dt.$$

Moreover we have

$$\psi = c_4 \theta^{1/2} \exp(-c_1/2) \int (1/\theta) dt \, .$$

Now we have five cases: (i) θ is a constant, (ii) θ is a linear function of t, (iii) the quadratic equation $\theta = 0$ has two different real roots, (iv) the quadratic equation $\theta = 0$ has the multiple root, (v) the quadratic equation $\theta = 0$ has imaginary roots. Here we obtain $P(t)$ more explicitly in every one of these cases.

In case of (i), we have $P(t) = \gamma e^{\sigma t}$ where $\gamma = c_2 c_4^{-\alpha} \theta^{-2-(\alpha/2)}$, $\sigma = \alpha c_1/2\theta$.

Suppose $\theta = qt + r$ in case of (ii). Then $P(t) = \gamma(qt+r)^{\sigma}$ where

$$\gamma = \zeta c_2 c_4^{-\alpha} \, , \qquad \sigma = -2 - (\alpha/2) + (\alpha c_1/2q) \, .$$

Here ζ is a constant with $|\zeta| = 1$ which appears in removing $|\ \ |$. It may be necessary to recall that the primitive function of $1/x$ is given by $\log|x|$ for real x.

If the case (iii) occurs, then we put

$$\theta = p(t-\xi)(t-\eta) \qquad (\xi \neq \eta) \, .$$

Thus we have

$$P(t) = \gamma(t-\xi)^{-2-(\alpha/2)+\mu}(t-\eta)^{-2-(\alpha/2)-\mu}$$

where $\gamma = \zeta c_2 c_4^{-\alpha} p^{-2-(\alpha/2)}$, $\mu = \alpha c_1/2p(\xi - \eta)$.

In the case (iv), we put $\theta = p(t-\xi)^2$. So we get

$$P(t) = \gamma(t-\xi)^{-4-\alpha} e^{\sigma/(t-\xi)}$$

where

$$\gamma = c_2 c_4^{-\alpha} p^{-2-(\alpha/2)} \, , \qquad \sigma = -\alpha c_1/2p \, .$$

Finally we consider the case (v). Then we put $\theta = p\{(t-\xi)^2 + \eta^2\}$. Therefore

$$P(t) = \gamma\{(t-\xi)^2 + \eta^2\}^{-2-(\alpha/2)} e^{\sigma \tan^{-1}(t-\xi)/\eta}$$

where

$$\gamma = c_2 c_4^{-\alpha} p^{-2-(\alpha/2)} \, , \qquad \sigma = \alpha c_1/2p\eta \, .$$

Let us determine the function K of (1.6) so that the transformation (1.4) turns (1.8) to a first order algebraic differential equation. First we let the term $(c/t)x'$ vanish. For this, we put $s = t^{1-c}$ if $c \neq 1$. Then from (1.8) we get

(2.12) $$d^2x/ds^2 + (1-c)^{-2} s^{2c/(1-c)} K x^{1+\alpha} = 0 \, .$$

If $c = 1$, then we put $s = \log t$. In this case, we have

(2.13) $$d^2x/ds^2 + e^{2s} K x^{1+\alpha} = 0 \, .$$

Since (2.12), (2.13) have the form contained in (1.1), we use Theorem A. Notice that $c=1$ is equivalent to $n=2$, because $c=n-1$. Then from the above discussions, (1.8) can be transformed to a first order algebraic differential equation through (1.4) if K has the following form: Corresponding to every case of (i) through (v), we have

(i) $K(t)=\gamma t^{\sigma-2}$ $\quad (n=2)$
$\quad =(n-2)^2\gamma t^{-2n+2}e^{\sigma t^{2-n}}$ $\quad (n\geqq 3)$,

(ii) $K(t)=\gamma t^{-2}(q\log t+r)^{\sigma}$ $\quad (n=2)$
$\quad =(n-2)^2\gamma t^{-2n+2}(qt^{2-n}+r)^{\sigma}$ $\quad (n\geqq 3)$,

(iii) $K(t)=\gamma t^{-2}(\log t-\xi)^{-2-(\alpha/2)+\mu}(\log t-\eta)^{-2-(\alpha/2)-\mu}$ $\quad (n=2)$
$\quad =(n-2)^2\gamma t^{-2n+2}(t^{2-n}-\xi)^{-2-(\alpha/2)+\mu}(t^{2-n}-\eta)^{-2-(\alpha/2)-\mu}$ $\quad (n\geqq 3)$,

(iv) $K(t)=\gamma t^{-2}(\log t-\xi)^{-4-\alpha}e^{\sigma/(\log t-\xi)}$ $\quad (n=2)$
$\quad =(n-2)^2\gamma t^{-2n+2}(t^{2-n}-\xi)^{-4-\alpha}e^{\sigma/(t^{2-n}-\xi)}$ $\quad (n\geqq 3)$,

(v) $K(t)=\gamma t^{-2}\{(\log t-\xi)^2+\eta^2\}^{-2-(\alpha/2)}e^{\sigma\tan^{-1}(\log t-\xi)/\eta}$ $\quad (n=2)$
$\quad =(n-2)^2\gamma t^{-2n+2}\{(t^{2-n}-\xi)^2+\eta^2\}^{-2-(\alpha/2)}e^{\sigma\tan^{-1}(t^{2-n}-\xi)/\eta}$ $\quad (n\geqq 3)$,

where $-\gamma$ is replaced by γ. If we change $n-1$ for c in these, then we obtain the form of K corresponding to (1.8) where c is any constant which is not necessarily a positive integer.

3. The application.

Let us consider the nonlinear differential equation (1.5) in the domain $0<t<\infty$, $0\leqq x<\infty$. First we transform (1.5) to (1.3). In (2.9), take

(3.1) $$c_2c_4^{-\alpha}=-1\,,$$

(3.2) $$p=r=0\,,\qquad q=1\,.$$

Then we have

$$P=-t^{-2-(\alpha/2)}\exp(\alpha c_1/2)\log t=-t^{(c_1-1)\alpha/2-2}\,.$$

Comparing this with (1.5), we take $(c_1-1)/2=\lambda$, namely

(3.3) $$c_1=2\lambda+1\,.$$

Moreover comparing (1.3) with (2.11), we find that c_1 is suitably chosen and

(3.4) $$c_2=c_3=\lambda(\lambda+1)\,.$$

From (3.2), we get $\theta=t$. Furthermore from (3.1), (3.4) we obtain $c_4^{-\alpha}=-1/\lambda(\lambda+1)$. Hence from (2.5)

$$\psi^{-\alpha}=(-1/\lambda(\lambda+1))t^{-\alpha/2}\exp(\alpha(2\lambda+1)/2)\log t=(-1/\lambda(\lambda+1))t^{\alpha\lambda}\,.$$

Now from (1.4) we have the transformation

$$y=(-1/\lambda(\lambda+1))t^{\alpha\lambda}\phi(t)^{\alpha}\,, \qquad z=ty'\,. \tag{3.5}$$

Also (2.10) is satisfied by (3.1)–(3.4). Hence this transforms (1.5) to (1.3).

Since $t>0$, we get $y<0$. It follows from (3.5) that we get

$$y^{-1}z=\alpha(\lambda+t\phi'/\phi)\,. \tag{3.6}$$

Moreover let $\phi(t)$ be any positive solution of (1.5) whose domain will be denoted by (ω', ω) where $0\leqq\omega'<\omega\leqq\infty$.

Now we consider the case $t\to\omega$. Then we have the following possibilities:

(i) $0<\omega<\infty$, $\lim_{t\to\omega}\phi(t)=\infty$,
(ii) $0<\omega<\infty$, $\lim_{t\to\omega}\phi(t)=0$,
(iii) $\omega=\infty$, $\lim_{t\to\omega}\phi(t)=\infty$,
(iv) $\omega=\infty$, $\lim_{t\to\omega}\phi(t)=c\ (>0)$,
(v) $\omega=\infty$, $\lim_{t\to\omega}\phi(t)=0$.

In the cases (i), (iii), (iv), we get $\lim_{t\to\omega}y=-\infty$. Furthermore in the case (ii), $\lim_{t\to\omega}y=0$.

If z is a fixed value in (1.3), then $dz/dy=0$ is valid on the discrete values of y. Therefore the solution $z=z(y)$ has the inverse function $y=y(z)$. If we put $\zeta=1/y$, then

$$\frac{d\zeta}{dz}=\frac{-\alpha\zeta^4 z}{-\lambda(\lambda+1)\alpha^2\zeta+(2\lambda+1)\alpha\zeta^2 z-(1-\alpha)\zeta^3 z^2+\lambda(\lambda+1)\alpha^2}\,. \tag{3.7}$$

Now suppose that y is unbounded. Then from (3.7) $\zeta\equiv 0$ i.e. $y\equiv\infty$. This is a contradiction. Thus y is bounded.

This means that the cases (i), (iii), (iv) cannot take place. Since $\lim_{t\to\omega}\phi(t)=0$ in the cases (ii), (v), there exists a positive constant T such that $\phi'(T)<0$. If $t>T$, then there exists a constant R such that $\phi'(t)<R<0$ since $\phi''(t)\leqq 0$. Consequently if $\phi(t)$ is continuable up to $t=\infty$, then we have a contradiction $\phi(t)-\phi(T)<R(t-T)\to-\infty$ as $t\to\infty$. Hence we obtain

$$0<\omega<\infty\,, \qquad \lim_{t\to\omega}\phi(t)=0\,.$$

Moreover since (ii) takes place, $\lim_{t\to\omega}y=0$.

As $t\to\omega'$, the solution $\phi(t)$ is bounded. Actually we get $\phi''(t)\leqq 0$. Hence if $t<\tau$, then $\phi'(t)\geqq\phi'(\tau)$. Integrating both sides of this from t to τ,

$$\phi(\tau)-\phi(t)\geqq\phi'(\tau)(\tau-t)\,.$$

However if $\phi(t)$ is unbounded as $t\to\omega'$, then $\phi(\tau)-\phi(t)$ is unbounded below. This contradicts the above inequality.

Here let us show the unique existence of a solution $\phi(t)$ of (1.5) such that

$$\lim_{t\to 0}\phi(t)=0\,,\qquad \lim_{t\to 0}\phi'(t)=b\,,\quad b>0\,. \tag{3.8}$$

Moreover let us obtain the analytical expression of this. If this exists, then we get

$$\lim_{t\to 0} t/\phi(t)=\lim_{t\to 0} 1/\phi'(t)=1/b$$

from l'Hospital's theorem. Therefore from (3.6)

$$\lim_{t\to 0} y=0\,,\quad \lim_{t\to 0} y^{-1}z=\alpha(\lambda+1)\,,\quad \lim_{t\to 0} z=0\,.$$

Thus we put $v=y^{-1}z-\alpha(\lambda+1)$. Hence we obtain

$$\frac{dv}{dy}=\frac{\lambda(\lambda+1)\alpha^2 y-\alpha v-v^2}{\alpha y\{v+\alpha(\lambda+1)\}} \tag{3.9}$$

or equivalently the 2-dimensional autonomous system

$$\begin{aligned} dy/ds&=(\lambda+1)\alpha^2 y+\alpha yv\\ dv/ds&=\lambda(\lambda+1)\alpha^2 y-\alpha v-v^2\,. \end{aligned} \tag{3.10}$$

The coefficient matrix of the linear terms of (3.10) is equal to

$$\begin{bmatrix} (\lambda+1)\alpha^2 & 0\\ \lambda(\lambda+1)\alpha^2 & -\alpha \end{bmatrix}.$$

Since α, λ are positive, the eigenvalues satisfy $-\alpha<0<(\lambda+1)\alpha^2$. Consequently $(y, v)=(0, 0)$ is a saddle point of (3.10). In this case (3.10) has only two solutions tending to $(0, 0)$ of the form

$$\begin{aligned} y&=a_1(Ce^{(\lambda+1)\alpha^2 s})+a_2(Ce^{(\lambda+1)\alpha^2 s})^2+\cdots\\ v&=b_1(Ce^{(\lambda+1)\alpha^2 s})+b_2(Ce^{(\lambda+1)\alpha^2 s})^2+\cdots \end{aligned} \tag{3.11}$$

as $s\to-\infty$, and

$$\begin{aligned} y&=a_1(Ce^{-\alpha s})+a_2(Ce^{-\alpha s})^2+\cdots\\ v&=b_1(Ce^{-\alpha s})+b_2(Ce^{-\alpha s})^2+\cdots \end{aligned} \tag{3.12}$$

as $s\to\infty$ where C is a constant. Substituting (3.11) into (3.10),

$$b_1/a_1=\lambda(\lambda+1)\alpha/\{(\lambda+1)\alpha+1\}\,.$$

Hence from (3.11)

$$v=[\lambda(\lambda+1)\alpha/\{(\lambda+1)\alpha+1\}]y+\cdots\,.$$

Returning to the original variables,

$$ty'=\alpha(\lambda+1)y[1+\{\lambda/((\lambda+1)\alpha+1)\}y+\cdots]\,.$$

Solving this, we get

$$-y=\sum_{n=1}^{\infty} a_n(Dt^{\alpha(\lambda+1)})^n\,, \qquad a_1=1$$

where D is a constant. Therefore from (3.5)

$$\phi(t)=\{\lambda(\lambda+1)\}^{1/\alpha}D^{1/\alpha}t\sum_{n=0}^{\infty}\tilde{a}_n(Dt^{\alpha(\lambda+1)})^n\,, \qquad \tilde{a}_0=1\,.$$

From this we get

$$\phi'(0)=\{\lambda(\lambda+1)\}^{1/\alpha}D^{1/\alpha}=b\,.$$

Namely we have $D=b^\alpha/\lambda(\lambda+1)$. Consequently

$$\phi(t)=bt\sum_{n=0}^{\infty}\tilde{a}_n\{(b^\alpha/\lambda(\lambda+1))t^{\alpha(\lambda+1)}\}^n\,, \qquad \tilde{a}_0=1\,. \tag{3.13}$$

Here notice that $y\equiv 0$, $v=b_1(Ce^{-\alpha s})+\cdots$ satisfies (3.10). Namely (3.12) is equal to this. Thus from (3.12) no solution of (3.9) can be obtained. Therefore the solution of (1.5) satisfying (3.8) is given only by (3.13).

Next, with the aid of the referee's comment, let us show that if

$$\lim_{t\to 0}\phi(t)=0\,, \tag{3.14}$$

then $\phi'(t)$ is bounded. From (3.14), there exists $T>0$ such that $\phi'(T)>0$. Since $\phi''(t)\leqq 0$, we get

$$\phi'(t)\geqq\phi'(T)>0 \tag{3.15}$$

if $0<t<T$. On the other hand, from (1.5) and (3.14) we get

$$-\phi''(t)=t^{\alpha\lambda-2}\phi(t)^{1+\alpha}<t^{\mu-2}\phi(t)^{1+\alpha}<C_1t^{\mu-2}$$

where C_1 is some positive constant and μ is a positive irrational constant such that $\mu<\min\{1, \alpha\lambda\}$. Hence for some positive constant δ we have

$$-\phi'(\delta)+\phi'(t)\leqq\frac{C_1}{1-\mu}(t^{\mu-1}-\delta^{\mu-1})$$

by integrating the above inequality from t to δ $(0<t<\delta)$. Hence there exists a positive constant C_1' such that $\phi'(t)\leqq C_1't^{\mu-1}$. Integrating this again, $\phi(t)\leqq C_1''t^\mu$ where C_1'' is a constant. Thus

$$-\phi''(t)<t^{\mu-2}\phi(t)^{1+\alpha}<C_2t^{2\mu-2}\,.$$

In case of $2\mu<1$, repeat the above procedure. Taking $m=[1/\mu]$, this procedure can be repeated m times, since $m\mu<1<(m+1)\mu$. Thus we obtain

$$\phi(t) \leqq C''_m t^{m\mu}, \qquad -\phi''(t) \leqq C_{m+1} t^{(m+1)\mu-2}$$

where C''_m, C_{m+1} are constants. Consequently for $0<t<\delta$

$$-\phi'(\delta)+\phi'(t) \leqq \frac{C_{m+1}}{(m+1)\mu-1}(\delta^{(m+1)\mu-1}-t^{(m+1)\mu-1}).$$

Hence $\phi'(t)$ is bounded as $t\to 0$.

From the above discussions, the asymptotic behavior of solutions of (1.5) has the following three possibilities:

(i) $\omega'>0$, $\lim_{t\to\omega'}\phi(t)=0$,
(ii) $\omega'=0$, $\lim_{t\to\omega'}\phi(t)=0$,
(iii) $\omega'=0$, $\lim_{t\to\omega'}\phi(t)=a>0$.

In every one of these cases, ω is finite and $\lim_{t\to\omega}\phi(t)=0$. Therefore from (3.6)

$$\lim_{t\to\omega} y=0, \qquad \lim_{t\to\omega} y^{-1}z=-\infty.$$

Actually $\phi'(\omega)=0$ is impossible, since $\phi''(t)\leqq 0$. Similarly in the case (i),

$$\lim_{t\to\omega'} y=0, \qquad \lim_{t\to\omega'} y^{-1}z=\infty.$$

As is stated above, in the case (ii)

$$\lim_{t\to\omega'} y=0, \quad \lim_{t\to\omega'} y^{-1}z=\alpha(\lambda+1), \quad \lim_{t\to\omega'} z=0.$$

In the case (iii)

$$\lim_{t\to\omega'} y=0, \quad \lim_{t\to\omega'} y^{-1}z=\alpha\lambda, \quad \lim_{t\to\omega'} z=0,$$

since if ϕ' diverges, then l'Hospital's theorem implies

$$\lim_{t\to 0} t\phi'=\lim_{t\to 0}\frac{\phi'}{1/t}=\lim_{t\to 0}\frac{\phi''}{-1/t^2}=\lim_{t\to 0}\frac{-t^{\alpha\lambda-2}\phi^{1+\alpha}}{-1/t^2}=\lim_{t\to 0} t^{\alpha\lambda}\phi^{1+\alpha}=0.$$

(y, z) is a solution of the 2-dimensional autonomous system

$$\begin{aligned} &dy/ds=\alpha yz \\ &dz/ds=-\lambda(\lambda+1)\alpha^2y^2+(2\lambda+1)\alpha yz-(1-\alpha)z^2+\lambda(\lambda+1)\alpha^2y^3. \end{aligned} \tag{3.16}$$

Orbits of this cannot intersect each other. If the point (y_0, z_0) of the yz-plane is given arbitrarily, then from (3.5) we obtain the solution $\phi(t)$ of (1.5) corresponding to (y_0, z_0). In any of the cases (i)–(iii), we have

$$\lim_{t\to\omega'} y=0, \qquad \lim_{t\to\omega} y=0.$$

Therefore every orbit of (3.16) intersects perpendicularly with the y-axis at a point. Moreover this tends to the nonnegative part of the z-axis as $s \to -\infty$ and to the nonpositive part of the z-axis as $s \to \infty$, in a monotonic way with respect to y, since z has the definite sign. As is shown above, there exists uniquely an orbit of (3.16) tending to the origin with $y^{-1}z \to \alpha(\lambda+1)$. Suppose that this orbit intersects with the y-axis at $(y_*, 0)$. Here give the initial condition

$$0 < T < \infty , \quad x(T) = A \ (>0) , \quad x'(T) = B$$

to (1.5). Then from (3.5), (3.6) we have

$$y(T) = (-1/\lambda(\lambda+1))T^{\alpha\lambda}A^{\alpha} , \qquad z(T) = \alpha y(T)(\lambda + TB/A) .$$

Fix T, A. Then $y(T)$ is fixed. Also $z(T)$ is a decreasing function of B. If $y_* < y(T) < 0$, then $0 < A < a_*$ where

$$a_* = \{\lambda(\lambda+1)\}^{1/\alpha} T^{-\lambda}(-y_*)^{1/\alpha} .$$

Draw a phase portrait of (3.16). If we put a point (y_0, z_0) inside the region surrounded by the unique orbit obtained in the case (ii) and by the z-axis, then the case (iii) takes place. If we put (y_0, z_0) outside this region, then the case (i) takes place. Consequently we get

THEOREM B. *If* $0 < A < a_*$, *then there exist* B_1, B_2 $(-\infty < B_1 < B_2 < \infty)$ *such that* $B < B_1$ *or* $B > B_2$ *implies*

$$\omega' > 0 , \qquad \lim_{t \to \omega'} \phi(t) = 0 ,$$

$B = B_1$ *or* $B = B_2$ *implies*

$$\omega' = 0 , \qquad \lim_{t \to \omega'} \phi(t) = 0 ,$$

and $B_1 < B < B_2$ *implies*

$$\omega' = 0 , \qquad \lim_{t \to \omega'} \phi(t) = a > 0 .$$

If $A = a_*$, *then there exists* B_3 *such that* $B \neq B_3$ *implies*

$$\omega' > 0 , \qquad \lim_{t \to \omega'} \phi(t) = 0 ,$$

$B = B_3$ *implies*

$$\omega' = 0 , \qquad \lim_{t \to \omega'} \phi(t) = 0 .$$

If $A > a_*$, *then*

$$\omega'>0\,, \qquad \lim_{t\to\omega'} \phi(t)=0\,.$$

For every (A, B), *we get*

$$0<\omega<\infty\,, \qquad \lim_{t\to\omega} \phi(t)=0\,.$$

Now let us obtain an analytical expression of $\phi(t)$ in the case (iii). Since $y^{-1}z\to\alpha\lambda$, we put $v=y^{-1}z-\alpha\lambda$. Then we get

$$\frac{dv}{dy}=\frac{\alpha v-v^2+\lambda(\lambda+1)\alpha^2 y}{\alpha y(v+\alpha\lambda)}\,.$$

In a neighborhood of $(y, v)=(0, 0)$, we get a Briot-Bouquet differential equation

$$y\frac{dv}{dy}=(\lambda+1)y+\frac{v}{\alpha\lambda}+\cdots\,.$$

If we put $\eta=-y$, then

$$\eta\frac{dv}{d\eta}=-(\lambda+1)\eta+\frac{v}{\alpha\lambda}+\cdots\,.$$

Therefore we obtain

$$v=\sum_{m+n>0} v_{mn}\eta^m\{\eta^{1/\alpha\lambda}(h\log\eta+\Gamma)\}^n\,, \qquad v_{01}=1 \tag{3.17}$$

where h, Γ are constants and $h=0$ if $1/\alpha\lambda$ is not an integer. Returning to the original variables, we get

$$y^{-1}z-\alpha\lambda=\sum_{m+n>0} v_{mn}\eta^m\{\eta^{1/\alpha\lambda}(h\log\eta+\Gamma)\}^n\,,$$

$$ty'=-t\eta'=-\eta[\alpha\lambda+\sum_{m+n>0} v_{mn}\eta^m\{\eta^{1/\alpha\lambda}(h\log\eta+\Gamma)\}^n]\,.$$

Thus we have

$$(1/\alpha\lambda)\left[(1/\eta)+\sum_{m+n>0} w_{mn}\eta^{m-1}\{\eta^{1/\alpha\lambda}(h\log\eta+\Gamma)\}^n\right]\eta'=1/t\,.$$

Integrating both sides, we obtain

$$\eta\left[1+\sum_{m+n>0} c_{mn}\eta^m\{\eta^{1/\alpha\lambda}(h\log\eta+\Gamma)\}^n\right]=(Dt)^{\alpha\lambda}$$

where D is a constant. Hence Smith's lemma (cf. Lemma 1 of [6], [7]) implies that

$$\eta=(Dt)^{\alpha\lambda}\left[1+\sum_{m+n>0}\hat{c}_{mn}(Dt)^{\alpha\lambda m}\{Dt(h\log(Dt)^{\alpha\lambda}+\Gamma)\}^n\right].$$

It follows from (3.5) that

$$\phi(t)=\{\lambda(\lambda+1)\}^{1/\alpha}t^{-\lambda}\eta^{1/\alpha} \tag{3.18}$$

$$=\{\lambda(\lambda+1)\}^{1/\alpha}D^{\lambda}\left[1+\sum_{m+n>0}\gamma_{mn}(Dt)^{\alpha\lambda m}\{Dt(h\log(Dt)^{\alpha\lambda}+\Gamma)\}^n\right].$$

Since $\phi(t)$ tends to a as $t\to 0$, we get $\{\lambda(\lambda+1)\}^{1/\alpha}D^{\lambda}=a$, i.e.,

$$D=a^{1/\lambda}/\{\lambda(\lambda+1)\}^{1/\alpha\lambda}. \tag{3.19}$$

Consequently we obtain

$$\phi(t)=a\left[1+\sum_{m+n>0}\gamma_{mn}(Dt)^{\alpha\lambda m}\{Dt(\hat{h}\log t+\hat{\Gamma})\}^n\right] \tag{3.20}$$

where D is given by (3.19), $\hat{h}=\alpha\lambda h$ and $\hat{\Gamma}=\alpha\lambda h\log D+\Gamma$. If $\alpha\lambda>1$, then the term of the lowest order of (3.17) is given by $\Gamma\eta^{1/\alpha\lambda}$. Therefore from (3.6) it follows that

$$\Gamma=\lim_{\eta\to 0}\eta^{-1/\alpha\lambda}v=\lim_{\eta\to 0}\eta^{-1/\alpha\lambda}(y^{-1}z-\alpha\lambda)$$

$$=\lim_{t\to 0}\{\lambda(\lambda+1)\}^{1/\alpha\lambda}t^{-1}\phi(t)^{-1/\lambda}\alpha t\phi'(t)/\phi(t)$$

$$=\{\lambda(\lambda+1)\}^{1/\alpha\lambda}\alpha b/a^{1+(1/\lambda)}$$

where $b=\lim_{t\to 0}\phi'(t)$. Since $1/\alpha\lambda$ is not an integer in this case, h is equal to 0. Hence

$$\phi(t)=a\left[1+\sum_{m+n>0}\gamma_{mn}(\{a^{\alpha}/\lambda(\lambda+1)\}t^{\alpha\lambda})^m((\alpha b/a)t)^n\right]. \tag{3.21}$$

Notice that these discussions are given in [5], [6] for the equation $x''=t^{\alpha\lambda-2}x^{1+\alpha}$.

Let us determine the analytical expression of $\phi(t)$ around $t=\omega$. From (3.6) we get

$$\lim_{t\to\omega}y=0, \qquad \lim_{t\to\omega}y^{-1}z=-\infty.$$

Thus we put $w=yz^{-1}$, $\eta=-y$. Then we obtain a Briot-Bouquet differential equation

$$\eta\frac{dw}{d\eta}=\frac{w}{\alpha}-(2\lambda+1)w^2+\lambda(\lambda+1)\alpha w^3+\lambda(\lambda+1)\alpha\eta w^3.$$

Hence

$$w=\sum_{m+n>0}w_{mn}\eta^m(D\eta^{1/\alpha})^n, \qquad w_{01}=1,\ w_{m0}=0$$

where D is a constant. Following the discussions of §5 of [6],

$$\phi(t)=C(\omega-t)\left[1+\sum_{m+n>0}\phi_{mn}(\omega-t)^m(\omega-t)^{\alpha n}\right] \tag{3.22}$$

where $\Gamma=-1/D$, $C=(\Gamma/\alpha\omega^{\lambda+1})\{\lambda(\lambda+1)\}^{1/\alpha}$.

Similarly if $0<\omega'<\infty$, $\lim_{t\to\omega'}\phi(t)=0$, then we obtain

$$\phi(t)=C(t-\omega')\left[1+\sum_{m+n>0}\phi_{mn}(t-\omega')^m(t-\omega')^{\alpha n}\right] \tag{3.23}$$

where $\Gamma=1/D$, $C=(\Gamma/\alpha\omega'^{\lambda+1})\{\lambda(\lambda+1)\}^{1/\alpha}$.

Namely the analytical expressions of $\phi(t)$ is given as follows:

THEOREM C. *If* $0<A<a_*$, $B<B_1$ *or* $0<A<a_*$, $B>B_2$ *or* $A=a_*$, $B\neq B_3$ *or* $A>a_*$, *then* (3.23) *is valid. If* $0<A<a_*$, $B=B_1$ *or* $0<A<a_*$, $B=B_2$ *or* $A=a_*$, $B=B_3$, *then* (3.13) *is valid. If* $0<A<a_*$, $B_1<B<B_2$, *then* (3.20) *is valid and especially in the case when* $\alpha\lambda>1$, (3.21) *is valid. For every* (A, B), (3.22) *is valid.*

Finally let us consider the case when $T=0$, namely when the initial condition is given as

$$x(0)=A(>0)\,,\qquad x'(0)=B\,. \tag{3.24}$$

If $\alpha\lambda>1$, then there exists the unique solution satisfying this for any (A, B). This is represented as (3.21) where $a=A$, $b=B$. Actually this is obtained from the orbit (y, z) of the 2-dimensional autonomous system (3.16) such that $(y, z)\to(0, 0)$, $y^{-1}z\to\alpha\lambda$. Suppose $0<\alpha\lambda<1$. From (3.20), we have

$$\phi(t)=a\{1+\gamma_{10}D^{\alpha\lambda}t^{\alpha\lambda}+\cdots\}\,.$$

Differentiating this, we see that

$$\phi'(t)=a\gamma_{10}D^{\alpha\lambda}\alpha\lambda t^{\alpha\lambda-1}+\cdots\,,$$

$$\phi''(t)=a\gamma_{10}D^{\alpha\lambda}\alpha\lambda(\alpha\lambda-1)t^{\alpha\lambda-2}+\cdots\,.$$

Substituting these into (1.5), we get

$$a\gamma_{10}D^{\alpha\lambda}\alpha\lambda(\alpha\lambda-1)=-a^{1+\alpha}\,.$$

Namely we obtain

$$\gamma_{10}D^{\alpha\lambda}=-a^{\alpha}/\alpha\lambda(\alpha\lambda-1)\,.$$

Thus $\phi'(t)$ tends to ∞ as $t\to0$, since

$$\phi'(t)=(a^{1+\alpha}/(1-\alpha\lambda))t^{\alpha\lambda-1}+\cdots\,.$$

Similarly if $\alpha\lambda=1$, then

$$\phi'(t) = -a^{1+\alpha}\log t + \cdots ,$$

since we have

$$\phi(t) = a[1 + \gamma_{01} Dt(\hat{h}\log t + \hat{\Gamma}) + \cdots]$$

from (3.20) in this case. Therefore $\phi'(t)$ tends to ∞ as $t \to 0$. Now we conclude that the initial value problem (1.5) with (3.24) has no solution if $0 < \alpha\lambda \leqq 1$.

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