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Disk/Band Surfaces of Spatial Graphs

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Abstract. In this paper, we show that, for any spatial embedding $\Gamma: G \to \mathbf{R}^3$ of a connected planar graph G, there exists a disk/band surface of $\Gamma(G)$ satisfying a certain linking condition. As an application of this result, it is proved that the homology class of $\Gamma(G)$ is determined only by the linking numbers of disjoint pairs in the set of boundary/outermost cycles with respect to a fixed planar embedding of G.

Introduction.

For any spatial embedding $\Gamma: G \to \mathbf{R}^3$ of a graph G, a *disk/band surface* S of $\Gamma(G)$ is a compact, orientable surface in $\mathbf{R}^3$ such that $\Gamma(G)$ is a deformation retract of S contained in $\mathrm{int}S$. In [1], Kauffman, Simon, Wolcott and Zhao studied the disk/band surfaces of spatial graphs, and showed that, if G is either the theta-curve or the K_4-graph, then any spatial graph $\Gamma(G)$ admits a unique (up to ambient isotopy) disk/band surface S with zero Seifert linking form. So, any topological invariant for the pair $(\mathbf{R}^3, S)$ can be regarded as an ambient-isotopy invariant for $\Gamma(G)$, e.g. various polynomial invariants for the link $(\mathbf{R}^3, \partial S)$. In [1], it was also proved that any connected, trivalent graph other than the theta-curve or the K_4-graph has a spatial embedding which admits no disk/band surfaces with zero Seifert linking form.

In this paper, we consider a certain condition weaker than the zero-Seifert-linking condition, and show that any spatial embedding of a connected graph G with a specified planar embedding has a disk/band surface S satisfying this condition. Furthermore, if G is prime and trivalent, then such an S is uniquely determined up to ambient isotopy. Thus, our disk/band surface for any prime, trivalent, planar graph G furnishes ambient-isotopy invariants for $\Gamma(G)$.

For any connected, planar graph G, we fix a planar embedding $\Gamma_0: G \to \mathbf{R}^2$ arbitrarily. The image $\Gamma_0(G)$ has complementary domains $D_1, D_2, \cdots, D_n$ that are bounded and one unbounded D_0. The preimage $c_i = \Gamma_0^{-1}(\partial D_i)$ is a 1-complex which can be viewed as a 1-cycle in $H_1(G; \mathbf{Z})$. We call c_i $(i \neq 0)$, c_0 respectively a *boundary cycle* and the *outermost cycle* in G with respect to Γ_0. Most of our results will be stated in

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terms of pairs of boundary/outermost cycles. Since the set $\{c_1, \cdots, c_n\}$ of boundary cycles generates $H_1(G; \mathbf{Z})$ (see [1, Lemma 2.5]), the Seifert linking form $\langle \ , \ \rangle_S$: $H_1(G; \mathbf{Z}) \times H_1(G; \mathbf{Z}) \to \mathbf{Z}$ is determined by the values of $\langle \Gamma(c_i), \Gamma(c_j) \rangle_S$ for i, j with $1 \leq i, j \leq n$.

THEOREM 1. *Suppose that G is a connected, planar graph, and $\Gamma_0: G \to \mathbf{R}^2$ is a planar embedding. Then, for any spatial embedding $\Gamma: G \to \mathbf{R}^3$, there exists a disk/band surface S of $\Gamma(G)$ that has the Seifert pairings satisfying the following equation* (0.1).

$$\langle \Gamma(c_i), \Gamma(c_j) \rangle_S = \begin{cases} -\operatorname{lk}(\Gamma(c_i), \Gamma(c_0)) - \sum_{c_i \cap c_k = \varnothing} \operatorname{lk}(\Gamma(c_i), \Gamma(c_k)) & \text{if } i=j \text{ and } c_i \cap c_0 = \varnothing \\ 0 & \text{if } i \neq j \text{ and } c_i \cap c_j \neq \varnothing \\ 0 & \text{if } i=j \text{ and } c_i \cap c_0 \neq \varnothing \\ \operatorname{lk}(\Gamma(c_i), \Gamma(c_j)) & \text{if } c_i \cap c_j = \varnothing, \end{cases} \tag{0.1}$$

where c_i, c_j, c_k are boundary cycles and c_0 is the outermost cycle with respect to Γ_0. Furthermore, if G is prime and trivalent, then S is determined uniquely up to ambient isotopy.

Our proof of Theorem 1 implies that, for a regular neighborhood S_0 of $\Gamma_0(G)$ in $\mathbf{R}^2$, there exists an embedding $f_\Gamma: S_0 \to \mathbf{R}^3$ extending $\Gamma \circ \Gamma_0^{-1}: \Gamma_0(G) \to \mathbf{R}^3$ such that the image $S = f_\Gamma(S_0)$ is a disk/band surface satisfying (0.1). One of the advantages of fixing planar embeddings is to reduce relatively the number of cycles in G to be considered for certain decisions.

The following straightforward corollary presents the necessary and sufficient condition for the existence of a disk/band surface with zero Seifert linking form.

COROLLARY. *Suppose that G is a connected graph admitting a planar embedding $\Gamma_0: G \to \mathbf{R}^2$. Then, for a spatial embedding $\Gamma: G \to \mathbf{R}^3$, $\Gamma(G)$ has a disk/band surface with zero Seifert linking form if and only if* $\operatorname{lk}(\Gamma(c_i), \Gamma(c_j)) = 0$ *for all disjoint pairs c_i, c_j in $\{c_1, \cdots, c_n, c_0\}$. In particular, if G has no disjoint cycles (e.g. the theta n-curve), then any spatial embedding of G admits a disk/band surface with zero Seifert linking form.*

In [3], Taniyama defined the spatial-graph homology, and in [4], he proved that two spatial embeddings $\Gamma_1, \Gamma_2: G \to \mathbf{R}^3$ of any graph G are homologous if and only if they have the same Wu invariant. The Wu invariant is given as an element of the certain 2-dimensional cohomology, see [4] for details.

Here, we will study the spatial-graph homology from a different point of view. It is well-known that the homology class of any link is determined by the linking numbers of all distinct component pairs. As an application of Theorem 1, we have the following theorem which implies that there is a similar situation also in the homology of spatial embeddings of planar graphs G. In this theorem, the planarity condition on G is crucial. In fact, there are spatial embeddings of certain non-planar graphs whose homology classes cannot be determined by the linking numbers of all disjoint cycle pairs, for example see [3, §4].

THEOREM 2. *Suppose that G is a connected graph admitting a planar embedding $\Gamma_0: G \to \mathbf{R}^2$. Then, two spatial embeddings $\Gamma_1, \Gamma_2: G \to \mathbf{R}^3$ of G are homologous if and only if* $\mathrm{lk}(\Gamma_1(c_i), \Gamma_1(c_j)) = \mathrm{lk}(\Gamma_2(c_i), \Gamma_2(c_j))$ *for all disjoint pairs c_i, c_j in the set $\{c_1, \cdots, c_n, c_0\}$ of boundary/outermost cycles with respect to Γ_0.*

Though the results similar to our theorems still hold without the assumption of G connected, we only consider the connected case to simplify the statements and proofs of theorems.

1. Preliminaries.

A *graph* G is a finite, 1-dimensional polyhedron. Each $x \in G$ has a small, open neighborhood $U(x) \subset G$ consisting of finitely many, half-open arcs ending at x. The *valence* of a point $x \in G$, denoted by $\mathrm{val}(x)$, is the number of such arcs. In this paper, graphs G are always assumed that $\mathrm{val}(x) \geq 2$ for any $x \in G$. A *vertex* is a point v in G with $\mathrm{val}(v) \geq 3$. Let $V(G)$ be the set of vertices of G. The closure of each component of $G - V(G)$ is called an *edge* of G.

A graph G is *planar* if there exists an embedding $\Gamma_0: G \to \mathbf{R}^2$. Then, the image $\Gamma_0(G)$ is called a *plane graph*. A connected, planar graph G is said to be *prime* if, for any spatial embedding $\Gamma_0: G \to \mathbf{R}^2$, there exists no simple closed curves C in $\mathbf{R}^2$ satisfying either the following (i) or (ii) (cf. [2], [1]), where A, B are the two components of $R^2 - C$.

(i) C meets $\Gamma_0(G)$ in a single point such that both $A \cap \Gamma_0(G)$ and $B \cap \Gamma_0(G)$ are non-empty.

(ii) C meets $\Gamma_0(G)$ in two points such that both $A \cap \Gamma_0(G)$, $B \cap \Gamma_0(G)$ are non-empty and not single open arcs.

Consider an oriented surface S in $\mathbf{R}^3$, and 1-cycles x, y on S. Let x^+ denote the result of the pushing x a very small amount into $\mathbf{R}^3 - S$ along the positive normal direction to S. The function $\langle\ ,\ \rangle_S: H_1(S; \mathbf{Z}) \times H_1(S; \mathbf{Z}) \to \mathbf{Z}$ defined by

$$\langle x, y \rangle_S = \mathrm{lk}(x^+, y) .$$

is called the *Seifert linking form* for S; $\langle x, y \rangle_S$ is called the *Seifert pairing* of x and y. It is well-defined, bilinear pairing, an invariant of the ambient isotopy class of the embedding $S \subset \mathbf{R}^3$. We say that S has a *zero Seifert linking form* if $\langle \alpha, \beta \rangle_S = 0$ for any $\alpha, \beta \in H_1(S; \mathbf{Z})$.

LEMMA 1. *With the notation as in Theorem 1, if a disk/band surface S of $\Gamma(G)$ satisfies*

$$\langle \Gamma(c_i), \Gamma(c_j) \rangle_S = \begin{cases} 0 & \text{if } i \neq j \text{ and } c_i \cap c_j \neq \emptyset \\ 0 & \text{if } i = j \text{ and } c_i \cap c_0 \neq \emptyset , \end{cases} \tag{1.1}$$

then it also satisfies

$$\langle\Gamma(c_i), \Gamma(c_j)\rangle_S = \begin{cases} -\mathrm{lk}(\Gamma(c_i), \Gamma(c_0)) - \sum_{c_i \cap c_k = \varnothing} \mathrm{lk}(\Gamma(c_i), \Gamma(c_k)) & \text{if } i=j \text{ and } c_i \cap c_0 = \varnothing \\ \mathrm{lk}(\Gamma(c_i), \Gamma(c_j)) & \text{if } c_i \cap c_j = \varnothing . \end{cases}$$

That is, the condition (1.1) *is equivalent to the condition* (0.1).

PROOF. We note that if $c_i \cap c_j = \varnothing$, then the value of $\langle\Gamma(c_i), \Gamma(c_j)\rangle_S$ is independent of S and equal to $\mathrm{lk}(\Gamma(c_i), \Gamma(c_j))$. Since $c_0 + c_1 + \cdots + c_n = 0$ in $H_1(S; \mathbf{Z})$ $(= H_1(G; \mathbf{Z}))$,

$$\langle\Gamma(c_i), \Gamma(c_0)\rangle_S + \sum_{k=1}^{n} \langle\Gamma(c_i), \Gamma(c_k)\rangle_S = \langle\Gamma(c_i), \Gamma(c_0 + c_1 + \cdots + c_n)\rangle_S = 0 .$$

Hence, by (1.1), we have

$$\langle\Gamma(c_i), \Gamma(c_0)\rangle_S + \sum_{c_i \cap c_k = \varnothing} \langle\Gamma(c_i), \Gamma(c_k)\rangle_S + \langle\Gamma(c_i), \Gamma(c_i)\rangle_S = 0 ,$$

for any c_i with $c_i \cap c_0 = \varnothing$. It follows that

$$\langle\Gamma(c_i), \Gamma(c_i)\rangle_S = -\mathrm{lk}(\Gamma(c_i), \Gamma(c_0)) - \sum_{c_i \cap c_k = \varnothing} \mathrm{lk}(\Gamma(c_i), \Gamma(c_k)) .$$

This completes the proof. □

2. Trivalent spatial graphs and equation systems.

Throughout the remainder of this paper, we always assume that (i) G is a connected graph admitting a planar embedding $\Gamma_0 \colon G \to \mathbf{R}^2$, (ii) $\{c_1, \cdots, c_n, c_0\}$ is the set of boundary/outermost cycles with respect to Γ_0, and (iii) S_0 is a regular neighborhood of $\Gamma_0(G)$ in $\mathbf{R}^2$.

In this section, we consider the special case where G is connected and trivalent. Here, G *trivalent* means that $\mathrm{val}(v) = 3$ for any $v \in V(G)$. Then, our proof will be completed by the argument similar to that in the proof of [1, Theorem 2.4]. In [1], they used equation systems associated to regular projections of $\Gamma(G)$ into $\mathbf{R}^2$ with variables representing the number of half twists of bands in disk/band surfaces of $\Gamma(G)$. We will use the same equation systems in essentials. However, our equation systems have variables representing the number of full twists of such bands, and they are defined directly without relying on regular projections of $\Gamma(G)$. A half-twisting argument will be needed later only for the proof of the uniqueness of a disk/band surface when G is prime and trivalent.

Let $\mathcal{N} = N_1 \cup \cdots \cup N_m$ be a regular neighborhood of $\Gamma_0(V(G))$ in $\mathbf{R}^2$, where N_i is the component of $\mathcal{N}$ containing $v_i \in \Gamma_0(V(G))$ and $m = \# V(G)$. We may assume that $N_i \cap \Gamma_0(G)$ is a star-shaped graph consisting of three edges with v_i a common vertex. It is easily seen that $\Gamma \circ \Gamma_0^{-1}$: $\Gamma_0(G) \to \mathbf{R}^3$ is extended to an embedding g_0: $\Gamma_0(G) \cup \mathcal{N} \to \mathbf{R}^3$. Let $\{\beta_1, \cdots, \beta_k, \gamma_1, \cdots, \gamma_{k'}\}$ be the set of closures of components of $\Gamma_0(G) - \mathcal{N}$ such that each $\Gamma_0(G) - \beta_s$ is connected and each $\Gamma_0(G) - \gamma_t$ is disconnected. Since G is trivalent, $2(k+k') = 3m$. Let B_s, C_t ($s=1, \cdots, k$; $t=1, \cdots, k'$) be mutually disjoint, thin bands in $\mathbf{R}^2 - \mathrm{int}\mathcal{N}$ with $B_s \supset \beta_s$, $C_t \supset \gamma_t$ and such that $B_s \cap \mathcal{N}$, $C_t \cap \mathcal{N}$ consist of two arcs. One can regard that $S_0 = \mathcal{N} \cup \mathcal{B} \cup \mathcal{C}$, where $\mathcal{B} = B_1 \cup \cdots \cup B_k$, $\mathcal{C} = C_1 \cup \cdots \cup C_{k'}$. Then, g_0: $\Gamma_0(G) \cup \mathcal{N} \to \mathbf{R}^3$ can be extended to an embedding f_0: $S_0 \to \mathbf{R}^3$. Let $\{d_1, \cdots, d_n, d_0\}$ be the set of components of ∂S_0 such that each d_i is homologous to ∂D_i in S_0, where D_i's are components of $\mathbf{R}^2 - \Gamma_0(G)$ given in Introduction. We also consider oriented simple loops d_i' ($i=1, \cdots, n$) in $\mathrm{int} S_0$ parallel to d_i in S_0. Note that, for any c_i, c_j with $i \neq j$, $\Gamma(c_i \cap c_j) \cap f_0(\mathcal{C}) = \varnothing$, and for any c_i, $\Gamma(c_i)$ is homologous in $f_0(S_0)$ to a sum of 1-cycles in $f_0(S_0 - \mathcal{C})$. This shows that both $\mathrm{lk}(f_0(d_i), f_0(d_j)) = \langle \Gamma(c_i), \Gamma(c_j) \rangle_{f_0(S_0)}$ and $\mathrm{lk}(f_0(d_i), f_0(d_i')) = \langle \Gamma(c_i), \Gamma(c_i) \rangle_{f_0(S_0)}$ are invariable under the modification of $f_0(S_0)$ by full twistings of any components of $f_0(\mathcal{C})$. For any $n_1, \cdots, n_k \in \mathbf{Z}$, let $f_{n_1, \cdots, n_k}$: $S_0 \to \mathbf{R}^3$ be the embedding extending $f_0|_{(S_0 - \mathcal{B}) \cup \Gamma_0(G)}$ such that, if $n_s \geq 0$ (resp. $n_s < 0$) for $s = 1, \cdots, k$, then $f_{n_1, \cdots, n_k}(B_s)$ is obtained by the right-hand (resp. left-hand) $|n_s|$-full twistings of $f_0(B_s)$ around $f_0(\beta_s) = \Gamma \circ \Gamma_0^{-1}(\beta_s)$, see Fig. 1.

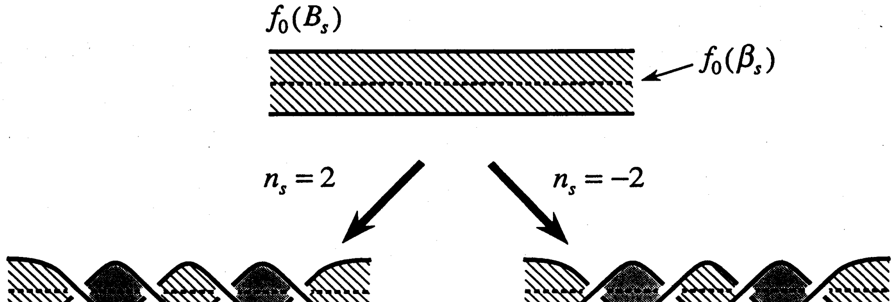

FIGURE 1

By Lemma 1, the Seifert linking form for $S = f_{n_1, \cdots, n_k}(S_0)$ satisfies the equation (0.1) if and only if $\{n_1, \cdots, n_k\}$ is an integral solution to the following linear equation system (2.1).

$$\begin{cases} \varepsilon_1^{(i,j)} n_1 + \cdots + \varepsilon_k^{(i,j)} n_k = \mathrm{lk}(f_0(d_i), f_0(d_j)) & \text{if } c_i \cap c_j \neq \varnothing \quad (i \neq j) \\ \varepsilon_1^{(i)} n_1 + \cdots + \varepsilon_k^{(i)} n_k = -\mathrm{lk}(f_0(d_i), f_0(d_i')) & \text{if } c_i \cap c_0 \neq \varnothing , \end{cases} \tag{2.1}$$

where $\varepsilon_s^{(i,j)}$ (resp. $\varepsilon_s^{(i)}$) is 1 if $\beta_s \subset \Gamma_0(c_i \cap c_j)$ (resp. $\beta_s \subset \Gamma_0(c_i)$), otherwise $\varepsilon_s^{(i,j)} = 0$ (resp. $\varepsilon_s^{(i)} = 0$).

PROPOSITION 1. *If G is prime and trivalent, then the equation* (2.1) *has a unique integral solution $\{n_1, \cdots, n_k\}$. Moreover, any disk/band surface of $\Gamma(G)$ satisfying* (0.1) *is ambient isotopic to $f_{n_1, \cdots, n_k}(S_0)$.*

PROOF. First, we note that the primeness of G implies $\mathscr{C}=\emptyset$. The argument similar to that in the proof of [1, Theorem 2.4] shows that the determinant of the coefficient matrix of (2.1) is 1. Hence, (2.1) has the unique integral solution. Let S be any disk/band surface of $\Gamma(G)$ of satisfying (0.1). Since $\{c_1, \cdots, c_n\}$ generates $H_1(G; \mathbf{Z})$, the condition (0.1) implies that, for any $\alpha, \beta \in H_1(G; \mathbf{Z})$,

$$\langle \Gamma(\alpha), \Gamma(\beta)\rangle_S = \langle \Gamma(\beta), \Gamma(\alpha)\rangle_S .$$

Since G is trivalent, S is ambient isotopic rel. $\Gamma(G)$ to the surface, still denoted by S, obtained from $f_0(S_0)$ by half twistings of components of $f_0(\mathscr{B})$. Suppose that S has a band defined by odd half-twistings of some component $f_0(B_s)$. Since G is prime and trivalent, for the c_i, c_j $(0 \leq i < j \leq n)$ with $\Gamma(c_i \cap c_j) \supset f_0(\beta_s)$, $c_i \cap c_j$ is homeomorphic to the theta-curve. Since the disk/band surface S is orientable by the definition, a regular neighborhood of $\Gamma(c_i \cup c_j)$ in S has genus one and the algebraic intersection number of $\Gamma(c_i)$ and $\Gamma(c_j)$ in S is

$$\Gamma(c_i) \cdot \Gamma(c_j) = \langle \Gamma(c_i), \Gamma(c_j)\rangle_S - \langle \Gamma(c_j), \Gamma(c_i)\rangle_S = 1 .$$

This contradiction shows that S is obtained from $f_0(S_0)$ by full twistings of components of $f_0(\mathscr{B})$. Then, the uniqueness of a solution to (2.1) proves that S is ambient isotopic rel. $\Gamma(G)$ to $f_{n_1,\cdots,n_k}(S_0)$. $\square$

PROPOSITION 2. *If G is a connected and trivalent, then* (2.1) *has an integral solution.*

PROOF. As in the proof of [1, Theorem 2.4-(2)], the orders of variables $n_1, n_2, \cdots, n_k$ associated with $\beta_1, \beta_2, \cdots, \beta_k$ can be arranged such that $\beta_s \cap c_0 = \emptyset$ for $1 \leq s \leq l$ and $\beta_s \cap c_0 \neq \emptyset$ for $l < s \leq k$. Set $S = f_{n_1,\cdots,n_k}(S_0)$. Then, the part of (2.1) corresponding to $\langle \Gamma(c_i), \Gamma(c_j)\rangle_S = 0$ for c_i, c_j $(i \neq j)$ with $c_i \cap c_j \neq \emptyset$ is represented as follows:

$$(B_{g\times l}\ O_{g\times(k-l)})\,{}^t(n_1\, n_2 \cdots n_k) = D ,$$

where $O_{p\times q}$ denotes the $p \times q$-zero matrix. If we represent the part of (2.1) corresponding to $\langle \Gamma(c_i), \Gamma(c_i)\rangle_S = 0$ for c_i with $c_i \cap c_0 \neq \emptyset$ by

$$(G_{h\times l}\ C_{h\times(k-l)})\,{}^t(n_1\, n_2 \cdots n_k) = H ,$$

then (2.1) is represented as follows:

$$\begin{pmatrix} B_{g\times l} & O_{g\times(k-l)} \\ G_{h\times l} & C_{h\times(k-l)} \end{pmatrix} {}^t(n_1\, n_2 \cdots n_k) = \begin{pmatrix} D \\ H \end{pmatrix} .$$

Our arrangement of the order of variables implies that (i) each entry of $B_{g\times l}$ and $C_{h\times(k-l)}$ is either 0 or 1, (ii) each row of them has at least one nonzero entry, and (iii) each column of them has just one nonzero entry. Consider any rows of $B_{g\times l}$ containing at least two nonzero entries, and divide the corresponding equations in (2.1) into some equations so that each row of the resulting matrix has just one nonzero entry 1, and any solution to the new equation system is also a solution to (2.1). For example, if some equation in (2.1) has the form as

$$1\cdot n_1+0\cdot n_2+0\cdot n_3+1\cdot n_4+1\cdot n_5+0\cdot n_6+0\cdot n_7+0\cdot n_8=9\,,$$

then we divide it into the three equations:

$$\begin{cases} 1\cdot n_1+0\cdot n_2+0\cdot n_3+0\cdot n_4+0\cdot n_5+0\cdot n_6+0\cdot n_7+0\cdot n_8=9\\ 0\cdot n_1+0\cdot n_2+0\cdot n_3+1\cdot n_4+0\cdot n_5+0\cdot n_6+0\cdot n_7+0\cdot n_8=0\\ 0\cdot n_1+0\cdot n_2+0\cdot n_3+0\cdot n_4+1\cdot n_5+0\cdot n_6+0\cdot n_7+0\cdot n_8=0\,. \end{cases}$$

By modifying $C_{h\times(k-l)}$ similarly and exchanging rows of the resulting coefficient matrix suitably, we have a new equation

$$\begin{pmatrix} I_{l\times l} & O_{l\times(k-l)} \\ G'_{(k-l)\times l} & I_{(k-l)\times(k-l)} \end{pmatrix} {}^t(n_1\, n_2 \cdots n_k)=\begin{pmatrix} D' \\ H' \end{pmatrix} \tag{2.2}$$

whose solution is also a solution to (2.1), where $I_{p\times p}$ denotes the unit matrix of order p. Since the determinant of the coefficient matrix of (2.2) is 1, the equation system (2.2) and hence (2.1) have an integral solution. □

3. Proofs of theorems.

Now we are ready to prove theorems.

PROOF OF THEOREM 1. Set $\mathrm{val}(G)=\max\{\mathrm{val}(v);\, v\in V(G)\}$, and let $V_{\max}(G)$ be the set of vertices $v\in V(G)$ with $\mathrm{val}(v)=\mathrm{val}(G)$. We will prove Theorem 1 by induction on the lexicographically ordered pair $(\mathrm{val}(G), \sharp V_{\max}(G))$.

If $\mathrm{val}(G)=3$, then the proof is completed by setting $S=f_{n_1,\cdots,n_k}(S_0)$ for an integral solution $\{n_1, \cdots, n_k\}$ to the equation system (2.1) given in Proposition 2.

Now, suppose that $\mathrm{val}(G)\geq 4$. Choose $v\in V_{\max}(G)$, and consider the set $\{c_{i_1}, c_{i_2}, \cdots, c_{i_u}\}$ of boundary/outermost cycles containing v, possibly $c_{i_s}=c_{i_t}$ for some s, t with $s\neq t$, for example see Figure 2. Here, we need to consider the following two cases.

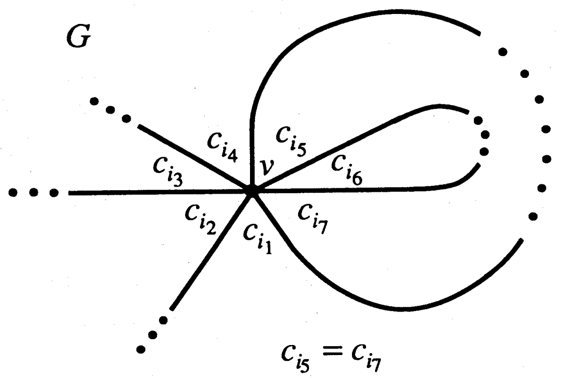

FIGURE 2

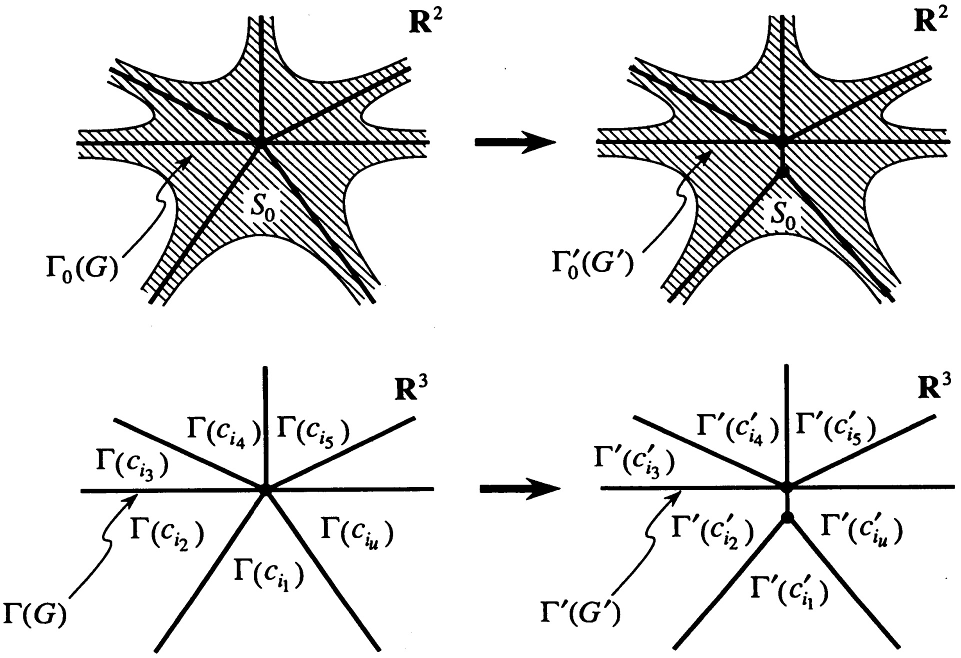

FIGURE 3

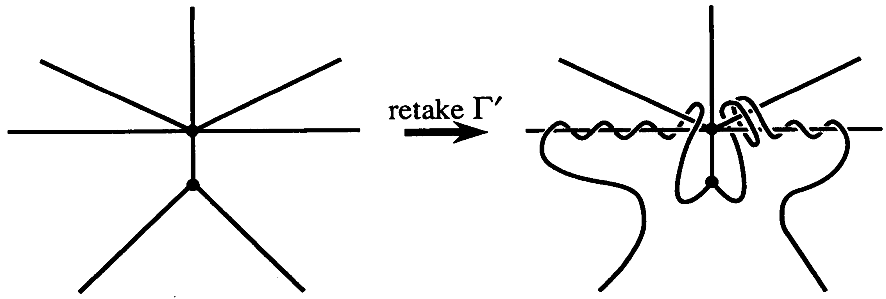

FIGURE 4

Case 1. The set $\{c_{i_1}, c_{i_2}, \cdots, c_{i_u}\}$ does not contain c_0. Then, we modify $\Gamma_0(G)$, $\Gamma(G)$ in small neighborhoods of $\Gamma_0(v)$, $\Gamma(v)$ to $\Gamma'_0(G')$, $\Gamma'(G')$ respectively as in Figure 3, so that $c'_{i_1} \cap c'_{i_2} \cap c'_{i_u}$ contains a vertex of valence$=3$, where c'_i (resp. c'_0) is the boundary cycle (resp. the outermost cycle) of G' with respect to Γ'_0 corresponding to c_i (resp. c_0) of G. Note that S_0 can be regarded as a regular neighborhood not only of $\Gamma_0(G)$ but also of $\Gamma'_0(G')$. If necessary retaking Γ' as illustrated in Figure 4, we may assume that $\mathrm{lk}(\Gamma'(c'_{i_1}), \Gamma'(c'_{i_t}))=0$ for any c'_{i_t} with $c'_{i_1} \cap c'_{i_t} = \emptyset$. Since $(\mathrm{val}(G'), \# V_{\max}(G')) < (\mathrm{val}(G), \# V_{\max}(G))$, by the hypothesis of our induction, we have a spatial embedding $f_{\Gamma'}: S_0 \to \mathbf{R}^3$ extending $\Gamma' \circ \Gamma_0'^{-1}$ and such that $f_{\Gamma'}(S_0)$ is a disk/band surface of $\Gamma'(G')$ satisfying (0.1). It is easily seen that $f_{\Gamma'}(\Gamma_0(G))$ is ambient isotopic to $\Gamma(G)$ in $\mathbf{R}^3$ (see Figure 5), so if necessary deforming $\Gamma(G)$ by ambient isotopy, $S = f_{\Gamma'}(S_0)$ can be regarded as a disk/band surface of $\Gamma(G)$. Since, then, each $\Gamma(c_i)$ $(i=0, 1, 2, \cdots, n)$ is homologous to $\Gamma'(c'_i)$ in S, we have $\langle \Gamma'(c'_i), \Gamma'(c'_j) \rangle_S = \langle \Gamma(c_i), \Gamma(c_j) \rangle_S$ for any i, j with

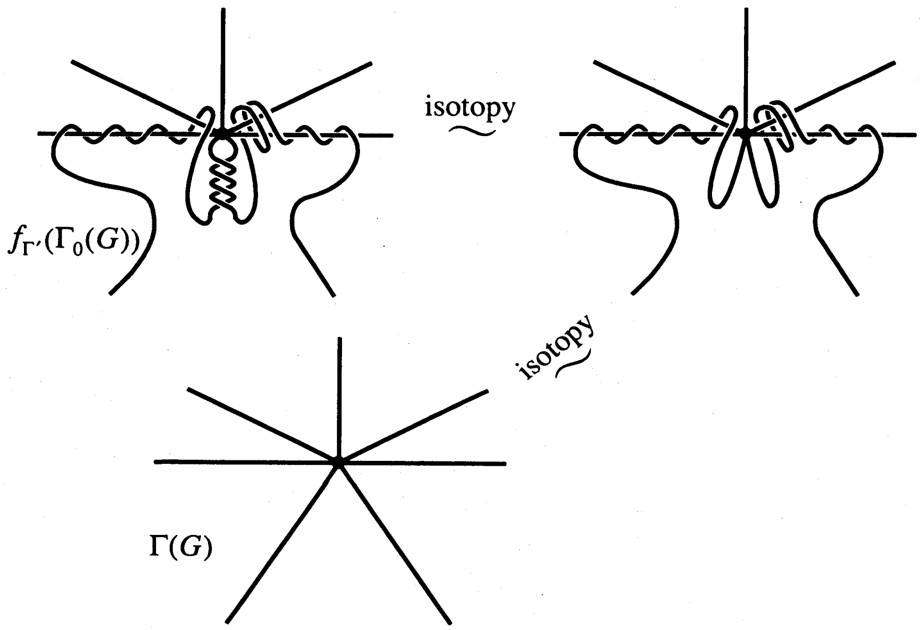

FIGURE 5

$1 \le i, j \le n$. Since $\{c_{i_1}, c_{i_2}, \cdots, c_{i_u}\}$ does not contain c_0, $c_i \cap c_0 \ne \varnothing$ if and only if $c'_i \cap c'_0 \ne \varnothing$. Thus, $\langle \Gamma(c_i), \Gamma(c_i)\rangle_S = \langle \Gamma'(c'_i), \Gamma'(c'_i)\rangle_S = 0$ for any boundary cycle c_i with $c_i \cap c_0 \ne \varnothing$. Consider any boundary cycles c_i, c_j $(i \ne j)$ of G with $c_i \cap c_j \ne \varnothing$, and the corresponding cycles c'_i, c'_j of G'. If $c'_i \cap c'_j \ne \varnothing$, then $\langle \Gamma(c_i), \Gamma(c_j)\rangle_S = \langle \Gamma'(c'_i), \Gamma'(c'_j)\rangle_S = 0$. If $c'_i \cap c'_j = \varnothing$, then one can set $c'_i = c'_{i_1}$, $c'_j = c'_{i_t}$ for some c'_{i_t} with $3 \le t \le u-1$, so our retaking of Γ' implies that $\langle \Gamma(c_i), \Gamma(c_j)\rangle_S = \mathrm{lk}(\Gamma'(c'_{i_1}), \Gamma'(c'_{i_t})) = 0$. As a result, for all these pairs c_i, c_j, we have $\langle \Gamma(c_i), \Gamma(c_j)\rangle_S = 0$. This shows that S satisfies (1.1) and hence (0.1) by Lemma 1.

Case 2. The set $\{c_{i_1}, c_{i_2}, \cdots, c_{i_u}\}$ contains c_0. Set $c_{i_u} = c_0$, and modify $\Gamma_0(G)$, $\Gamma(G)$ to $\Gamma'_0(G')$, $\Gamma'(G')$ as in Case 1. Since $c'_{i_t} \cap c'_0 \ne \varnothing$ for $1 \le t \le u-1$, $c_i \cap c_0 \ne \varnothing$ if and only if $c'_i \cap c'_0 \ne \varnothing$. Thus, the proof in this case is completed by the argument similar to that in Case 1. □

Two spatial embeddings $\Gamma_1, \Gamma_2 \colon G \to \mathbf{R}^3$ $(k = 1, 2)$ are said to be *homologous* if there is a locally flat embedding $\Phi \colon (G \times I) \# \bigcup_{i=1}^n F_i \to \mathbf{R}^3 \times I$ between Γ_1 and Γ_2, where $\{F_i\}_{i=1}^n$ is a finite set of mutually disjoint, closed, orientable surfaces, and $\#$ denotes the connected sum. More precisely, for each F_i, there is just one edge e in G such that F_i is attached to an open disk $\mathrm{int}(e \times I)$ by the usual connected sum of surfaces.

PROOF OF THEOREM 2. The "only if" part is easily verified. In fact, if Γ_1 is homologous to Γ_2, then for any c_i, c_j with $c_i \cap c_j = \varnothing$, the restriction $\Phi|_{((c_i \cup c_j) \times I) \# \bigcup_{j=1}^m F'_j}$ of the function Φ presenting the homology between Γ_1 and Γ_2 defines a link-homology between $\Gamma_1|_{c_i \cup c_j}$ and $\Gamma_2|_{c_i \cup c_j}$, where $\{F'_j\}_{j=1}^m$ is the set of elements of $\{F_i\}_{i=1}^n$ attached to $(c_i \cup c_j) \times I$. This shows that $\mathrm{lk}(\Gamma_1(c_i), \Gamma_1(c_j)) = \mathrm{lk}(\Gamma_2(c_i), \Gamma_2(c_j))$.

Now, we prove the "if" part. If $\mathrm{lk}(\Gamma_1(c_i), \Gamma_1(c_j)) = \mathrm{lk}(\Gamma_2(c_i), \Gamma_2(c_j))$ for any c_i, c_j in $\{c_1, \cdots, c_n, c_0\}$ with $c_i \cap c_j = \emptyset$, then by the proof of Theorem 1, there exsit embeddings $f_{\Gamma_k}: S_0 \to \mathbf{R}^3$ $(k=1, 2)$ extending $\Gamma_k \circ \Gamma_0^{-1}: \Gamma_0(G) \to \mathbf{R}^3$ such that $\langle \Gamma_1(c_s), \Gamma_1(c_t) \rangle_{S_1} = \langle \Gamma_2(c_s), \Gamma_2(c_t) \rangle_{S_2}$ for all boundary-cycle pairs c_s, c_t, where $S_k = f_{\Gamma_k}(S_0)$. Since $\{c_1, \cdots, c_n\}$ generates $H_1(G; \mathbf{Z})$, for any $\alpha, \beta \in H_1(G; \mathbf{Z})$,

$$\langle \Gamma_1(\alpha), \Gamma_1(\beta) \rangle_{S_1} = \langle \Gamma_2(\alpha), \Gamma_2(\beta) \rangle_{S_2}\,. \tag{3.1}$$

Let T be a spanning tree of $\Gamma_0(G)$ and $\{e_1, e_2, \cdots, e_n\}$ the set of the closures of components of $\Gamma_0(G) - T$. Let α_i $(i=1, 2, \cdots, n)$ be the 1-cycle in G represented by the subgraph $\Gamma_0^{-1}(T \cup e_i)$ of G. For a regular neighborhood S_T of T in S_0, we may assume that $f_{\Gamma_1}|_{S_T} = f_{\Gamma_2}|_{S_T}$, and $D_T = f_{\Gamma_k}(S_T)$ $(k=1, 2)$ is embedded in $\mathbf{R}^2 = \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$, as illustraded in Figure 6. Let B_T be a regular neighborhood of $f_{\Gamma_1}(T) = f_{\Gamma_2}(T)$ in $\mathbf{R}^3$ with

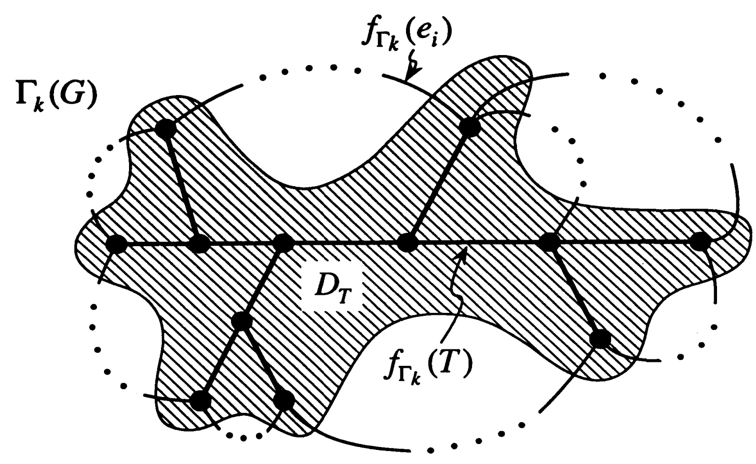

FIGURE 6

$B_T \cap \mathbf{R}^2 = D_T$, and $\Gamma_1(G) \cap B_T = \Gamma_2(G) \cap B_T \subset D_T$. It is not hard to see that there are mutually disjoint embeddings $b_i: I \times I \to B_T$ $(i=1, 2, \cdots, n)$ such that, for $k=1, 2$,

(i) $b_i(I \times I) \cap \Gamma_k(G) = b_i(I \times \partial I) \subset f_{\Gamma_k}(e_i)$, and

(ii) $(\Gamma_k(G) - \bigcup_{i=1}^n b_i(I \times \partial I)) \cup (\bigcup_{i=1}^n b_i(\partial I \times I))$ is the union of a graph $\Gamma'_{0k}(G)$ with $\Gamma'_{01}(G) = \Gamma'_{02}(G)$ and an n-component link $\Gamma'_k(L_1 \cup \cdots \cup L_n)$ such that $\Gamma'_{0k}(G) \cup \Gamma'_k(L_1 \cup \cdots \cup L_n)$ is ambient isotopic in $\mathbf{R}^3$ to $\bar{\Gamma}'_{0k}(G) \cup \bar{\Gamma}'_k(L_1 \cup \cdots \cup L_n)$ with $\bar{\Gamma}'_{0k}(G) \subset D_T$ and $\bar{\Gamma}'_k(L_1 \cup \cdots \cup L_n) \cap B_T = \emptyset$, see Figure 7. Then, $\mathrm{lk}(\Gamma'_k(L_i), \Gamma'_k(L_j)) = \langle \Gamma_k(\alpha_i), \Gamma_k(\alpha_j) \rangle_{S_k}$ for $k=1, 2$. Since, by (3.1), $\langle \Gamma_1(\alpha_i), \Gamma_1(\alpha_j) \rangle_{S_1} = \langle \Gamma_2(\alpha_i), \Gamma_2(\alpha_j) \rangle_{S_2}$, we have $\mathrm{lk}(\Gamma'_1(L_i), \Gamma'_1(L_j)) = \mathrm{lk}(\Gamma'_2(L_i), \Gamma'_2(L_j))$. This implies that $\Gamma'_{01}(G) \cup \Gamma'_1(L_1 \cup \cdots \cup L_n)$ is homologous to $\Gamma'_{02}(G) \cup \Gamma'_2(L_1 \cup \cdots \cup L_n)$. Let E be a 2-complex which is locally-flatly and properly embedded in $\mathbf{R}^3 \times [1/3, 2/3]$ and realizes this homology. In particular, E satisfies $E \cap (\mathbf{R}^3 \times \{1/3\}) = \Gamma'_{01}(G) \cup \Gamma'_1(L_1 \cup \cdots \cup L_n)$ and $E \cap (\mathbf{R}^3 \times \{2/3\}) = \Gamma'_{02}(G) \cup \Gamma'_2(L_1 \cup \cdots \cup L_n)$. Consider the 2-complexes $E_1 \subset \mathbf{R}^3 \times [0, 1/3]$, $E_2 \subset \mathbf{R}^3 \times [2/3, 1]$ defined as follows:

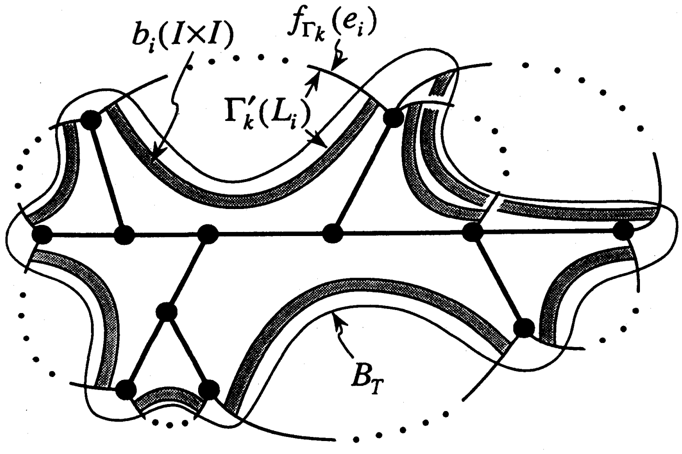

FIGURE 7

$$E_1 \cap (\mathbf{R}^3 \times \{t\}) = \begin{cases} \Gamma_1(G) & \text{if } 0 \le t < 1/6 \\ \Gamma_1(G) \cup \bigcup_{i=1}^n b_i(I \times I) & \text{if } t = 1/6 \\ \Gamma'_{01}(G) \cup \Gamma'_1(L_1 \cup L_2 \cup \cdots \cup L_n) & \text{if } 1/6 < t \le 1/3, \end{cases}$$

$$E_2 \cap (\mathbf{R}^3 \times \{t\}) = \begin{cases} \Gamma'_{02}(G) \cup \Gamma'_2(L_1 \cup L_2 \cup \cdots \cup L_n) & \text{if } 2/3 \le t < 5/6 \\ \Gamma_2(G) \cup \bigcup_{i=1}^n b_i(I \times I) & \text{if } t = 5/6 \\ \Gamma_2(G) & \text{if } 5/6 < t \le 1. \end{cases}$$

Then, the union $E_1 \cup E \cup E_2$ in $\mathbf{R}^3 \times [0, 1]$ determines the homology between Γ_1 and Γ_2. This completes the proof. □

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