TOKYO J. MATH.
VOL. 20, NO. 2, 1997

On the Variance of the Feasible Weighted Least Squares Estimator

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(Communicated by T. Suzuki)

1. Introduction.

Let X_{ij} $(j=1, 2, \cdots, n_i)$ be mutually independent random variables distributed according to $N(\theta, \sigma_i^2)$ $(i=1, \cdots, k)$, where θ and $\{\sigma_i^2\}_{i=1}^k$ are unknown parameters. We consider an unbiased estimator of θ defined by

$$\hat{\theta}_k=\left\{\sum_{i=1}^k c_i n_i \hat{\sigma}_i^{-2}\bar{X}_i\right\}\left\{\sum_{i=1}^k c_i n_i \hat{\sigma}_i^{-2}\right\}^{-1}, \tag{1.1}$$

where $\bar{X}_i=\sum_{j=1}^{n_i} X_{ij}/n_i$, $\hat{\sigma}_i^2=\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2/(n_i-1)$ $(i=1, \cdots, k)$ and $\{c_i\}_{i=1}^k$ is a sequence of positive constants.

For $k\geq 3$ and $n_i\geq 6$ $(i=1, \cdots, k)$, Shinozaki (1978) proved that $V[\hat{\theta}_k]\leq \min_{1\leq i\leq k} V[\bar{X}_i]$ holds if and only if $c_q c_p^{-1}\leq 2(n_p-1)(n_q-5)(n_p+1)^{-1}(n_q-1)^{-1}$ for any $p\neq q$. For example, if we choose $c_i=(n_i-5)(n_i-1)^{-1}$ $(i=1, \cdots, k)$, this condition is equivalent to $n_i\geq 11$ $(i=1, \cdots, k)$. It follows from this that the combined estimator $\hat{\theta}_k$ is preferable to each $\bar{X}_i$ when $c_i=(n_i-5)(n_i-1)^{-1}$ and $n_i\geq 11$ $(i=1, \cdots, k)$.

Now we consider the accuracy of $\hat{\theta}_k$ when $n_i\geq 6$ $(i=1, \cdots, k)$. Though we would like to evaluate the variance of $\hat{\theta}_k$, it seems to be difficult to obtain its exact expression. When $k\to +\infty$, however, Takeuchi (1994) indicated that the ratio of $V[\hat{\theta}_k]$ to the Cramér-Rao lower bound $m^{-1}\{\sum_{i=1}^k \sigma_i^{-2}\}^{-1}$ is greater than or equal to $(m-3)(m-5)^{-1}$ when $n_i=m\geq 6$ $(i=1, \cdots, k)$.

In this paper, the limiting variance of $\hat{\theta}_k$-type estimator is obtained. The lists of notations and conditions are given in Section 2.1. In Section 2.2, the asymptotic properties of $V[\hat{\theta}_k]$ are described. Theorem 2.1 asserts that the ratio of $V[\hat{\theta}_k]$ to the Cramér-Rao lower bound satisfies some limit relation. Theorem 2.2 ensures the existence of the limiting variance of $\hat{\theta}_k$ and Theorem 2.3 gives the optimal estimator in the sense that it attains the minimum of the limiting variance in some class of estimators. In Section 2.3, three auxiliary lemmas are proved. The proofs of Theorem 2.1–Theorem

Received April 19, 1996
Revised September 13, 1996

2.3 are carried out in Section 2.4. In Section 3, the results of Section 2.2 are applied to the feasible weighted least squares estimator which is given by putting $c_i=1$ $(i=1, \cdots, k)$ in (1.1).

2. Some asymptotic properties of $\hat{\theta}_k$.

2.1. Notations and conditions. We use the following notations and conditions.

NOTATIONS.

$d_l=\Gamma^{-1}(l/2)$ (where $\Gamma(\cdot)$ denotes the gamma function and $l>0$),
$\alpha_i=(n_i-5)^{-1}(n_i-3)^{-1}$, $\beta_i=c_i^{-2}n_i^{-1}(n_i-1)^{-2}\sigma_i^2$, $\gamma_i=c_in_i(n_i-1)$,
$a_k=\sum_{i=1}^k n_i\sigma_i^{-2}$, $b_{i,k}=2^{-1}(\alpha_i+\beta_ia_k)^{-1/2}$, $c_i^*=(n_i-5)(n_i-1)^{-1}$,
$B_i=(n_i-5)(n_i-3)^{-1}n_i\sigma_i^{-2}$,
$\{\hat{\sigma}_i^{-2p}\}_{i=1}^k$ $(p=1,2)$: a sequence of independent random variables defined by

$$\hat{\sigma}_i^{-2p}=\begin{cases}(\hat{\sigma}_i^2)^{-p} & \text{if } \hat{\sigma}_i^2>0\,,\\ 1 & \text{if } \hat{\sigma}_i^2=0\,,\end{cases}$$

$\hat{\theta}_k^*=\{\sum_{i=1}^k c_i^*n_i\hat{\sigma}_i^{-2}\bar{X}_i\}\{\sum_{i=1}^k c_i^*n_i\hat{\sigma}_i^{-2}\}^{-1}$, $\hat{\theta}_k^c=\{\sum_{i=1}^k n_i\hat{\sigma}_i^{-2}\bar{X}_i\}\{\sum_{i=1}^k n_i\hat{\sigma}_i^{-2}\}^{-1}$,
$\{Y_i\}_{i=1}^k$: a sequence of independent inverse chi-square random variables with n_i-1 degrees of freedom defined by $Y_i=(n_i-1)^{-1}\sigma_i^2\hat{\sigma}_i^{-2}$,
$\{Z_i\}_{i=1}^k$: a sequence of independent random variables defined by $Z_i=\beta_i^{-1}(Y_i^2-\alpha_i)$,
$\{W_{1k}\}_{k\in N}$: a sequence of random variables defined by $W_{1k}=a_k^{-1}\sum_{i=1}^k c_i^2n_i\sigma_i^2\hat{\sigma}_i^{-4}=a_k^{-1}\sum_{i=1}^k\beta_i^{-1}Y_i^2$,
$\{W_{2k}\}_{k\in N}$: a sequence of random variables defined by $W_{2k}=\{a_k^{-1}\sum_{i=1}^k c_in_i\hat{\sigma}_i^{-2}\}^2$,
$E[\cdot]$, $V[\cdot]$: the expectation and the variance with respect to the product probability measure when θ and $\{\sigma_i^2\}_{i\in N}$ are given,
$A_{1k}=E[W_{1k}]=a_k^{-1}\sum_{i=1}^k(c_i/c_i^*)^2B_i=a_k^{-1}\sum_{i=1}^k\alpha_i\beta_i^{-1}$,
$A_{2k}=\{E[W_{2k}^{1/2}]\}^2=\{a_k^{-1}\sum_{i=1}^k(c_i/c_i^*)B_i\}^2$,
$\xrightarrow{P}$: convergence in probability.

REMARK 1. By the definition of $\{\hat{\sigma}_i^{-2p}\}_{i=1}^k$ $(p=1,2)$, the random variables related to them are well-defined for any value of X_{ij} $(j=1,2,\cdots,n_i;\, i=1,\cdots,k)$. Since $\bigcup_{i=1}^k\{\hat{\sigma}_i^2=0\}$ is the set of Lebesgue measure zero for all $k\in N$, essentially we regard $\{\hat{\sigma}_i^{-2p}\}_{i=1}^k$ as $\{(\hat{\sigma}_i^2)^{-p}\}_{i=1}^k$ $(p=1,2)$ from now on.

CONDITIONS.

(Bdd) There exist positive constants n_U, c_L, σ_L^2 and σ_U^2 such that
$$6\le n_i\le n_U,\quad c_L\le c_i\le 1\quad\text{and}\quad \sigma_L^2\le\sigma_i^2\le\sigma_U^2\qquad\text{for all } i\in\{1,\cdots,k\}\,.$$

(C.1) $n_i\ge 6$ $(i=1,\cdots,k)$.

(C.2) There exists a positive constant c_L such that

$$c_L \le c_i \le 1 \qquad \text{for all} \quad i \in \{1, \cdots, k\} .$$

(C.3) $a_k \to +\infty \qquad (\text{as } k \to +\infty)$.

(C.4) $\sum_{i=1}^{k} d_{n_i+1} b_{i,k}^{(n_i-1)/2} = o(1) \qquad (\text{as } k \to +\infty)$.

(C.5) $\sum_{i=1}^{k} \{\alpha_i^2 I_{\{n_i \ge 10\}}(n_i) + I_{\{n_i=9\}}(n_i) \cdot \log b_{i,k}^{-1} + I_{\{6 \le n_i \le 8\}}(n_i) \cdot b_{i,k}^{(n_i-9)/2}\} \beta_i^{-2} = o(a_k^2)$

(as $k \to +\infty$).

(C.6) $\max_{1 \le i \le k} \beta_i^{-1} = o(a_k) \qquad (\text{as } k \to +\infty)$.

(C.7) There exist α (>1), p (>1) and q (>1) such that

$$4\alpha p < \min_{1 \le i \le k} (n_i - 1) , \quad p^{-1} + q^{-1} = 1 \quad \text{and}$$

$$\sup_{k \in N} \left[k^{-1-\alpha} a_k^{\alpha} \left\{ \sum_{i=1}^{k} \beta_i^{-\alpha p} d_{n_i-1} d_{n_i-1-4\alpha p}^{-1} \right\}^{1/p} \left\{ \sum_{i=1}^{k} \gamma_i^{-2\alpha q} \sigma_i^{4\alpha q} d_{n_i-1} d_{n_i-1+4\alpha q}^{-1} \right\}^{1/q} \right] < +\infty .$$

REMARK 2. The condition (Bdd) implies (C.1)–(C.7).

REMARK 3. The conditions (C.1)–(C.7) don't necessarily require the boundedness of $\{\sigma_i^2\}_{i \in N}$. For example, we consider the simple case $n_i = m \ge 6$ and $c_i = 1$ $(i = 1, \cdots, k)$. Let δ (>0), α (>1) and p (>1) be constants such that $\delta < 1$ and $\alpha p < (m-1)/4$. And we put $\sigma_i^2 = i^{\delta}$ $(i = 1, \cdots, k)$. Then (C.3)–(C.6) are satisfied. And moreover, if $\delta < (\alpha p)^{-1}$, (C.7) is satisfied.

2.2. Main theorems.

THEOREM 2.1. *Assume that* (C.1)–(C.6) *hold. Then*

$$\liminf_{k \to +\infty} \{a_k V[\hat{\theta}_k] - A_{1k} A_{2k}^{-1}\} \ge 0 . \tag{2.1}$$

REMARK 4. Since we can regard $a_k V[\hat{\theta}_k]$ as the ratio of $V[\hat{\theta}_k]$ to the Cramér-Rao lower bound a_k^{-1}, we know that the assertion of Theorem 2.1 is analogous to the one indicated by Takeuchi (1994).

THEOREM 2.2. *Assume that* (C.1)–(C.7) *hold. Then*

$$V[\hat{\theta}_k] = a_k^{-1} A_{1k} A_{2k}^{-1} + o(a_k^{-1}) \qquad (\textit{as } k \to +\infty) . \tag{2.2}$$

REMARK 5. In the relation (2.2), $a_k^{-1} A_{1k} A_{2k}^{-1}$ represents the limiting variance of $\hat{\theta}_k$ as $k \to +\infty$.

THEOREM 2.3. *Assume that the condition* (Bdd) *holds. Then the limit relation* (2.2) *holds and*

(2.3) $$a_k^{-1}A_{1k}A_{2k}^{-1} \geq \{\sum_{i=1}^k B_i\}^{-1}.$$

In (2.3), *the equality holds if and only if* $c_1/c_1^* = \cdots = c_k/c_k^*$. *In this case*, (2.2) *is written as*

(2.4) $$V[\hat{\theta}_k^*] = \{\sum_{i=1}^k B_i\}^{-1} + o(a_k^{-1}) \qquad (\text{as } k \to +\infty).$$

REMARK 6. Under the condition (Bdd), we can consider the class of estimators $\{\hat{\theta}_k\}$ generated by $\{c_i\}_{i=1}^k$. From Theorem 2.3 we know that $\hat{\theta}_k^*$ is asymptotically optimal in the sense that it attains the minimum of the limiting variance in the class $\{\hat{\theta}_k\}$ above.

REMARK 7. As a by-product of Theorem 2.3, under the condition (Bdd) we can show

$$\liminf_{k \to +\infty} \{V[\hat{\theta}_k] - V[\hat{\theta}_k^*]\} \geq 0.$$

In fact, from (2.2), (2.3) and (2.4) we obtain

$$\begin{aligned}
&\liminf_{k \to +\infty} \{V[\hat{\theta}_k] - V[\hat{\theta}_k^*]\} \\
&\geq \liminf_{k \to +\infty} \{V[\hat{\theta}_k] - a_k^{-1}A_{1k}A_{2k}^{-1}\} + \liminf_{k \to +\infty} [a_k^{-1}A_{1k}A_{2k}^{-1} - \{\sum_{i=1}^k B_i\}^{-1}] \\
&\quad + \liminf_{k \to +\infty} [\{\sum_{i=1}^k B_i\}^{-1} - V[\hat{\theta}_k^*]] \\
&= \liminf_{k \to +\infty} [a_k^{-1}A_{1k}A_{2k}^{-1} - \{\sum_{i=1}^k B_i\}^{-1}] \geq 0.
\end{aligned}$$

2.3. Auxiliary lemmas. To prove the theorems, we make use of the following lemmas.

LEMMA 2.4. *Under the condition* (C.1), $A_{1k}A_{2k}^{-1}$ *attains its minimum with respect to* $(c_1, \cdots, c_k)$ *if and only if*

$$c_1/c_1^* = \cdots = c_k/c_k^*.$$

PROOF. Putting $f((c_1, \cdots, c_k)) = a_k^{-1}A_{1k}A_{2k}^{-1}$, we regard $a_k^{-1}A_{1k}A_{2k}^{-1}$ as a function of $(c_1, \cdots, c_k)$. Then, by using the Schwarz inequality, we have

$$\begin{aligned}
f((c_1, \cdots, c_k)) &= \frac{\sum_{i=1}^k (c_i/c_i^*)^2 B_i}{\{\sum_{i=1}^k (c_i/c_i^*) B_i\}^2} \geq \frac{\sum_{i=1}^k (c_i/c_i^*)^2 B_i}{\{\sum_{i=1}^k (c_i/c_i^*)^2 B_i\} \sum_{i=1}^k B_i} \\
&= \{\sum_{i=1}^k B_i\}^{-1} = f((c_1^*, \cdots, c_k^*)).
\end{aligned}$$

The equality holds if and only if $c_1/c_1^* = \cdots = c_k/c_k^*$. □

LEMMA 2.5. *Assume that* (C.1)–(C.6) *hold. Then*

$$W_{1k}-A_{1k}\xrightarrow{p}0 \qquad (as\ k\to+\infty)\,.$$

PROOF. Let ε (<1) be an arbitrary fixed positive number. By (C.6), there exists a positive integer k_0 such that

$$\min_{1\le i\le k}\beta_i a_k>4\varepsilon^{-4} \qquad \text{for all} \quad k\ge k_0\,. \tag{2.5}$$

In the following, we assume $k\ge k_0$ and note that this assumption ensures $b_{i,k}^{-1}>4$.

Now, to prove the lemma, noting (C.3) is satisfied, we verify that the following conditions are satisfied [see Petrov (1995), page 131]:

(P.1) $\sum_{i=1}^{k} P(|Z_i|\ge a_k)=o(1) \qquad (\text{as } k\to+\infty)$,

(P.2) $\sum_{i=1}^{k}\left\{\int_{|z|<a_k} z^2dF_i(z)-\left(\int_{|z|<a_k} zdF_i(z)\right)^2\right\}=o(a_k^2) \qquad (\text{as } k\to+\infty)$,

(P.3) $\sum_{i=1}^{k}\int_{|z|<a_k} zdF_i(z)=o(a_k) \qquad (\text{as } k\to+\infty)$,

where F_i denotes the distribution of Z_i. In practice, we follow the next steps:
Step 1. we show that (P.1) is satisfied under (C.1), (C.4) and (C.6).
Step 2. we show that (P.2) is satisfied under (C.1), (C.5) and (C.6).
Step 3. we show that (P.3) is satisfied under (C.1), (C.2) and (C.6).

Step 1. For any $c>0$, we can show that

$$\begin{aligned} P(Y_i\ge c)&=\int_c^{\infty} 2d_{n_i-1}\left(\frac{1}{2y}\right)^{(n_i-1)/2+1}\exp\left(-\frac{1}{2y}\right)dy \\ &=d_{n_i-1}\int_0^{1/2c} x^{(n_i-1)/2-1}e^{-x}dx\le d_{n_i-1}\int_0^{1/2c} x^{(n_i-1)/2-1}dx \\ &=2(n_i-1)^{-1}d_{n_i-1}(2c)^{-(n_i-1)/2}=d_{n_i+1}(2c)^{-(n_i-1)/2}\,. \end{aligned} \tag{2.6}$$

Therefore, from (2.5) and (2.6), we see for all $i\in\{1,\cdots,k\}$ that

$$\begin{aligned} P(|Z_i|\ge a_k)&=P(|Y_i^2-\alpha_i|\ge\beta_i a_k) \\ &=P(Y_i^2\ge\alpha_i+\beta_i a_k)+P(Y_i^2\le\alpha_i-\beta_i a_k) \\ &=P(Y_i^2\ge\alpha_i+\beta_i a_k)=P(Y_i\ge 2^{-1}b_{i,k}^{-1}) \\ &\le d_{n_i+1}\cdot(2\cdot 2^{-1}b_{i,k}^{-1})^{-(n_i-1)/2}=d_{n_i+1}\cdot b_{i,k}^{(n_i-1)/2}\,, \end{aligned}$$

and accordingly we have

$$\sum_{i=1}^{k}P(|Z_i|\ge a_k)=o(1) \qquad (\text{as } k\to+\infty)\,.$$

Step 2. First of all, we show

$$\int_{|z|<a_k} z^2 dF_i(z) < \beta_i^{-2}\left\{\alpha_i^2 + 16^{-1} d_{n_i-1}\int_{b_{i,k}}^{\infty} x^{(n_i-11)/2} e^{-x}dx\right\}. \tag{2.7}$$

$$\begin{aligned}
&\int_{|z|<a_k} z^2 dF_i(z)\\
&=\int_{|z|<a_k} z^2\cdot 2d_{n_i-1}\beta_i\{2^{-1}(\alpha_i+\beta_i z)^{-1/2}\}^{(n_i+3)/2}\exp\{-2^{-1}(\alpha_i+\beta_i z)^{-1/2}\} I_{\{z>-\beta_i^{-1}\alpha_i\}}(z)dz\\
&=2d_{n_i-1}\beta_i\int_{-\beta_i^{-1}\alpha_i}^{a_k} z^2\{2^{-1}(\alpha_i+\beta_i z)^{-1/2}\}^{(n_i+3)/2}\exp\{-2^{-1}(\alpha_i+\beta_i z)^{-1/2}\}dz\\
&=d_{n_i-1}\beta_i^{-2}\int_{b_{i,k}}^{\infty}(4^{-1}x^{-2}-\alpha_i)^2 x^{(n_i-3)/2}e^{-x}dx\\
&=d_{n_i-1}\beta_i^{-2}\int_{b_{i,k}}^{\infty}(16^{-1}x^{(n_i-11)/2}-2^{-1}\alpha_i x^{(n_i-7)/2}+\alpha_i^2 x^{(n_i-1)/2-1})e^{-x}dx\\
&<d_{n_i-1}\beta_i^{-2}\left\{\alpha_i^2\int_{b_{i,k}}^{\infty}x^{(n_i-1)/2-1}e^{-x}dx+16^{-1}\int_{b_{i,k}}^{\infty}x^{(n_i-11)/2}e^{-x}dx\right\}\\
&<d_{n_i-1}\beta_i^{-2}\left\{\alpha_i^2 d_{n_i-1}^{-1}+16^{-1}\int_{b_{i,k}}^{\infty}x^{(n_i-11)/2}e^{-x}dx\right\}\\
&=\beta_i^{-2}\left\{\alpha_i^2+16^{-1}d_{n_i-1}\int_{b_{i,k}}^{\infty}x^{(n_i-11)/2}e^{-x}dx\right\}.
\end{aligned}$$

Next we verify the following inequality:

$$\int_{b_{i,k}}^{\infty} x^{(n_i-11)/2}e^{-x}dx \le \begin{cases} d_{n_i-9}^{-1} & \text{if } n_i\ge 10,\\ \exp(-b_{i,k})\log(1+b_{i,k}^{-1}) & \text{if } n_i=9,\\ 2(9-n_i)^{-1}b_{i,k}^{(n_i-9)/2}\exp(-b_{i,k}) & \text{if } 6\le n_i\le 8. \end{cases} \tag{2.8}$$

For $n_i\ge 10$, we have

$$\int_{b_{i,k}}^{\infty} x^{(n_i-11)/2}e^{-x}dx \le \int_0^{\infty} x^{(n_i-11)/2}e^{-x}dx = d_{n_i-9}^{-1}.$$

For $n_i=9$, noting that

$$\log x \le \log(b_{i,k}+1)+(b_{i,k}+1)^{-1}x-1 \qquad \text{for all} \quad x>0,$$

we obtain

$$\int_{b_{i,k}}^{\infty} x^{-1}e^{-x}dx = \exp(-b_{i,k})\log b_{i,k}^{-1} + \int_{b_{i,k}}^{\infty} e^{-x}\log x\,dx$$

$$\leq \exp(-b_{i,k})\log b_{i,k}^{-1} + \int_{b_{i,k}}^{\infty} e^{-x}\{\log(b_{i,k}+1)+(b_{i,k}+1)^{-1}x-1\}dx$$
$$= \exp(-b_{i,k})\log b_{i,k}^{-1} + \exp(-b_{i,k})\log(b_{i,k}+1)$$
$$= \exp(-b_{i,k})\log(1+b_{i,k}^{-1}) .$$

For $6 \leq n_i \leq 8$, we have

$$\int_{b_{i,k}}^{\infty} x^{(n_i-11)/2} e^{-x}dx \leq \exp(-b_{i,k}) \int_{b_{i,k}}^{\infty} x^{(n_i-11)/2}dx = 2(9-n_i)^{-1} b_{i,k}^{(n_i-9)/2} \exp(-b_{i,k}) .$$

The following is the immediate consequence of (2.7) and (2.8), where M_1, M_2 and M_3 are positive constants independent of $i \in \{1, \cdots, k\}$:

$$\int_{|z|<a_k} z^2 dF_i(z) < \begin{cases} M_1 \cdot \alpha_i^2 \beta_i^{-2} & \text{if } n_i \geq 10 , \\ M_2 \cdot \beta_i^{-2} \log b_{i,k}^{-1} & \text{if } n_i = 9 , \\ M_3 \cdot \beta_i^{-2} b_{i,k}^{(n_i-9)/2} & \text{if } 6 \leq n_i \leq 8 . \end{cases}$$

Thus we know that (P.2) is satisfied.

Step 3. First of all, we show

$$\left| \int_{b_{i,k}}^{\infty} x^{(n_i-1)/2-1} e^{-x}dx - d_{n_i-1}^{-1} \right| < \left| \int_{b_{i,k}}^{\infty} x^{(n_i-5)/2-1} e^{-x}dx - d_{n_i-5}^{-1} \right| < \varepsilon . \tag{2.9}$$

By (2.5), we have $b_{i,k} < \varepsilon^2/4 < 1$. Hence we obtain

$$\left| \int_{b_{i,k}}^{\infty} x^{(n_i-1)/2-1} e^{-x}dx - d_{n_i-1}^{-1} \right| = \int_0^{b_{i,k}} x^{(n_i-1)/2-1} e^{-x}dx \leq b_{i,k}^2 \int_0^{b_{i,k}} x^{(n_i-5)/2-1} e^{-x}dx$$
$$< \left| \int_{b_{i,k}}^{\infty} x^{(n_i-5)/2-1} e^{-x}dx - d_{n_i-5}^{-1} \right| = \int_0^{b_{i,k}} x^{(n_i-5)/2-1} e^{-x}dx$$
$$\leq \int_0^{b_{i,k}} x^{(n_i-5)/2-1}dx = 2(n_i-5)^{-1} b_{i,k}^{(n_i-5)/2} \leq 2b_{i,k}^{1/2} < \varepsilon .$$

Now, in the same way as (2.7), we have

$$\int_{|z|<a_k} z dF_i(z) = d_{n_i-1}\beta_i^{-1} \int_{b_{i,k}}^{\infty} (4^{-1}x^{-2} - \alpha_i) x^{(n_i-1)/2-1} e^{-x}dx$$
$$= d_{n_i-1}\beta_i^{-1} \left\{ 4^{-1} \int_{b_{i,k}}^{\infty} x^{(n_i-5)/2-1} e^{-x}dx - \alpha_i \int_{b_{i,k}}^{\infty} x^{(n_i-1)/2-1} e^{-x}dx \right\} .$$

From this and (2.9), we obtain

$$d_{n_i-1}^{-1}\beta_i \int_{|z|<a_k} z dF_i(z) < 4^{-1}(d_{n_i-5}^{-1}+\varepsilon) - \alpha_i(d_{n_i-1}^{-1}-\varepsilon)$$

$$=4^{-1}\left\{\Gamma\left(\frac{n_i-5}{2}\right)+\varepsilon-\Gamma\left(\frac{n_i-5}{2}\right)+4\alpha_i\varepsilon\right\}=4^{-1}(1+4\alpha_i)\varepsilon$$

and

$$d_{n_i-1}^{-1}\beta_i\int_{|z|<a_k} zdF_i(z)>-4^{-1}(1+4\alpha_i)\varepsilon\,,$$

and accordingly we have

$$a_k^{-1}\sum_{i=1}^{k}\left|\int_{|z|<a_k} zdF_i(z)\right|<a_k^{-1}\sum_{i=1}^{k}4^{-1}(1+4\alpha_i)d_{n_i-1}\beta_i^{-1}\varepsilon$$

$$=\varepsilon\cdot a_k^{-1}\sum_{i=1}^{k}(1+4\alpha_i)\cdot d_{n_i-5}\cdot\alpha_i(n_i-1)^2\cdot c_i^2\cdot n_i\sigma_i^{-2}\leq\varepsilon M_4\cdot a_k^{-1}\cdot a_k=\varepsilon M_4\,,$$

where M_4 is a positive constant independent of $i\in\{1,\cdots,k\}$. Thus we know that (P.3) is satisfied and we have just completed the proof. □

LEMMA 2.6. *Assume that* (C.1), (C.2), (C.5) *and* (C.6) *hold. Then*

$$W_{2k}-A_{2k}\xrightarrow{p}0\qquad(as\ k\to+\infty)\,.$$

PROOF. Let ε and k_0 be the same as in the proof of Lemma 2.5. And we assume $k\geq k_0$ in the following.

Now, by using the Chebyshev inequality, we have

$$P\{|W_{2k}^{1/2}-A_{2k}^{1/2}|>\varepsilon\}=P\left\{\left|a_k^{-1}\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i-a_k^{-1}\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}(n_i-3)^{-1}\right|>\varepsilon\right\}$$

$$\leq\varepsilon^{-2}V\left[a_k^{-1}\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right]=\varepsilon^{-2}a_k^{-2}\sum_{i=1}^{k}\gamma_i^2\sigma_i^{-4}V[Y_i]$$

$$=2\varepsilon^{-2}a_k^{-2}\sum_{i=1}^{k}\gamma_i^2\sigma_i^{-4}(n_i-3)^{-2}(n_i-5)^{-1}\leq\varepsilon^{-2}M_5\cdot a_k^{-2}\sum_{i=1}^{k}\alpha_i^2\beta_i^{-2}$$

$$\leq\varepsilon^{-2}M_5\cdot a_k^{-2}\sum_{i=1}^{k}\{\alpha_i^2I_{\{n_i\geq10\}}(n_i)+I_{\{n_i=9\}}(n_i)\cdot\log b_{i,k}^{-1}+I_{\{6\leq n_i\leq8\}}(n_i)\cdot b_{i,k}^{(n_i-9)/2}\}\beta_i^{-2}$$

$$=o(1)\qquad(\text{as } k\to+\infty)\,,$$

where M_5 is a positive constant independent of $i\in\{1,\cdots,k\}$. Hence we obtain $W_{2k}^{1/2}-A_{2k}^{1/2}\xrightarrow{p}0$ (as $k\to+\infty$). From this, noting that (C.1) and (C.2) ensure $A_{2k}^{1/2}=O(1)$ (as $k\to+\infty$), we can conclude that $W_{2k}-A_{2k}\xrightarrow{p}0$ (as $k\to+\infty$). □

2.4. Proofs of the main theorems. Before proceeding to the proofs we show

$$V[\hat\theta_k]=E\left[\left\{\sum_{i=1}^{k}c_i^2n_i\sigma_i^2\hat\sigma_i^{-4}\right\}\left\{\sum_{i=1}^{k}c_in_i\hat\sigma_i^{-2}\right\}^{-2}\right].\tag{2.10}$$

Using the Schwarz inequality, we have

$$|\hat{\theta}_k|^2=\left\{\sum_{i=1}^{k} c_i n_i \hat{\sigma}_i^{-2}\bar{X}_i\right\}^2\left\{\sum_{i=1}^{k} c_i n_i \hat{\sigma}_i^{-2}\right\}^{-2}=\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\bar{X}_i\right\}^2\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-2}$$
$$\le\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\bar{X}_i^2\right\}\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-1}.$$

Since $\bar{X}_i$'s and Y_i's are mutually independent, we see

$$E[|\hat{\theta}_k|^2]\le E\left[\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-1}\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_iE[\bar{X}_i^2\mid Y_1,\cdots,Y_k]\right]$$
$$=E\left[\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-1}\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i(n_i^{-1}\sigma_i^2+\theta^2)\right]$$
$$=\theta^2+E\left[\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-1}\sum_{i=1}^{k}\gamma_i n_i^{-1}Y_i\right].$$

By using the Hölder inequality, we can verify the finiteness of $E[|\hat{\theta}_k|^2]$. In fact,

$$E\left[\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-1}\sum_{i=1}^{k}\gamma_i n_i^{-1}Y_i\right]\le k^{-2}E\left[\left\{\sum_{i=1}^{k}\gamma_i^{-1}\sigma_i^2Y_i^{-1}\right\}\sum_{i=1}^{k}\gamma_i n_i^{-1}Y_i\right]$$
$$\le k^{-2}\left\{E\left[\left\{\sum_{i=1}^{k}\gamma_i^{-1}\sigma_i^2Y_i^{-1}\right\}^a\right]\right\}^{1/a}\left\{E\left[\left\{\sum_{i=1}^{k}\gamma_i n_i^{-1}Y_i\right\}^b\right]\right\}^{1/b}$$
$$\le k^{-2}\left\{E\left[k^{a-1}\sum_{i=1}^{k}\gamma_i^{-a}\sigma_i^{2a}Y_i^{-a}\right]\right\}^{1/a}\left\{E\left[k^{b-1}\sum_{i=1}^{k}\gamma_i^b n_i^{-b}Y_i^b\right]\right\}^{1/b}$$
$$=k^{-1}\left\{\sum_{i=1}^{k}\gamma_i^{-a}\sigma_i^{2a}d_{n_i-1}d_{n_i-1+2a}^{-1}\right\}^{1/a}\left\{\sum_{i=1}^{k}\gamma_i^b n_i^{-b}d_{n_i-1}d_{n_i-1-2b}^{-1}\right\}^{1/b}<+\infty,$$

where $a>1$, $a^{-1}+b^{-1}=1$ and $0<2b<\min_{1\le i\le k}(n_i-1)$. Hence we have

(2.11) $$V[\hat{\theta}_k]=V[E[\hat{\theta}_k\mid Y_1,\cdots,Y_k]]+E[V[\hat{\theta}_k\mid Y_1,\cdots,Y_k]].$$

Noting that

$$E[\hat{\theta}_k\mid Y_1,\cdots,Y_k]=\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-1}\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\cdot E[\bar{X}_i\mid Y_1,\cdots,Y_k]=\theta$$

and

$$V[\hat{\theta}_k\mid Y_1,\cdots,Y_k]=E[|\hat{\theta}_k|^2\mid Y_1,\cdots,Y_k]-\theta^2$$
$$=\left\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\right\}^{-2}\left\{\sum_{i=1}^{k}\gamma_i^2\sigma_i^{-4}Y_i^2\cdot E[|\bar{X}_i|^2\mid Y_1,\cdots,Y_k]\right.$$

$$+\sum_{i\neq l}\gamma_i\gamma_l\sigma_i^{-2}\sigma_l^{-2}Y_iY_l\cdot E[\bar{X}_i|Y_1,\cdots,Y_k]\cdot E[\bar{X}_l|Y_1,\cdots,Y_k]\Big\}-\theta^2$$

$$=\Big\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\Big\}^{-2}\Big\{\sum_{i=1}^{k}\gamma_i^2\sigma_i^{-4}Y_i^2(n_i^{-1}\sigma_i^2+\theta^2)+\sum_{i\neq l}\gamma_i\gamma_l\sigma_i^{-2}\sigma_l^{-2}Y_iY_l\cdot\theta^2\Big\}-\theta^2$$

$$=\Big\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\Big\}^{-2}\Big\{\sum_{i=1}^{k}\gamma_i^2n_i^{-1}\sigma_i^{-2}Y_i^2\Big\},$$

(2.11) is rewritten as

$$V[\hat{\theta}_k]=E\Big[\Big\{\sum_{i=1}^{k}\gamma_i\sigma_i^{-2}Y_i\Big\}^{-2}\Big\{\sum_{i=1}^{k}\gamma_i^2n_i^{-1}\sigma_i^{-2}Y_i^2\Big\}\Big]$$

$$=E\Big[\Big\{\sum_{i=1}^{k}c_i^2n_i\sigma_i^2\hat{\sigma}_i^{-4}\Big\}\Big\{\sum_{i=1}^{k}c_in_i\hat{\sigma}_i^{-2}\Big\}^{-2}\Big].$$

REMARK 8. In the following proofs of the theorems, we make use of (2.10) in the form $V[\hat{\theta}_k]=a_k^{-1}E[W_{1k}W_{2k}^{-1}]$.

PROOF OF THEOREM 2.1. We note that (C.1) and (C.2) ensure $A_{1k}A_{2k}^{-1}=O(1)$ (as $k\to+\infty$) and $A_{2k}>c_L^2>0$. Therefore, by Lemma 2.5 and Lemma 2.6, we have

$$W_{1k}W_{2k}^{-1}-A_{1k}A_{2k}^{-1}\xrightarrow{p}0\qquad(\text{as }k\to+\infty)\,. \tag{2.12}$$

We can show (2.1) as follows.

Let $W_k=W_{1k}W_{2k}^{-1}$ and $A_k=A_{1k}A_{2k}^{-1}$. Since A_k is bounded upwards, there exists a positive constant L such that $\sup_{k\in N}A_k<L$. Now, let $W_k^{(L)}$ be a random variable defined by

$$W_k^{(L)}=\begin{cases}W_k & \text{if}\quad W_k\le L\,,\\ L & \text{if}\quad W_k>L\,.\end{cases}$$

Since $W_k^{(L)}-A_k\xrightarrow{p}0$ (as $k\to+\infty$) by (2.12) and $|W_k^{(L)}-A_k|<2L$ by the definition of $W_k^{(L)}$, we have

$$E[W_k^{(L)}-A_k]\to0\qquad(\text{as }k\to+\infty)\,. \tag{2.13}$$

From (2.10) and (2.13), we obtain

$$\begin{aligned}\liminf_{k\to+\infty}\{a_kV[\hat{\theta}_k]-A_k\}&=\liminf_{k\to+\infty}\{E[W_k]-A_k\}\\&\ge\liminf_{k\to+\infty}E[W_k-W_k^{(L)}]+\liminf_{k\to+\infty}E[W_k^{(L)}-A_k]\\&=\liminf_{k\to+\infty}E[W_k-W_k^{(L)}]\ge0\,.\end{aligned}$$

□

PROOF OF THEOREM 2.2. Let W_k, A_k and L be the same as in the proof of Theorem 2.1. We know that (C.7) guarantees the uniform integrability of $\{W_k - A_k\}_{k\in N}$ as follows.

Let α, p and q be the positive numbers such that $\alpha > 1$, $p > 1$, $4\alpha p < \min_{1\le i\le k}(n_i - 1)$ and $p^{-1} + q^{-1} = 1$. Then, by using the Hölder inequality, we have

$$
\begin{aligned}
a_k^{-\alpha} E[|W_k|^\alpha] &= E\left[\left\{\sum_{i=1}^k \beta_i^{-1} Y_i^2\right\}^\alpha \left\{\sum_{i=1}^k \gamma_i \sigma_i^{-2} Y_i\right\}^{-2\alpha}\right] \\
&\le k^{-4\alpha} E\left[\left\{\sum_{i=1}^k \beta_i^{-1} Y_i^2\right\}^\alpha \left\{\sum_{i=1}^k \gamma_i^{-1} \sigma_i^2 Y_i^{-1}\right\}^{2\alpha}\right] \\
&\le k^{-4\alpha} \left\{E\left[\left\{\sum_{i=1}^k \beta_i^{-1} Y_i^2\right\}^{\alpha p}\right]\right\}^{1/p} \left\{E\left[\left\{\sum_{i=1}^k \gamma_i^{-1} \sigma_i^2 Y_i^{-1}\right\}^{2\alpha q}\right]\right\}^{1/q} \\
&\le k^{-4\alpha} \left\{k^{\alpha p - 1} \sum_{i=1}^k \beta_i^{-\alpha p} E[Y_i^{2\alpha p}]\right\}^{1/p} \left\{k^{2\alpha q - 1} \sum_{i=1}^k \gamma_i^{-2\alpha q} \sigma_i^{4\alpha q} E[Y_i^{-2\alpha q}]\right\}^{1/q} \\
&= k^{-4\alpha} \left\{k^{\alpha p - 1} \sum_{i=1}^k \beta_i^{-\alpha p} \cdot 2^{-2\alpha p} \cdot d_{n_i - 1} \cdot d_{n_i - 1 - 4\alpha p}^{-1}\right\}^{1/p} \\
&\qquad \times \left\{k^{2\alpha q - 1} \sum_{i=1}^k \gamma_i^{-2\alpha q} \cdot \sigma_i^{4\alpha q} \cdot 2^{2\alpha q} \cdot d_{n_i - 1} \cdot d_{n_i - 1 + 4\alpha q}^{-1}\right\}^{1/q} \\
&= k^{-1-\alpha} \left\{\sum_{i=1}^k \beta_i^{-\alpha p} \cdot d_{n_i - 1} \cdot d_{n_i - 1 - 4\alpha p}^{-1}\right\}^{1/p} \\
&\qquad \times \left\{\sum_{i=1}^k \gamma_i^{-2\alpha q} \cdot \sigma_i^{4\alpha q} \cdot d_{n_i - 1} \cdot d_{n_i - 1 + 4\alpha q}^{-1}\right\}^{1/q} .
\end{aligned}
$$

From this and (C.7), we have $\sup_{k\in N} E[|W_k|^\alpha] < +\infty$. Hence we obtain

$$
\begin{aligned}
\sup_{k\in N} E[|W_k - A_k|^\alpha] &\le 2\left\{\sup_{k\in N} E[|W_k|^\alpha] + \sup_{k\in N} |A_k|^\alpha\right\} \\
&\le 2\left\{\sup_{k\in N} E[|W_k|^\alpha] + L^\alpha\right\} < +\infty ,
\end{aligned}
$$

and accordingly we know that $\{W_k - A_k\}_{k\in N}$ is uniformly integrable.

Now that we have (2.12) and the uniform integrability of $\{W_k - A_k\}_{k\in N}$, we can conclude that $a_k V[\hat\theta_k] - A_k = E[W_k - A_k] \to 0$ (as $k \to +\infty$). □

PROOF OF THEOREM 2.3. By Remark 2 and Theorem 2.2, we obtain (2.2). The inequality (2.3) and the necessary and sufficient condition for $a_k^{-1} A_{1k} A_{2k}^{-1} = \{\sum_{i=1}^k B_i\}^{-1}$ are immediate consequences of Lemma 2.4. Hence we have only to verify (2.4). By the definition of c_i^*, we have $1/5 \le c_i^* < 1$ $(i = 1, \cdots, k)$. From this, we know that the choice $c_i = c_i^*$ $(i = 1, \cdots, k)$ does not break the present assumption concerning $\{c_i\}_{i=1}^k$. Consequently we obtain the limiting variance of $\hat\theta_k^*$ and the limit relation (2.4). □

3. The application to the feasible weighted least squares estimator.

Let $f((c_1, \cdots, c_k))$ be the same as in the proof of Lemma 2.4. And we put $A_k^c = a_k f((1, \cdots, 1))$. We now obtain the following corollaries of Theorem 2.1–Theorem 2.3.

COROLLARY 3.1. *Assume that* (C.1) *and* (C.3) *hold and that* (C.4)–(C.6) *hold for* $c_i = 1$ $(i=1, \cdots, k)$. *Then*

$$\liminf_{k\to+\infty} \{a_k V[\hat{\theta}_k^c] - A_k^c\} \geq 0 .$$

COROLLARY 3.2. *Assume that* (C.1) *and* (C.3) *hold and that* (C.4)–(C.7) *hold for* $c_i = 1$ $(i=1, \cdots, k)$. *Then*

$$V[\hat{\theta}_k^c] = a_k^{-1} A_k^c + o(a_k^{-1}) \qquad (as\ k \to +\infty) . \tag{3.1}$$

COROLLARY 3.3. *Assume that* (Bdd) *holds except the condition concerning* $\{c_i\}_{i=1}^k$. *Then* (2.4) *and* (3.1) *hold and*

$$a_k^{-1} A_k^c \geq \{\textstyle\sum_{i=1}^k B_i\}^{-1} . \tag{3.2}$$

In (3.2), *the equality holds if and only if* $n_1 = \cdots = n_k$.

REMARK 9. When $n_1 = \cdots = n_k$, $\hat{\theta}_k^*$ reduces to $\hat{\theta}_k^c$. Therefore, in this case $\hat{\theta}_k^c$ is not asymptotically improved by $\hat{\theta}_k$-type estimator.

REMARK 10. Let $n_i = m \geq 6$ and $c_i = 1$ $(i=1, \cdots, k)$. And assume that (C.3)–(C.6) hold. Then it follows from Theorem 2.1 that

$$\liminf_{k\to+\infty} \{a_k V[\hat{\theta}_k^c] - (m-3)(m-5)^{-1}\} \geq 0 . \tag{3.3}$$

If (C.7) is also satisfied, it follows from Theorem 2.2 that

$$V[\hat{\theta}_k^c] = a_k^{-1}(m-3)(m-5)^{-1} + o(a_k^{-1}) \qquad (\text{as } k \to +\infty) . \tag{3.4}$$

From (3.3) (or (3.4)), $\hat{\theta}_k^c$ seems to be asymptotically improved. In fact, if the condition (Bdd) is satisfied and $\sigma_U^2/\sigma_L^2 < (m-3)/(m-5)$, we have

$$\limsup_{k\to+\infty} \frac{V[\bar{X}_k]}{V[\hat{\theta}_k^c]} \leq \frac{\sigma_U^2/\sigma_L^2}{(m-3)/(m-5)} < 1 ,$$

where $\bar{X}_k$ denotes the grand mean $\sum_{i=1}^k \bar{X}_i/k$.

References

[1] V. V. PETROV, *Limit Theorems of Probability Theory*, Oxford Univ. Press (1995).

[2] N. SHINOZAKI, A note on estimating the common mean of k normal distribution and Stein problem, Comm. Statist. Theory Methods **A7(15)** (1978), 1421–1432.

[3] K. Takeuchi, Tôkei-teki suisokuriron no syomondai, *Theory and Applications of Mathematical Statistics* (in Japanese), Univ. Tokyo Press (1994), 81–101.

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