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Graph Labelings in Elementary Abelian 2-Groups

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Abstract. Let $n\geq 2$ be an integer. We show that if G is a graph such that every component of G has order at least 3, and $|V(G)|\leq 2^n$ and $|V(G)|\neq 2^n-2$, then there exists an injective mapping φ from $V(G)$ to an elementary abelian 2-group of order 2^n such that for every component C of G, the sum of $\varphi(x)$ as x ranges over $V(C)$ is o.

1. Introduction.

Let $n\geq 2$ be an integer, and let E_{2^n} denote an elementary abelian 2-group of order 2^n (the operation is written additively).

Let G be a graph with no isolated vertex (by a graph, we mean a simple undirected graph). Suppose that there exists a mapping ψ from the edge set $E(G)$ of G to E_{2^n} such that if we define a mapping φ from the vertex set $V(G)$ of G to E_{2^n} by

$$\varphi(x)=\sum_{\substack{e\in E(G)\\ e\text{ is incident with }x}}\varphi(e)\qquad(x\in V(G))\,,$$

then φ is injective. In this situation, we say that G is realizable in E_{2^n}. We easily see that if G is realizable in E_{2^n}, then every component of G has order at least 3 (recall that we are assuming G has no isolated vertex). It also follows that

$$\begin{array}{l}G\text{ is realizable in }E_{2^n}\text{ if and only if there exists an injective}\\ \text{mapping }\varphi\text{ from }V(G)\text{ to }E_{2^n}\text{ such that }\sum_{x\in V(C)}\varphi(x)=o\\ \text{for every component }C\text{ of }G\end{array}\tag{1.1}$$

(see [1, Lemma 4]). We let $g(n)$ denote the maximum of those integers m for which every graph G of order at most m such that all components of G have order at least 3 is realizable in E_{2^n}.

A subset S of E_{2^n} is called a zero-sum subset if $\sum_{v\in S}v=o$. Let a, b, c be nonnegative integers, and let Z be a subset of E_{2^n}. Let K be a family of zero-sum subsets of Z, and

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suppose that $S \cap T = \varnothing$ for all $S, T \in K$ with $S \neq T$, that $3 \leq |S| \leq 5$ for all $S \in K$, and that $a = |\{S \mid |S| = 3\}|$, $b = |\{S \mid |S| = 4\}|$ and $c = |\{S \mid |S| = 5\}|$. In this situation, we say that K realizes (a, b, c) in Z. If there exists a family realizing (a, b, c) in Z, we say that (a, b, c) is realizable in Z. We let $f(n)$ denote the largest integer such that every triple (a, b, c) of nonnegative integers with $3a + 4b + 5c \leq f(n)$ is realizable in E_{2^n}. It follows from (1.1) that $f(n) = g(n)$ (see the first paragraph of the proof of Theorem 3 in [1]).

In [1], Aigner and Triesch proved that $f(n) \geq 2^{n-2}$ for all $n \geq 2$, and conjectured that $f(n) \geq 2^{n-1}$. In [5], Tuza settled this conjecture for large values of n by proving $\lim_{n \to \infty} f(n)/2^n = 1$ by a probabilistic method. In this paper, we settle the conjecture completely by proving the following theorem (it is easy to see that $f(2) = 4$):

THEOREM 1. *Let $n \geq 3$ be an integer. Then $f(n) = 2^n - 3$.*

We in fact give a constructive proof of the following stronger result:

THEOREM 2. *Let $n \geq 2$ be an integer, and let a, b, c be nonnegative integers with $3a + 4b + 5c = 2^n - 1$. Then (a, b, c) is realizable in $E_{2^n} \backslash \{o\}$.*

COROLLARY 3. *Let $n \geq 2$ be an integer, and let a, b, c be nonnegative integers such that $3a + 4b + 5c \leq 2^n$ and $3a + 4b + 5c \neq 2^n - 2$. Then (a, b, c) is realizable in E_{2^n}.*

REMARK. From $n \geq 2$, we see that E_{2^n} itself is a zero-sum subset. Since no subset of E_{2^n} having cardinality 2 is a zero-sum subset, this implies that no triple (a, b, c) with $3a + 4b + 5c = 2^n - 2$ is realizable in E_{2^n}.

In view of the above remark, it is straightforward to verify that Corollary 3 implies Theorem 1. For completeness, we here include a description of how Corollary 3 follows from Thorem 2. Let a, b, c be as in Corollary 3. By replacing a, b, c by suitable larger integers (if necessary), we may assume that $3a + 4b + 5c = 2^n - 1$ or 2^n. If $3a + 4b + 5c = 2^n - 1$, the desired conclusion immediately follows from Theorem 2. Thus we may assume $3a + 4b + 5c = 2^n$. Since 2^n is not a multiple of 3, we have $b > 0$ or $c > 0$. Assume first that $b > 0$. Then by Theorem 2, there exists a family K realizing $(a+1, b-1, c)$ in $E_{2^n} \backslash \{o\}$. Take $S \in K$ with $|S| = 3$. Then the family $(K \backslash \{S\}) \cup \{S \cup \{o\}\}$ realizes (a, b, c). If $c > 0$, then we can simmilarly get a family realizing (a, b, c) from a family realizing $(a, b+1, c-1)$ in $E_{2^n} \backslash \{o\}$.

We prove several preliminary results in Sections 2 and 3, and prove Theorem 2 in Sections 4 and 5. We conclude this section with related results. Let $n \geq 2$ be an integer. A graph G with no isolated vertex is said to be embeddable in a set A of cardinality n if there exists a mapping ψ from $E(G)$ to the set of all subsets of A such that the mapping φ defined by

$$\varphi(x) = \bigcup_{\substack{e \in E(G) \\ e \text{ is incident with } x}} \psi(e) \qquad (x \in V(G))$$

is injective. We let $h(n)$ denote the maximum of those integers m for which every graph G of order at most m such that all components of G have order at least 3 is embeddable in a set of cardinality n. In [1], Aigner and Triesch proved that $h(n) \geq 2^{n-1}$ for all $n \geq 2$, and it has recently been proved in [3] that $h(n) = 2^n$ for all $n \geq 2$.

REMARK. In [2], Caccetta and Jia have recently obtained the same result as Theorem 2.

2. Nonnegative integers.

In this section and the following section, we prove a number of preliminary results which we use in the proof of Theorem 2 (readers not interested in technical details may skip Sections 2 through 4, and proceed to Section 5). We start with lemmas concerning nonnegative integers.

LEMMA 2.1. *Let a, b, c be nonnegative integers such that*

$$3a + 4b + 5c \geq 57\,, \tag{2.1}$$

and suppose that we have $a \geq 3$ or $c \geq 1$. Then there exist nonnegative integers x, y, z such that

$$3x + 4y + 5z = 45\,, \qquad x \leq a\,, \quad y \leq b\,, \quad z \leq c\,. \tag{2.2}$$

PROOF. Let $d = \min\{b, c\}$. If $d \geq 5$, (2.2) holds with $(x, y, z) = (0, 5, 5)$. Thus we may assume $d \leq 4$. If $a + 3d \geq 15$, (2.2) holds with $(x, y, z) = (15 - 3d, d, d)$. Thus we may assume

$$a + 3d < 15\,. \tag{2.3}$$

We first consider the case where $b \geq c$. If $c \geq 1$, let $a_0 = 0$ and $c_0 = 1$; if $c = 0$ (so $a \geq 3$ by assumption), let $a_0 = 3$ and $c_0 = 0$. Then $a_0 \leq a$, $c_0 \leq c$ and

$$3a_0 + 9c_0 = 9\,. \tag{2.4}$$

Also we easily see that there exist nonnegative integers p, r with $p \leq a - a_0$, $r \leq c - c_0$ and

$$p + r \leq 3 \tag{2.5}$$

such that

$$3p + 5r \equiv 3(a - a_0) + 5(c - c_0) \qquad (\mathrm{mod}\, 4)\,. \tag{2.6}$$

Since $3p + 5r \leq 15$ by (2.5), we obtain

$$3(a - p) + 4b + 5(c - r) \geq 42 \tag{2.7}$$

by (2.1). On the other hand, we get

$$3(a - p) + 5(c - r) \leq 3(a + 3c) = 3(a + 3d) < 45 \tag{2.8}$$

from (2.3). Since $45-(3(a-p)+5(c-r))=36+4c_0-(3(a-a_0-p)+5(c-c_0-r))$ is a multiple of 4 by (2.4) and (2.6), it follows from (2.7) and (2.8) that there exists a positive integer y such that (2.2) holds with $x=a-p$ and $z=c-r$.

We now consider the case where $b<c$. In this case, we take nonnegative integers p, q with $p\leq a$, $q\leq b$ and $p+q\leq 4$ such that $3p+4q\equiv 3a+4b \pmod 5$. Then we have $3(a-p)+4(b-q)+5c\geq 41$ and $3(a-p)+4(b-q<45$, and $45-(3(a-p)+4(b-q))$ is a multiple of 5. Consequently, there exists a positive integer z such that (2.2) holds with $x=a-p$ and $y=b-q$.

LEMMA 2.2. *Let r be a nonnegative integer. Let a, b, c, be nonnegative integers such that* $3a+4b+5c\geq 45r+12$ *and*

$$[(b-1)/9]\leq (a/3)+c\,. \tag{2.9}$$

Then there exist nonnegative integers x_1, y_1, z_1; x_1, y_2, z_2;$\cdots$; x_r, y_y, z_r *such that* $3x_i+4y_i+5z_i=45$ *for all* i, $\sum x_i\leq a$, $\sum y_i\leq b$ *and* $\sum z_i\leq c$.

PROOF. If $r=0$, the lemma trivially holds. Thus let $r\geq 1$, and assume that the lemma is proved for $r-1$. It suffices to show that there exist nonnegative integers x, y, z satisfying (2.2) such that

$$[((b-y)-1)/9]\leq (a-x)/3+(c-z)\,. \tag{2.10}$$

Assume first that $b\geq 10$. Then $a/3+c\geq 1$ by (2.9), and hence $a\geq 3$ or $c\geq 1$. If $a\geq 3$, let $(x, y, z)=(3, 9, 0)$; if $a<3$ (so $c\geq 1$), let $(x, y, z)=(0, 10, 1)$. Then (2.10) easily follows from (2.9). Assume now that $b\leq 9$. Then $3a+5c\geq (45+12)-36$, and hence we have $a>3$ or $c>1$. Consequently, it follows from Lemma 2.1 that there exist nonnegative integers x, y, z satisfying (2.2), and (2.10) clearly holds because $(b-y)-1\leq b-1<9$.

LEMMA 2.3. *Let* b, c, t *be nonnegative integers such that* $4b+5c\geq t+12$, *and suppose that one of the folloiwng holds*:

(i) *t is a multiple of* 4 *and* $b\geq 4$; *or*

(ii) *t is a multiple of* 5 *and* $c\geq 3$.

Then there exist nonnegative integers y, z *such that*

$$4y+5z=t\,, \qquad y\leq b\,, \quad z\leq c\,. \tag{2.11}$$

PROOF. We first consider the case where (i) holds. Clearly we may assume $b<t/4$. Also there exists a nonnegative integer q with $q\leq 4$ such that $t-4(b-q)$ is a multiple of 5. Then $4(b-q)+5c\geq t-4$, and hence there exists a positive integer z such that (2.11) holds with $y=b-q$. We now consider the case where (ii) holds. We may assume $c<t/5$. Also there exists a nonnegative integer r with $r\leq 3$ such that $t-5(c-r)$ is a multiple of 4. Then $4b+5(c-r)\geq t-3$, and hence there exists a nonnegative integer y such that (2.11) holds with $z=c-r$.

3. Realizable triples.

Throughout this section, we let $n \geq 2$ be an integer, and let X be an elementary abelian 2-group of order 2^n. For a subset S of X, we let $\langle S \rangle$ denote the subgroup of X generated by S. For subsets S, T of X, we let $S+T=\{u+v \mid u \in S, v \in T\}$.

LEMMA 3.1. *Let S be a subset of X and let Q be a zero-sum subset of cardinality 4 of X, and suppose that $\langle S \rangle \cap \langle Q \rangle = \{o\}$. Then $(0, |S|, 0)$ is realizable in $S+Q$.*

PROOF. The family $\{u+Q \mid u \in S\}$ realizes $(0, |S|, 0)$.

LEMMA 3.2. *Let S, P be zero-sum subsets of cardinality 3 such that $\langle S \rangle \cap \langle P \rangle = \{o\}$. Then $(0, 1, 1)$ and $(3, 0, 0)$ are realizable in $S+P$.*

PROOF. Write $S=\{s_1, s_2, s_3\}$ and $P=\{v_1, v_2, v_3\}$, and let $P_k=\{s_i+v_{i+k} \mid 1 \leq i \leq 3\}$ for each $1 \leq k \leq 3$ (subscripts of the letter v are to be read modulo 3). Then $\{P_1, P_2, P_3\}$ realizes $(3, 0, 0)$, and

$$\{\{s_1, s_3\}+\{v_1, v_3\}, \qquad (S+P) \setminus (\{s_1, s_3\}+\{v_1, v_3\})\}$$

realizes $(0, 1, 1)$.

LEMMA 3.3. *Let S and R be zero-sum subsets of cardinality 3 and 5, respectively, such that $\langle S \rangle \cap \langle R \rangle = \{o\}$. Then $(0, 0, 3)$ is realizable in $S+R$.*

PROOF. Write $S=\{s_1, s_2, s_3\}$ and $R=\{v_1, v_2, v_3, p, q\}$ and define P_k as in Lemma 3.2. Then $\{P_k \cup \{s_k+p, s_k+q\} \mid 1 \leq k \leq 3\}$ realizes $(0, 0, 3)$.

LEMMA 3.4. *Let S, P, Q be zero-sum subsets of cardinality 3 such that $P \cap Q = \varnothing$ and $\langle S \rangle \cap \langle P \cup Q \rangle = \{o\}$. Then $(2, 3, 0)$ and $(1, 0, 3)$ are realizable in $S+(P \cup Q)$.*

PROOF. Write $S=\{s_1, s_2, s_3\}$, $P=\{v_1, v_2, v_3\}$ and $Q=\{w_1, w_2, w_3\}$ (subscripts are to be read modulo 3). For each $1 \leq k \leq 3$, let

$$Q_k=\{s_{k+1}, s_{k+2}\}+\{v_k, w_k\},$$
$$R_k=\{s_k+v_i \mid 1 \leq i \leq 3\} \cup \{s_{k+1}+w_k, s_{k+2}+w_k\}.$$

Let

$$P_1=\{s_i+v_i \mid 1 \leq i \leq 3\}, \qquad P_2=\{s_i+w_i \mid 1 \leq i \leq 3\}.$$

Then $\{P_1, P_2\} \cup \{Q_k \mid 1 \leq k \leq 3\}$ realizes $(2, 3, 0)$, and $\{P_2\} \cup \{R_k \mid 1 \leq k \leq 3\}$ realizes $(1, 0, 3)$.

LEMMA 3.5. *Let T be a zero-sum subset of cardinality 5 and let P, Q be zero-sum subsets of cardinality 3, and suppose that $P \cap Q = \varnothing$ and $\langle T \rangle \cap \langle P \cup Q \rangle = \{o\}$. Then $(0, 5, 2)$ is realizable in $T+(P \cup Q)$.*

PROOF. Write $T=\{s_1, s_2, s_3, t, u\}$, $P=\{v_1, v_2, v_3\}$ and $Q=\{w_1, w_2, w_3\}$. Define

P_1, P_2, Q_1, Q_2, Q_3 as in Lemma 3.4, and let

$$R_1 = P_1 \cup \{t+v_2, u+v_2\}, \qquad R_2 = P_2 \cup \{t+w_2, u+w_2\},$$
$$Q_4 = \{t, u\} + \{v_1, v_3\}, \qquad Q_5 = \{t, u\} + \{w_1, w_3\}.$$

Then $\{R_1, R_2\} \cup \{Q_k \mid 1 \leq k \leq 5\}$ realizes (0, 5, 2).

LEMMA 3.6. *Let S be a zero-sum subset of cardinality 3 and let Q, R be zero-sum subsets of cardinality 5, and suppose that $Q \cap R = \emptyset$ and $\langle S \rangle \cap \langle Q \cup R \rangle = \{o\}$. Then (0, 5, 2) is realizable in $S + (Q \cup R)$.*

PROOF. Write $S = \{s_1, s_2, s_3\}$, $Q = \{v_1, v_2, v_3, p, q\}$, $R = \{w_1, w_2, w_3, x, y\}$. Define P_1, P_2, Q_1, Q_2, Q_3 as in Lemma 3.4, and let

$$R_1 = P_1 \cup \{s_2+p, s_2+q\}, \qquad R_2 = P_2 \cup \{s_2+x, s_2+y\},$$
$$Q_4 = \{s_1, s_3\} + \{p, q\}, \qquad Q_5 = \{s_1, s_3\} + \{x, y\}.$$

Then $\{R_1, R_2\} \cup \{Q_k \mid 1 \leq k \leq 5\}$ realizes (0, 5, 2).

LEMMA 3.7. *Let S, P be zero-sum subsets of cardinality 3 and let R be a zero-sum subset of cardinality 5, and suppose that $P \cap R = \emptyset$ and $\langle S \rangle \cap \langle P \cup R \rangle = \{o\}$. Then (1, 4, 1) is realizable in $S + (P \cup R)$.*

PROOF. Write $S = \{s_1, s_2, s_3\}$, $P = \{v_1, v_2, v_3\}$ and $R = \{w_1, w_2, w_3, x, y\}$, and let P_1, R_2, Q_1, Q_2, Q_3, Q_5 be as in Lemma 3.6. Then they form a family realizing (1, 4, 1).

LEMMA 3.8. *Let W be a subgroup of order 2^3 of X and let P be a zero-sum subset of cardinality 3 of X, and suppose that $W \cap \langle P \rangle = \{o\}$. Then (0, 1, 4), (3, 0, 3) and (8, 0, 0) are realizable in $W + P$.*

PROOF. Let Z be a subgroup of order 2^2 of W. By Lemma 3.2, (0, 1, 1) and (3, 0, 0) are realizable in $(Z \backslash \{o\}) + P$. By Lemma 3.3, (0, 0, 3) is realizable in $((W \backslash Z) \cup \{o\}) + P$. Consequently, (0, 1, 4) and (3, 0, 3) are realizable in $W + P$. Now write $W = \langle v_1, v_2, v_3 \rangle$ and $P = \{p_1, p_2, p_3\}$ (subscripts are to be read modulo 3). For each $1 \leq k \leq 3$, let

$$P_k = \{p_k, v_k + p_{k+1}, v_k + p_{k+2}\},$$
$$S_k = \{v_k + p_k, v_{k+1} + v_{k+2} + p_{k+2}, v_k + v_{k+1} + v_{k+2} + p_{k+1}\}.$$

For each $1 \leq l \leq 2$, let

$$T_l = \{v_i + v_{i+1} + p_{i+1+l} \mid 1 \leq i \leq 3\}.$$

Then $\{P_k, S_k, T_l\} \mid 1 \leq k \leq 3, 1 \leq l \leq 2\}$ realizes (8, 0, 0).

LEMMA 3.9. *Let W be a subgroup of order 2^3 and let R be a zero-sum subset of cardinality 5, and suppose that $W \cap \langle R \rangle = \{o\}$. Then (0, 0, 8) is realizable in $W + R$.*

PROOF. Write $W = \langle v_1, v_2, v_3 \rangle$ and $P = \{p_1, p_2, p_3, q, r\}$ (subscripts are to be read modulo 3). Define P_k, S_k, T_l as in Lemma 3.8, and let

$$R_k = P_k \cup \{v_k + q, v_k + r\},$$

$$U_k = S_k \cup \{v_{k+1} + v_{k+2} + q, v_{k+1} + v_{k+2} + r\},$$

$$V_1 = T_1 \cup \{q, r\}, \qquad V_2 = T_2 \cup \{v_1 + v_2 + v_3 + q, v_1 + v_2 + v_3 + r\}.$$

Then $\{R_k, U_k, V_l \mid 1 \le k \le 3, 1 \le l \le 2\}$ realizes (0, 0, 8).

LEMMA 3.10. *Let W be a subgroup of order 2^3 and let R be a zero-sum subset of cardinality 5 with $o \notin R$, and suppose that $W \cap \langle R \rangle = \{o\}$. Then* (0, 3, 7) *is realizable in* $(W + (R \cup \{o\})) \setminus \{o\}$.

PROOF. Under the notation of Lemma 3.9, let

$$V_3 = \{v_2 + v_3 + p_2, v_1 + v_3, v_1 + v_3 + p_2, v_1 + v_3 + p_3, v_1 + v_2 + p_3\},$$

$$Q_1 = \{v_1, v_2, v_3, v_1 + v_2 + v_3\},$$

$$Q_2 = \{v_2 + v_3, v_1 + v_2\} + \{o, p_1\},$$

$$Q_3 = \{o, v_1 + v_2 + v_3\} + \{q, r\}.$$

Then $\{Q_k, R_k, U_k, V_3 \mid 1 \le k \le 3\}$ realizes (0, 3, 7).

LEMMA 3.11. *If n is odd, let W denote a subgroup of order 2^3; if n is even, let $W = \{o\}$. Then* $((|X| - |W|)/3, 0, 0)$ *is realizable in* $X \setminus W$.

PROOF. We proceed by induction on n. It is easy to verify the lemma for $n = 2, 3$. Thus let $n \ge 4$, and assume that the lemma is proved for $n - 2$. Take subgroups U and V of order 2^{n-2} and 2^2, respectively, so that $U \supseteq W$ and $U \cap V = \{o\}$. By the induction hypothesis, there exists a family L realizing $((|U| - |W|)/3, 0, 0)$ in $U \setminus W$. It follows from Lemma 3.2 that for each $P \in L$, there exists a family M_P realizing (3, 0, 0) in $P + (V \setminus \{o\})$. Furthermore, there exists a family N realizing $(|W|, 0, 0)$ in $W + (V \setminus \{o\})$ (if $W = \{o\}$, this is trivial; if $|W| = 8$, this follows from Lemma 3.8). Thus the family $(\bigcup_{P \in L} M_P) \cup N \cup L$ realizes $((|X| - |W|)/3, 0, 0)$ in $X \setminus W$.

LEMMA 3.12. *Suppose that $n \ge 3$, and let U be a subgroup of order 2^{n-1}. Let a, b, c be nonnegative integers with $3a + 4b + 5c = 2^n - 1$ and $b \ge 2^{n-3}$, and suppose that there exists a family K realizing $(a, b - 2^{n-3}, c)$ in $U \setminus \{o\}$. Then (a, b, c) is realizable in $X \setminus \{o\}$.*

PROOF. Let W be a subgroup of order 2^2 of U. Then the family L consisting of those cosets of W which are disjoint from U realizes $(0, 2^{n-3}, 0)$, and hence $K \cup L$ realizes (a, b, c).

The following lemma shows that Theorem 2 holds for $2 \le n \le 4$:

LEMMA 3.13. *Suppose that $2 \le n \le 4$, and let a, b, c be nonnegative integers with $3a + 4b + 5c = 2^n - 1$. Then (a, b, c) is realizable in $X \setminus \{o\}$.*

PROOF. If $n=2$ or 3, the lemma clearly holds. Thus we may assume $n=4$. In view of Lemmas 3.11 and 3.12, we may assume $b+c\neq 0$ and $b\leq 1$. Thus $(a, b, c)=(2, 1, 1)$ or $(0, 0, 3)$. Let U, V be subgroups of order 2^2 such that $U\cap V=\{o\}$. Then by Lemma 3.2, there exists a family K realizing $(0, 1, 1)$ in $(U\backslash\{o\})+(V\backslash\{o\})$, and hence $K\cup\{U\backslash\{o\}, V\backslash\{o\}\}$ realizes $(2, 1, 1)$. Now write $U\backslash\{o\}=\{u_1, u_2, u_3\}$ and $V\backslash\{o\}=\{v_1, v_2, v_3\}$ (subscripts are to be read modulo 3). Then the family

$$\{\{u_k, v_k, u_k+v_{k+1}, u_{k+1}+v_k, u_{k+1}+v_{k+1}\}\mid 1\leq k\leq 3\}$$

realizes $(0, 0, 3)$.

We prove four more technical results.

LEMMA 3.14. *Let r be a nonnegative integer, and let $P_1, \cdots, P_r, S$ be zero-sum subsets of cardinality* 3 *such that $P_i\cap P_j=\varnothing$ for all i, j with $i\neq j$ and $\langle S\rangle\cap\langle P_1\cup\cdots\cup P_r\rangle=\{o\}$. Let x, y, z be nonnegative integers with $3x+4y+5z=9r$, and let $d=\min\{y, z\}$ and $e=\max\{y, z\}$. Suppose that $e-d\leq 3(r-d)/2$. Then (x, y, z) is realizable in $S+(P_1\cup\cdots\cup P_r)$.*

PROOF. We proceed by induction on r. If $r=0$, the lemma trivially holds. Thus assume $r\geq 1$. If $d\geq 1$, then $(x, y-1, z-1)$ is realizable in $S+(P_1\cup\cdots\cup P_{r-1})$ by the induction hypothesis, and $(0, 1, 1)$ is realizable in $S+P_r$ by Lemma 3.2, and hence (x, y, z) is realizable in $S+(P_1\cup\cdots\cup P_r)$. Thus we may assume $d=0$. If $e=0$, then $y=z=0$ and $x=3r$, and hence the desired conclusion immediately follows from Lemma 3.2. Thus we may assume $e>0$. Then either $y=0$ and $z>0$, or $z=0$ and $y>0$.

Assume first that $y=0$ and $z>0$. Then since $3x+5z=9r$, z is a multiple of 3, and hence $z\geq 3$ and $r\geq 2$. Since $z=e-d\leq 3(r-d)/2=3r/2$, we also get $x=(9r-5z)/3\geq r/2\geq 1$. Thus by the induction hypothesis, $(x-1, 0, z-3)$ is realizable in $S+(P_1\cup\cdots\cup P_{r-2})$. Since $(1, 0, 3)$ is realizable in $S+(P_{r-1}\cup P_r)$ by Lemma 3.4, this implies that $(x, 0, z)$ is realizable in $S+(P_1\cup\cdots\cup P_r)$.

Assume now that $z=0$ and $y>0$. Then $y\geq 3$, $r\geq 2$ and $x\geq 2$. Thus by the induction hypothesis, $(x-2, y-3, 0)$ is realizable in $S+(P_1\cup\cdots\cup P_{r-2})$. Since $(2, 3, 0)$ is realizable in $S+(P_{r-1}\cup P_r)$ by Lemma 3.4, this implies that $(x, y, 0)$ is realizable in $S+(P_1\cup\cdots\cup P_r)$.

LEMMA 3.15. *Let V be a subgroup of order 2^4, and let S be a zero-sum subset of cardinality* 3 *such that $\langle S\rangle\cap V=\{o\}$. Let x, y, z be nonnegative integers with $3x+4y+5z=45$. Then (x, y, z) is realizable in $S+(V\backslash\{o\})$.*

PROOF. Let $d=\min\{y, z\}$ and $e=\max\{y, z\}$. By Lemma 3.13, we can partition $V\backslash\{o\}$ into five zero-sum subsets of cardinality 3. Consequently, if $e-d\leq 3(5-d)/2$, then the desired conclusion immediately follows from Lemma 3.14. Thus we may assume $e-d>3(5-d)/2$. Then $(x, y, z)=(0, 0, 9)$, $(0, 10, 1)$, $(1, 8, 2)$ or $(3, 9, 0)$. By Lemma 3.13, we can partition $V\backslash\{o\}$ into three zero-sum subsets of cardinality 5, and hence it follows from Lemma 3.3 that $(0, 0, 9)$ is realizable in $S+(V\backslash\{o\})$. By Lemma 3.13, we can partition $V\backslash\{o\}$ into four zero-sum subsets P, Q_1, Q_2, Q_3 such that $|P|=3$ and $|Q_1|=|Q_2|=|Q_3|=4$. Then $(0, 1, 1)$ and $(3, 0, 0)$ are realizable in $S+P$ by Lemma 3.2, and $(0, 3, 0)$ is realizable in $S+Q_i$ by Lemma 3.1. Consequently, $(0, 10, 1)$ and $(3, 9, 0)$ are realizable in $S+(V\backslash\{o\})$. Now by Lemma 3.13, we can partition $V\backslash\{o\}$ into four zero-sum subsets P_1, P_2, Q, R such that $|P_1|=|P_2|=3$, $|Q|=4$ and $|R|=5$. Then $(1, 4, 1)$ is realizable in $S+(P_1 \cup R)$ by Lemma 3.7, $(0, 1, 1)$ is realizable in $S+P_2$ by Lemml 3.2, and $(0, 3, 0)$ is realizable in $S+Q$ by Lemma 3.1. Consequently, $(1, 8, 2)$ is realizable in $S+(V\backslash\{o\})$.

LEMMA 3.16. *Let V be a subgroup of order 2^4, and S be a zero-sum subset of cardinality 3 such that $\langle S\rangle \cap V=\{o\}$. Let x, y, z be nonnegative integers with $3x+4y+5z=48$. Then (x, y, z) is realizable in $S+V$.*

PROOF. If $x\geq 1$, the desired conclusion immediately follows from Lemma 3.15. Thus we may assume $x=0$. Since $(1, 3, 0)$ is realizable in $V\backslash\{o\}$ by Lemma 3.13, $(0, 4, 0)$ is realizable in V. Consequently, it follows from Lemma 3.1 that $(0, 12, 0)$ is realizable in $S+V$. Thus we may assume $z>0$, and hence $(y, z)=(2, 8)$ or $(7, 4)$. Since $(2, 1, 1)$ is realizable in $V\backslash\{o\}$ by Lemma 3.13, we can partition V into two zero-sum sets P_1, P_2 of cardinality 3 and two zero-sum subsets R_1, R_2 of cardinality 5. By Lemma 3.2, $(0, 1, 1)$ is realizable in $S+P_1$ and $S+P_2$. By Lemmas 3.3 and 3.6, $(0, 0, 6)$ and $(0, 5, 2)$ are realizable in $S+(R_1 \cup R_2)$. Consequently, $(0, 2, 8)$ and $(0, 7, 4)$ are realizable in $S+V$.

LEMMA 3.17. *Let V be a subgroup of order 2^4, and let T be a zero-sum subset of cardinality 5 such that $\langle T\rangle \cap V=\{o\}$. Let y, z be nonnegative integers with $4y+5z=75$. Then $(0, y, z)$ is realizable in $T+(V\backslash\{o\})$.*

PROOF. By Lemma 3.13, we can partition $V\backslash\{o\}$ into a zero-sum subset P of cardinality 3 and three zero-sum subsets Q_1, Q_2, Q_3 of cardinality 4. By Lemma 3.3, $(0, 0, 3)$ is realizable in $T+P$. By Lemma 3.1, $(0, 5, 0)$ is realizable in $T+Q_i$ for each i. Consequently, $(0, 15, 3)$ is realizable in $T+(V\backslash\{o\})$. Thus we may assume $z>3$, and hence $(y, z)=(0, 15)$, $(5, 11)$ or $(10, 7)$. By Lemma 3.13, we can partition $V\backslash\{o\}$ into five zero-sum subsets $P_1, \cdots, P_5$ of cardinality 3. By Lemmas 3.3 and 3.5, $(0, 0, 6)$ and $(0, 5, 2)$ are realizable in $T+(P_{2i-1} \cup P_{2i})$ for each $1\leq i\leq 2$. By Lemma 3.3, $(0, 0, 3)$ is realizable in $T+P_5$. Consequently, $(0, 0, 15)$, $(0, 5, 11)$ and $(0, 10, 7)$ are realizable in $T+(V\backslash\{o\})$.

4. Small case.

Let $n\geq 2$ be an integer, and let X be an elementary abelian 2-group of order 2^n. In this section, we consider the case where $5\leq n\leq 7$.

LEMMA 4.1. *Suppose that $n=5$, and let a, b, c be nonnegative integers with $3a+4b+5c=31$. Then (a, b, c) is realizable in $X\setminus\{o\}$.*

PROOF. By Lemmas 3.12 and 3.13, we may assume $b\leq 3$. Thus $(a, b, c)=(1, 2, 4)$, $(2, 0, 5)$, $(3, 3, 2)$, $(4, 1, 3)$, $(6, 2, 1)$, $(7, 0, 2)$ or $(9, 1, 0)$. Take subgroups U and V of order 2^3 and 2^2, respectively, so that $U\cap V=\{o\}$, and let W be a subgroup of order 2^2 of U.

Case 1. $(a, b, c)=(1, 2, 4)$, $(4, 1, 3)$ or $(9, 1, 0)$. By Lemma 3.8, $(0, 1, 4)$, $(3, 0, 3)$ and $(8, 0, 0)$ are realizable in $U+(V\setminus\{o\})$. Since $\{W\setminus\{o\}, U\setminus W\}$ realizes $(1, 1, 0)$, it follows that $(1, 2, 4)$, $(4, 1, 3)$ and $(9, 1, 0)$ are realizable in $X\setminus\{o\}$. This completes the discussion for Case 1.

Throughout the rest of the proof of the lemma, we write $V\setminus\{o\}=\{p, q, r\}$ and $W\setminus\{o\}=\{u, v, w\}$, and fix $z\in U\setminus W$.

Case 2. $(a, b, c)=(6, 2, 1)$. Since $\{V\setminus\{o\}, W\setminus\{o\}, U\setminus W\}$ realizes $(2, 1, 0)$ in $(U\cup V)\setminus\{o\}$, it suffices to show that $(4, 1, 1)$ is realizable in $X\setminus(U\cup V)$. Let

$$\begin{aligned}
P_1&=\{p+z, q+u, r+u+z\},\\
P_2&=\{q+z, r+v, p+v+z\},\\
P_3&=\{r+z, p+w, q+w+z\},\\
P&=\{p+u, q+v, r+w\},\\
Q&=\{p+v, q+w, p+w+z, q+v+z\},\\
R&=\{r+u, p+u+z, q+u+z, r+v+z, r+w+z\}.
\end{aligned}$$

Then $\{P_1, P_2, P_3, P, Q, R\}$ realizes $(4, 1, 1)$.

Case 3. $(a, b, c)=(3, 3, 2)$. Let

$$\begin{aligned}
P&=\{p+u, q+v+z, r+w+z\},\\
Q_1&=\{r, q+z\}+\{u, w\},\\
Q_2&=\{p+z, r+z\}+\{u, v\},\\
R_1&=\{p+v, q+u, q+v, q+w, r+v\},\\
R_2&=\{p+z, q+z, r+z, p+w, p+w+z\}.
\end{aligned}$$

Then $\{P, Q_1, Q_2, R_1, R_2\}$ realizes $(1, 2, 2)$ in $X\setminus(U\cup V)$.

Case 4. $(a, b, c)=(7, 0, 2)$. We show that $(5, 0, 2)$ is realizable in $X\setminus(V\cup W)$. Let

$$\begin{aligned}
P_1&=\{p+z, u+z, p+u\},\\
P_2&=\{q+z, v+z, q+v\},\\
P_3&=\{r+z, w+z, r+w\},\\
S_1&=\{q+u, p+w+z, r+v+z\},
\end{aligned}$$

$$S_2=\{q+w, p+v+z, r+u+z\},$$

$$R_1=\{z, p+v, p+w, r+v, r+w+z\},$$

$$R_2=\{r+u, p+u+z, q+u+z, q+v+z, q+w+z\}.$$

Then P_1, P_2, P_3, S_1, S_2, R_1, R_2 realizes (5, 0, 2) in $X\backslash(V\cup W)$.

Case 5. $(a, b, c)=(2, 0, 5)$. Let

$$R_1=\{p+z, q+z, r+u, r+v, r+w\},$$

$$R_2=\{u+z, v+z, w+z, q+w, q+w+z\},$$

$$R_3=\{p+u, p+v, p+w, q+v+z, r+v+z\},$$

$$R_4=\{q+u, p+u+z, p+v+z, p+w+z, r+u+z\},$$

$$R_5=\{z, r+z, q+v, q+u+z, r+w+z\}.$$

Then $\{R_i \mid 1\le i\le 5\}$ realizes (0, 0, 5) in $X\backslash(V\cup W)$.

LEMMA 4.2. *Suppose that $n=6$, and let a, b, c be nonnegative integers with $3a+4b+5c=63$. Then (a, b, c) is realizable in $X\backslash\{o\}$.*

PROOF. By Lemmas 3.12 and 4.1, we may assume $b\le 7$, and hence we have $a>5$ or $c>3$. If $a>5$, let $a_1=5$ and $c_1=0$; if $a\le 5$ (so $c>3$), let $a_1=0$ and $c_1=3$. Let U, V be subgroups of order 2^4 and 2^2 such that $U\cap V=\{o\}$. Then $(a_1, 0, c_1)$ is realizable in $U\backslash\{o\}$ by Lemma 3.13, and $(a-a_1, b, c-c_1)$ is realizable in $U+(V\backslash\{o\})$ by Lemma 3.16, and hence (a, b, c) is realizable in $X\backslash\{o\}$.

LEMMA 4.3. *Suppose that $n=7$, and let a, b, c be nonnegative integers with $3a+4b+5c=127$. Then (a, b, c) is realizable in $X\backslash\{o\}$.*

PROOF. By Lemmas 3.12 and 4.2, we may assume

$$b\le 15. \tag{4.1}$$

We divide the proof into four cases.

Case 1. $a=0$. By (4.1), we have $(b, c)=(3, 23)$, $(8, 19)$ or $(13, 15)$.

Subcase 1.1. $(b, c)=(3, 23)$. Let U, V be subgroups of order 2^3 and 2^4 such that $U\cap V=\{o\}$. By Lemma 3.13, we can partition $V\backslash\{o\}$ into three zero-sum subsets R_1, R_2, R_3 of cardinality 5. It follows from Lemma 3.9 that (0, 0, 8) is realizable in $U+R_1$ and $U+R_2$, and it follows from Lemma 3.10 that (0, 3, 7) is realizable in $(U+(R_3\cup\{o\}))\backslash\{o\}$, and hence (0, 3, 23) is realizable in $X\backslash\{o\}$.

Subcase 1.2. $(b, c)=(8, 19)$ or $(13, 15)$. Let U, V be subgroups of order 2^5 and 2^2 such that $U\cap V=\{o\}$. Since (4, 1, 3) is realizable in $U\backslash\{o\}$ by Lemma 4.1, we can partition U into four zero-sum subsets $P_1, \cdots, P_4$ of cardinality 3 and four zero-sum subsets $R_1, \cdots, R_4$ of cardinality 5. By Lemma 3.2, (0, 1, 1) is realizable in $P_i+(V\backslash\{o\})$ for each $1 \leq i \leq 4$. By Lemmas 3.3 and 3.6, $(0, 0, 6)$ and $(0, 5, 2)$ are realizable in $(R_{2i-1} \cup R_{2i})+(V\setminus\{o\})$ for each $1 \leq i \leq 2$. Since $(0, 4, 3)$ is realizable in $U\setminus\{o\}$ by Lemma 4.1, it now follows that $(0, 8, 19)$ and $(0, 13, 15)$ (and $(0, 18, 11)$) are realizable in $X\setminus\{o\}$.

Case 2. $1 \leq a \leq 11$ and $b=0$. We have $(a, c)=(4, 23)$ or $(9, 20)$. Let U, V be subgroups of order 2^5 and 2^2 such that $U \cap V=\{o\}$. By Lemma 4.1, we can partition $U\setminus\{o\}$ into two zero-sum subsets P_1, P_2 of cardinality 3 and five zero-sum subsets $R_1, \cdots, R_5$ of cardinality 5. By Lemmas 3.2 and 3.4, $(6, 0, 0)$ and $(1, 0, 3)$ are realizable in $(P_1 \cup P_2)+(V\setminus\{o\})$. By Lemma 3.3, $(0, 0, 3)$ is realizable in $R_i+(V\setminus\{o\})$ for each i. Since $\{V\setminus\{o\}, P_1, P_2\} \cup \{R_i \mid 1 \leq i \leq 5\}$ realizes $(3, 0, 5)$, it now follows that $(9, 0, 20)$ and $(4, 0, 23)$ are realizable in $X\setminus\{o\}$.

Case 3. $1 \leq a \leq 11$ and $b \geq 1$. Let U, V be subgroups of order 2^3 and 2^4 such that $U \cap V=\{o\}$, and let W be a subgroup of order 2^2 of U. Then $\{W\setminus\{o\}, U\setminus W\}$ realizes $(1, 1, 0)$ in $U\setminus\{o\}$. We aim at showing that we can write $b-1=b_1+b_2$ and $c=c_1+c_2$ so that $(0, b_1, c_1)$ and $(a-1, b_2, c_2)$ are realizable in $((U\setminus W) \cup \{o\})+(V\setminus\{o\})$ and $(W\setminus\{o\})+(V\setminus\{o\})$, respectively. Since $a \leq 11$, we get $4(b-1)+5c \geq 127-33-4=75+15$, and we also get $c \geq 7$ from (4.1). Hence by Lemma 2.3, there exist nonnegative integers b_1, c_1 with $b_1 \leq b-1$ and $c_1 \leq c$ such that $4b_1+5c_1=75$. Let $b_2=b-1-b_1$ and $c_2=c-c_1$. Then $3(a-1)+4b_2+5c_2=45$. It now follows from Lemma 3.17 that $(0, b_1, c_1)$ is realizable in $((U\setminus W) \cup \{o\})+(V\setminus\{o\})$, and it follows from Lemma 3.15 that $(a-1, b_2, c_2)$ is realizable in $(W\setminus\{o\})+(V\setminus\{o\})$, and hence (a, b, c) is realizable in $X\setminus\{o\}$.

Case 4. $a \geq 12$. Let U, V be subgroups of order 2^5 and 2^2 such that $U \cap V=\{o\}$, and let W be a subgroup of order 2^3 of U. We aim at showing that we can write $a=a_1+a_2+a_3$, $b=b_1+b_2+b_3$ and $c=c_1+c_2+c_3$ so that (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3) are realizable in $W+(V\setminus\{o\})$, $(U\setminus W)+(V\setminus\{o\})$ and $U\setminus\{o\}$.

We first take up $W+(V\setminus\{o\})$. If $12 \leq a \leq 16$, let $(a_1, b_1, c_1)=(3, 0, 3)$; if $a \geq 17$, let $(a_1, b_1, c_1)=(8, 0, 0)$. Note that in the case where $12 \leq a \leq 16$, it follows from (4.1) that $c \geq (127-48-60)/5$, i.e., $c \geq 4$. Thus in either case, we have

$$3a_1+4b_1+5c_1=24\,, \tag{4.2}$$

$a-a_1 \geq 9$, $b \geq b_1$ and $c \geq c_1$. Moreover, (a_1, b_1, c_1) is realizable in $W+(V\setminus\{o\})$ by Lemma 3.8.

We now consider $(U\setminus W)+(V\setminus\{o\})$. We choose nonnegative integers a_2, b_2, c_2 as follows so that they satisfy

$$8 \leq a_2 \leq a-a_1\,, \tag{4.3}$$

$$b_2 \leq b-b_1\,, \quad c_2 \leq c-c_1\,, \quad 3a_2+4b_2+5c_2=72\,. \tag{4.4}$$

If $a-a_1 \geq 24$, we simply let $(a_2, b_2, c_2)=(24, 0, 0)$. Thus assume that $a-a_1 \leq 23$. Then $4(b-b_1)+5(c-c_1) \geq 34$ by (4.2), and hence we have $b-b_1>4$ or $c-c_1>3$. We first consider the case where $b-b_1>4$. In this case, we let a_2 be the largest integer with $a_2 \leq a-a_1$ such that $24-a_2$ is a multiple of 4. Then

$$(a-a_1)-a_2 \leq 3 \tag{4.5}$$

and, from $a-a_1 \geq 9$, we obtain $a_2 \geq 8$. Since we get $4(b-b_1)+5(c-c_1)=3(24-a_2)+31-3((a-a_1)-a_2) \geq 3(24-a_2)+22$ from (4.2) and (4.5), it follows from Lemma 2.3 that there exist nonnegative integers b_2, c_2 satisfying (4.4). We now consider the case where $b-b_2 \leq 4$ (so $c-c_1>3$). In this case, we let a_2 be the largest integer with $a_2 \leq a-a_1$ such that $24-a_2$ is a multiple of 5. Then we get $a_2 \geq 9$ and $4(b-b_1)+5(c-c_1) \geq 3(24-a_2)+19$, and hence by Lemma 2.3, there exist nonnegative integers b_2, c_2 satisfying (4.4). Now in any case, we have (4.3) and (4.4). By Lemma 3.11, we can partition $U \setminus W$ into 8 zero-sum subsets of cardinality 3. Since (4.3) and (4.4) imply $\max\{b_2, c_2\}+(\min\{b_2, c_2\})/2 \leq b_2+c_2 \leq (72-24)/4=(3/2)\cdot 8$, we have $\max\{b_2, c_2\}-\min\{b_2, c_2\} \leq (3/2)(8-\min\{b_2, c_2\})$, and hence it now follows from Lemma 3.14 that (a_2, b_2, c_2) is realizable in $(U \setminus W)+(V \setminus \{o\})$.

Finally, let $a_3=a-a_1-a_2$, $b_3=b-b_1-b_2$ and $c_3=c-c_1-c_2$. Then by (4.2), (4.3) and (4.4), a_3, b_3, c_3 are nonnegative integers and $3a_3+4b_3+5c_3=31$, and hence by Lemma 4.1, (a_3, b_3, c_3) is realizable in $U \setminus \{o\}$. Consequently, (a, b, c) is realizable in $X \setminus \{o\}$.

5. Proof of Theorem 2.

In this section, we complete the proof of Theorem 2. Let n, a, b, c be as in Theorem 2 and, as in the preceding section, let X denote an elementary abelian 2-group of order 2^n.

We proceed by induction on n. The theorem holds for $n \leq 7$ by Lemmas 3.13, 4.1, 4.2 and 4.3. Thus let $n \geq 8$, and assume that the theorem is proved for smaller values of n. By Lemma 3.12, we may assume

$$b<2^{n-3}, \tag{5.1}$$

and hence

$$3a+5c>2^{n-1}. \tag{5.2}$$

Let U, V be subgroups of order 2^{n-4} and 2^4 such that $U \cap V=\{o\}$. If n is odd, let W be a subgroup of order 8 of U; if n is even, let $W=\{o\}$. Since $n \geq 8$, we have

$$|W| \leq 2^{n-6}. \tag{5.3}$$

We aim at showing that we can write $a=a_1+a_2+a_3$, $b=b_1+b_2+b_3$ and $c=c_1+c_2+c_3$ so that (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3) are realizable in $W+(V \setminus \{o\})$, $(U \setminus W)+(V \setminus \{o\})$ and $U \setminus \{o\}$.

We first take up $W+(V \setminus \{o\})$. By (5.2), we have $3a>2^{n-2}$ or $5c>2^{n-2}$. Assume first that $3a>2^{n-2}$. In this case, we let $(a_1, b_1, c_1)=(5|W|, 0, 0)$. By (5.3), we have $a_1<a$. By Lemma 3.13, we can partition $V \setminus \{o\}$ into five zero-sum subsets $P_1, \cdots, P_5$ of cardinality 3. Then for each $1 \leq i \leq 5$, there is a family K_i of subsets of $W+P_i$ realizing

$(|W|, 0, 0)$ (in the case where n is odd, we here use Lemma 3.8). Consequently, the family $K=\bigcup_{1\le i\le 5} K_i$ realizes (a_1, b_1, c_1) in $W+(V\setminus\{o\})$. Assume now that $3a\le 2^{n-2}$, so $5c>2^{n-2}$. In this case, we let $(a_1, b_1, c_1)=(0, 0, 3|W|)$. By (5.3), we have $c_1<c$. By Lemma 3.13, we can partition $V\setminus\{o\}$ into three zero-sum subsets R_1, R_2, R_3 of cardinality 5. Then we see from Lemma 3.9 that for each $1\le i\le 3$, there is a family K_i of subsets of $W+R_i$ realizing $(0, 0, |W|)$. Consequently, the family $K=\bigcup_{1\le i\le 3} K_i$ realizes (a_1, b_1, c_1) in $W+(V\setminus\{o\})$.

We now consider $(U\setminus W)+(V\setminus\{o\})$ and $U\setminus\{o\}$. Let $r=(|U|-|W|)/3$. Then

$$\begin{aligned} &3(a-a_1)+4(b-b_1)+5(c-c_1) \\ &\quad =|U\setminus W|\cdot|V\setminus\{o\}|+|U\setminus\{o\}|=45r+|U\setminus\{o\}|. \end{aligned} \tag{5.4}$$

We also have

$$\begin{aligned} [((b-b_1)-1)/9]<b/9<2^{n-3}/9 \qquad &\text{(by (5.1))} \\ <(2^{n-1}-15|W|)/9 \qquad &\text{(by (5.3))} \\ =(2^{n-1}-(3a_1+5c_1))/9 \qquad & \\ <((3a+5c)-(3a_1+5c_1))/9 \qquad &\text{(by (5.2))} \\ \le(a-a_1)/3+(c-c_1). \qquad & \end{aligned}$$

Since (5.4) implies $3(a-a_1)+4(b-b_1)+5(c-c_1)=45r+(2^{n-4}-1)\ge 45r+15$, it now follows from Lemma 2.2 that there exist nonnegative integers $x_1, y_1, z_1; x_2, y_2, z_2; \cdots; x_r, y_r, z_r$ such that

$$3x_i+4y_i+5z_i=45 \tag{5.5}$$

for all i, $\sum x_i\le a-a_1$, $\sum y_i\le b-b_1$ and $\sum z_i\le c-c_1$. Let $a_2=\sum x_i$, $b_2=\sum y_i$, $c_2=\sum z_i$, $a_3=a-a_1-a_2$, $b_3=b-b_1-b_2$, $c_3=c-c_1-c_2$. By the induction hypothesis, it follows from (5.4) and (5.5) that there exists a family L of subsets of $U\setminus\{o\}$ realizing (a_3, b_3, c_3). By Lemma 3.11, we can partition $U\setminus W$ into r zero-sum subsets $S_1, \cdots, S_r$ of cardinality 3, and we see from Lemma 3.15 that for each $1\le i\le r$, there exists a family N_i of subsets of $S_i+(V\setminus\{o\})$ realizing (x_i, y_i, z_i). Then $\bigcup_{1\le i\le r} N_i$ realizes (a_2, b_2, c_2) in $(U\setminus W)+(V\setminus\{o\})$. Consequently, the family $K\cup L\cup(\bigcup_{1\le i\le r} N_i)$ realizes (a, b, c) in $X\setminus\{o\}$.

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