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The Integral Representations of Harmonic Polynomials in the Case of $\mathfrak{su}(p, 1)$

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Introduction.

Let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the complexification of a Cartan decomposition of $\mathfrak{g}_{\mathbf{R}}$, where $\mathfrak{g}_{\mathbf{R}}$ is a noncompact real form of $\mathfrak{g}$. Kostant-Rallis [3] showed that polynomials on $\mathfrak{p}$ are expressed as the tensor product of harmonic polynomials and K-invariant polynomials, where $K=\exp \operatorname{ad} \mathfrak{k}$. Related to this result, we showed in [14] that the set of common zero points of all K-invariant polynomials on $\mathfrak{p}$ is a uniqueness set of holomorphic functions on $\mathfrak{p}$ (see Proposition 1.1).

On the other hand, for classical harmonic functions on $\mathbf{C}^p$ and functions on the sphere, there are many studies (see, [2], [4], [5], [6], [7], [10], [12], [15], etc.). For example, it is known that harmonic functions on $\mathbf{C}^p$ are represented by an integral on some $O(p)$-orbits, and the reproducing kernels of these formulas are expressed by the Legendre polynomials (cf. Lemma 1.2). For details, see [7] Lemma 7 and [15] Theorem 2.4. In the Lie algebraic viewpoint, classical harmonic functions on $\mathbf{C}^p$ correspond to harmonic functions on $\mathfrak{p}$ for the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{so}(p, 1)$, and we can easily rewrite the classical integral formulas in Lemma 1.2 in the Lie algebraic form (A.1)–(A.4) in Appendix.

Our purpose of this paper is to obtain integral representation formulas of harmonic polynomials in the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{su}(p, 1)$. Our main results in this paper are described in Theorem 2.2, in which we obtain the similar results to the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{so}(p, 1)$. In the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{so}(p, 1)$ harmonic functions are expressed in the form of integral on some simple $K_{\mathbf{R}}$-orbits, where $K_{\mathbf{R}}=\exp \operatorname{ad} \mathfrak{k}_{\mathbf{R}}$. But in the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{su}(p, 1)$ we express the formulas in the form of double integrals on some family of $K_{\mathbf{R}}$-orbits.

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1. Preliminaries.

In this section we fix the notations and review known results. For details, see [2],

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[3], [5], [7], and [15].

Let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathfrak{g}_{\mathbf{R}}$ be a noncompact real form of $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbf{R}}=\mathfrak{k}_{\mathbf{R}}+\mathfrak{p}_{\mathbf{R}}$ be a Cartan decomposition of $\mathfrak{g}_{\mathbf{R}}$ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the direct sum obtained by complexifying $\mathfrak{k}_{\mathbf{R}}$ and $\mathfrak{p}_{\mathbf{R}}$. In this paper, for a Lie algebra $\mathfrak{h}$, we denote by $\exp \operatorname{ad}\mathfrak{h}$ the adjoint group of $\mathfrak{h}$. We put $G=\exp\operatorname{ad}\mathfrak{g}$ and $K_{\theta}=\{a\in G;\ \theta a=a\theta\}$, where $\theta\colon \mathfrak{g}\to\mathfrak{g}$ is the Lie algebra automorphism of order 2 defined by $\theta=1$ on $\mathfrak{k}$, $\theta=-1$ on $\mathfrak{p}$. Let K be the identity component of K_{θ}. Then we have $K=\exp\operatorname{ad}\mathfrak{k}$. Furthermore, we put $K_{\mathbf{R}}=\exp\operatorname{ad}\mathfrak{k}_{\mathbf{R}}$ which acts on the space $\mathfrak{p}$. Then we have $K_{\mathbf{R}}=K\cap\exp\operatorname{ad}\mathfrak{g}_{\mathbf{R}}$. We denote by S the symmetric algebra on $\mathfrak{p}$ and we put $J=\{u\in S;\ au=u \text{ for any } a\in K_{\theta}\}$ and $J_{+}=\{u\in J;\ \partial(u)1=0\}$. We denote by J' the ring of K-invariant polynomials on $\mathfrak{p}$ and we put $J'_{+}=\{f\in J';\ f(0)=0\}$. Let S' be the ring of all polynomials on $\mathfrak{p}$ and let S'_n be the space of homogeneous polynomials on $\mathfrak{p}$ of degree n. For $f\in S'$ and $a\in K_{\theta}$, $af\in S'$ is defined by $(af)(x)=f(a^{-1}x)$ $(x\in\mathfrak{p})$. It is known that any element of J' is invariant under K_{θ} ([3] Proposition 10). It is also known that J' has homogeneous generators $P_1,\cdots,P_r$, where $r=\dim\mathfrak{a}_{\mathbf{R}}$ and $\mathfrak{a}_{\mathbf{R}}$ is a maximal abelian subalgebra of $\mathfrak{p}_{\mathbf{R}}$. Let $\mathscr{H}=\{f\in S';\ \partial(u)f=0 \text{ for any } u\in J_{+}\}$ be the space of harmonic polynomials on $\mathfrak{p}$. We put $\mathscr{H}_n=S'_n\cap\mathscr{H}$ and $J'_n=S'_n\cap J'$. Let $\mathcal{O}(\mathfrak{p})$ and $\mathcal{O}_0(\mathfrak{p})=\{f\in\mathcal{O}(\mathfrak{p});\ \partial(u)f=0 \text{ for any } u\in J_{+}\}$ be the space of holomorphic functions and the space of harmonic functions on $\mathfrak{p}$, respectively. We put $\mathfrak{N}=\{x\in\mathfrak{p};\ h(x)=0 \text{ for any } h\in J'_{+}\}$ and denote by $\mathcal{O}(\mathfrak{N})$ the space of holomorphic functions on the analytic set $\mathfrak{N}$. By the Oka-Cartan Theorem we have $\mathcal{O}(\mathfrak{N})=\mathcal{O}(\mathfrak{p})|_{\mathfrak{N}}$. We put $(J'_{+}S')_n=J'_{+}S'\cap S'_n$, and $\mathbf{Z}_{+}=\{0,1,2,3,\cdots\}$. Then the following proposition is known.

PROPOSITION 1.1 ([3], [8], [14]). (i) *For any $n\in\mathbf{Z}_{+}$ we have*

$$S'_n=(J'_{+}S')_n\oplus\mathscr{H}_n\,.$$

(ii) *The restriction mapping $f\to f|_{\mathfrak{N}}$ is a bijection from $\mathcal{O}_0(\mathfrak{p})$ onto $\mathcal{O}(\mathfrak{N})$.*

$H_n(\mathbf{C}^p)$ denotes the space of homogeneous harmonic polynomials of degree n on $\mathbf{C}^p$ and $H_{n,p}$ denotes the space of spherical harmonics of degree n on S^{p-1} $(p\geqq 3)$. It is well known that the restriction mapping $\gamma\colon f\to f|_{S^{p-1}}$ is a bijection from $H_n(\mathbf{C}^p)$ onto $H_{n,p}$ (cf. [5], [7], etc.). For spherical harmonics, harmonic functions on $\mathbf{C}^p$ and functions on S^{p-1}, we refer the reader to [5], [6], [7], etc.

$P_{n,p}$ denotes the Legendre polynomial of degree n and dimension p. For z, $w\in\mathbf{C}^p$ we put $z\cdot w={}^tzw$. We put $N^{p-1}=\{z\in\mathbf{C}^p;\ z\cdot z=0,\ z\cdot\bar{z}=2\}$. Then the following lemma is known.

LEMMA 1.2 ([2], [5], [6], [7], [15]). (i) *We put $h_a(x)=P_{n,p}(x\cdot a)$ and $g_b(x)=(x\cdot b)^n$ $(x,a,b\in\mathbf{C}^p)$. Then $H_{n,p}$ is generated by the set $\{h_a|_{S^{p-1}};\ a\in S^{p-1}\}$ and $H_n(\mathbf{C}^p)$ is generated by the set $\{g_b;\ b\in N^{p-1}\}$.*

(ii) *For any $f\in H_m(\mathbf{C}^p)$ and $g\in H_n(\mathbf{C}^p)$ it is valid that*

$$(1.1)\qquad \dim H_{n,p}\int_{S^{p-1}} f(s)P_{n,p}(s\cdot a)ds=\delta_{m,n}f(a)\qquad (a\in S^{p-1}),$$

$$(1.2)\qquad \dim H_{n,p}\int_{N^{p-1}} f(z)(\bar{z}\cdot w)^n dN(z)=2^n\delta_{m,n}f(w)\qquad (w\in \mathbf{C}^p).$$

$$(1.3)\qquad \int_{S^{p-1}} f(s)\overline{g(s)}ds=2^{-2n}\frac{n!\Gamma(p/2)}{\Gamma(n+p/2)}\dim H_{n,p}\int_{N^{p-1}} f(z)\overline{g(z)}dN(z),$$

where ds and dN denote the unique $O(p)$-invariant measures on S^{p-1} and on N^{p-1} such that $\int_{S^{p-1}} 1 ds=\int_{N^{p-1}} 1 dN(z)=1$, respectively, and

$$\dim H_{n,p}=\frac{(2n+p-2)(n+p-3)!}{n!(p-2)!}.$$

(iii) *For any $f\in\mathcal{O}_0(\mathbf{C}^p)$ we have*

$$(1.4)\qquad f(x)=\int_{N^{p-1}} f(z)(1+(x\cdot\bar{z})/2)(1-(x\cdot\bar{z})/2)^{-p+1}dN(z).$$

For $z\in\mathbf{C}^p$ and $a\in S^{p-1}$ we put

$$\tilde{P}_{n,p}(z,a)=P_{n,p}\left(\frac{z\cdot a}{\sqrt{z\cdot z}}\right)(z\cdot z)^{n/2}.$$

Then $\tilde{P}_{n,p}(\ ,a)$ belongs to $H_n(\mathbf{C}^p)$ and $\tilde{P}_{n,p}(s,a)=P_{n,p}(s\cdot a)$ for any $s\in S^{p-1}$.

2. The case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{su}(p,1)$.

In this section we obtain integral representations of harmonic polynomials on some $K_{\mathbf{R}}$-orbits in the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{su}(p,1)$ $(p\in\mathbf{Z}_+, p\geq 2)$.

We put $\mathfrak{g}=\mathfrak{sl}(p+1,\mathbf{C})$ and

$$\mathfrak{g}_{\mathbf{R}}=\mathfrak{su}(p,1)=\left\{\begin{pmatrix} A & x\\ {}^t\bar{x} & \alpha\end{pmatrix};\ A\in\mathfrak{u}(p),\ \alpha\in\mathfrak{u}(1),\ \mathrm{Tr}A+\alpha=0,\ x\in\mathbf{C}^p\right\}.$$

In this case we have

$$\mathfrak{k}_{\mathbf{R}}=\left\{\begin{pmatrix} A & 0\\ 0 & \alpha\end{pmatrix};\ A\in\mathfrak{u}(p),\ \alpha\in\mathfrak{u}(1),\ \mathrm{Tr}A+\alpha=0\right\},$$

$$\mathfrak{p}_{\mathbf{R}}=\left\{\begin{pmatrix} 0 & x\\ {}^t\bar{x} & 0\end{pmatrix};\ x\in\mathbf{C}^p\right\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix} A & 0\\ 0 & \alpha\end{pmatrix};\ A\in M(p,\mathbf{C}),\ \mathrm{Tr}A+\alpha=0\right\},$$

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & x\\ {}^t y & 0\end{pmatrix};\ x,y\in\mathbf{C}^p\right\}.$$

And we get $G=\mathrm{Ad}\,SL(p+1, \mathbf{C})$, K is the identity component of $\left\{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix};\ |A|b=1\right\}$, and

$$K_{\mathbf{R}}=\mathrm{Ad}\,S(U(p)\times U(1))=\left\{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix};\ A\in U(p),\ b\in U(1),\ |A|b=1\right\}.$$

We can also express $K_{\mathbf{R}}$ as $K_{\mathbf{R}}=\left\{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix};\ A\in U(p)\right\}$. For $X=\begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}\in\mathfrak{p}$ and $g=\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\in K_{\mathbf{R}}$ ($A\in U(p)$) we have $gX=\begin{pmatrix} 0 & Ax \\ {}^t(\bar{A}y) & 0 \end{pmatrix}$. For this Lie algebra $\mathfrak{g}$ the Killing form $B(X, Y)$ equals $(2p+2)\mathrm{Tr}(XY)$. The generator of J' is $B(X, X)$. We put $P(X)=4x\cdot y=2\,\mathrm{Tr}(X^2)$. Then $\mathscr{H}_n=\{f\in S_n';\ P(D)f(X)=0\}$, where $P(D)=4\sum_{j=1}^{p}\partial^2/\partial x_j\partial y_j$. Furthermore we have $\mathfrak{N}=\left\{\begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}\in\mathfrak{p};\ x\cdot y=0\right\}$. Here we put

$$\Sigma=\left\{\begin{pmatrix} 0 & x \\ {}^t\bar{x} & 0 \end{pmatrix};\ x\in\mathbf{C}^p,\ x\cdot\bar{x}=1\right\},$$

$$N(r)=\left\{\begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}\in\mathfrak{p};\ x\cdot\bar{x}=r^2,\ y\cdot\bar{y}=1-r^2\right\}\subset\mathfrak{N}\qquad(0\le r\le 1).$$

We put

$$E_0=\begin{pmatrix} 0 & e_1 \\ {}^t e_1 & 0 \end{pmatrix},\qquad \tilde{E}_r=\begin{pmatrix} 0 & re_1 \\ (1-r^2)^{1/2}\,{}^t e_2 & 0 \end{pmatrix},$$

where $e_1={}^t(1\ 0\cdots 0)\in\mathbf{C}^p$ and $e_2={}^t(0\ 1\ 0\cdots 0)\in\mathbf{C}^p$. It is easy to show that $\Sigma=K_{\mathbf{R}}E_0$ and $N(r)=K_{\mathbf{R}}\tilde{E}_r$. We denote by K_0 the isotropy group of E_0. Then we have

$$K_0=\left\{\mathrm{Ad}\begin{pmatrix} 1 & & \\ & B & \\ & & 1 \end{pmatrix};\ B\in U(p-1)\right\}.$$

For $X=\begin{pmatrix} 0 & x \\ {}^t y & 0 \end{pmatrix}\in\mathfrak{p}$ we define the mapping $\Psi:\ \mathfrak{p}\to\mathbf{C}^{2p}$ by $\Psi(X)=\frac{1}{2}\begin{pmatrix} x+y \\ -i(x-y) \end{pmatrix}$. Then the mapping $\Psi|_{\Sigma}:\ \Sigma\to S^{2p-1}$ is bijective and $f\in\mathscr{H}_n$ if and only if $f\circ\Psi^{-1}\in H_{n,2p}$. Therefore, it is clear that $\dim\mathscr{H}_n=\dim H_{n,2p}$. We put $\langle X, Y\rangle=\mathrm{Tr}({}^t X\bar{Y})$ and $Q_{n,p}(X, Z)=\tilde{P}_{n,2p}(\Psi(X), \Psi(Z))$ for $X, Y\in\mathfrak{p}$ and $Z\in\Sigma$. Then

$$Q_{n,p}(X, Z)=2^{-n}P_{n,2p}\left(\frac{\langle X, Z\rangle}{\sqrt{P(X)}}\right)P(X)^{n/2}.$$

Then we see that $\mathscr{H}_n$ is generated by $\{Q_{n,p}(\ , Z);\ Z\in\Sigma\}$. It is also known that $\mathscr{H}_n$ is generated by $\{\langle\ , Z\rangle^n;\ Z\in\mathfrak{N}\}$ (see [3]). Let $d\mu_{\Sigma}$ and $d\mu_r$ be the unique $K_{\mathbf{R}}$-invariant measures on Σ and on $N(r)$ such that $\int_{\Sigma}1\,d\mu_{\Sigma}=\int_{N(r)}1\,d\mu_r=1$, respectively. For $g=\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\in K_{\mathbf{R}}$ we obtain

(2.1) $$\Psi(gX)=M(g)\Psi(X),$$

where

(2.2) $$M(g)=\frac{1}{2}\begin{pmatrix} A+\bar{A} & i(A-\bar{A}) \\ -i(A-\bar{A}) & A+\bar{A} \end{pmatrix} \qquad (A\in U(p)).$$

Now we define the measure $d\mu$ on Σ by

(2.3) $$\int_{\Sigma} f(X)d\mu=\int_{S^{2p-1}} f\circ\Psi^{-1}(s)ds,$$

where ds is the unique $O(2p)$-invariant measure on S^{2p-1} such that $\int_{S^{2p-1}} 1ds=1$. Since A belongs to $U(p)$, we see $M(g)\in O(2p)$ from (2.2). From (2.1) we have for any $g\in K_{\mathbf{R}}$

$$\int_{\Sigma} f(gX)d\mu=\int_{S^{2p-1}} f\circ\Psi^{-1}(M(g)s)ds$$
$$=\int_{S^{2p-1}} f\circ\Psi^{-1}(s)ds=\int_{\Sigma} f(X)d\mu.$$

Hence we see that $d\mu=d\mu_{\Sigma}$. Therefore, the following proposition is clear from (2.3) and (1.1).

PROPOSITION 2.1. *For $f\in\mathscr{H}_m$ and $X\in\Sigma$ it is valid*

(2.4) $$\dim\mathscr{H}_n\int_{\Sigma} f(Y)Q_{n,p}(Y,X)d\mu_{\Sigma}(Y)=\delta_{m,n}f(X).$$

Next we define the function ρ on $[0, 1]$ by

(2.5) $$\rho(r)=2^{2p-2}\frac{\Gamma(p-1/2)}{\pi^{1/2}\Gamma(p-1)}r^{2p-3}(1-r^2)^{p-2}.$$

Under these notations, we state our main theorem of this section.

THEOREM 2.2. (i) *For any $X\in\mathfrak{p}$ and $Y\in\Sigma$ it is valid that*

(2.6) $$\int_0^1 \rho(r)\left(\int_{N(r)}\langle Z,Y\rangle^m\overline{\langle Z,X\rangle}^n d\mu_r(Z)\right)dr=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}Q_{n,p}(X,Y).$$

(ii) *For any $f\in\mathscr{H}_m$ and any $X\in\mathfrak{p}$ we have*

(2.7) $$\int_0^1 \rho(r)\left(\int_{N(r)} f(Z)\overline{\langle Z,X\rangle}^n d\mu_r(Z)\right)dr=(\dim\mathscr{H}_n)^{-1}\delta_{m,n}f(X).$$

(iii) *For any $f\in\mathscr{H}_m$ and $g\in\mathscr{H}_n$ we have*

(2.8) $$\int_{\Sigma} f(X)\overline{g(X)}d\mu_{\Sigma}(X)=\frac{n!\Gamma(p)\dim\mathscr{H}_n}{\Gamma(n+p)}\int_0^1 \rho(r)\left(\int_{N(r)} f(X)\overline{g(X)}d\mu_r(X)\right)dr.$$

To prove Theorem 2.2, we need some lemmas.

LEMMA 2.3 ([2], [5], [7], [15]). (i) *For* $\theta\in\mathbf{R}$ *we have*

$$P_{n,2p}(\cos\theta)=\int_0^1\{\cos\theta+(2r^2-1)i\sin\theta\}^n\rho(r)dr\,. \tag{2.9}$$

(ii) *Let* $\alpha,\ \beta\in\mathbf{C}^{2p}$, $\alpha\cdot\alpha=\beta\cdot\beta=0$. *Then we have*

$$\int_{S^{2p-1}}(s\cdot\alpha)^m\overline{(s\cdot\beta)^n}ds=\frac{2^{-n}n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}(\alpha\cdot\bar{\beta})^n\,. \tag{2.10}$$

(iii) *If* X *and* Y *belong to* $\mathfrak{N}$, *we get*

$$\int_\Sigma\langle X,Z\rangle^m\overline{\langle Y,Z\rangle^n}d\mu_\Sigma(Z)=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}\langle X,Y\rangle^n\,. \tag{2.11}$$

PROOF. (i) It is well known that the following equation holds for $0\le t\le 1$:

$$P_{n,2p}(t)=\frac{\Gamma(p-1/2)}{\sqrt{\pi}\,\Gamma(p-1)}\int_{-1}^1\{t\pm i(1-t^2)^{1/2}x\}^n(1-x^2)^{p-2}dx$$

(see [5], [7]). From this formula and (2.5) we get (2.9) by putting $r=\sqrt{(x+1)/2}$.

(ii) Suppose $\alpha\cdot\alpha=\beta\cdot\beta=0$. Then we have $(z\cdot\alpha)^m\in H_m(\mathbf{C}^{2p})$ and $\overline{(z\cdot\beta)^n}\in H_n(\mathbf{C}^{2p})$. Hence (2.10) follows from (1.2) and (1.3).

(iii) We put $X=\begin{pmatrix}0 & a_1\\ {}^ta_2 & 0\end{pmatrix}$, $Y=\begin{pmatrix}0 & b_1\\ {}^tb_2 & 0\end{pmatrix}\in\mathfrak{N}$ and $Z=\begin{pmatrix}0 & x\\ {}^t\bar{x} & 0\end{pmatrix}\in\Sigma$ $(x=z+iw,\ z,\ w\in\mathbf{R}^p)$. Then we have by (2.10)

$$\begin{aligned}
&\int_\Sigma\langle X,Z\rangle^m\overline{\langle Y,Z\rangle^n}d\mu_\Sigma(Z)\\
&=\int_{S^{2p-1}}\left\{\begin{pmatrix}a_1\\ a_2\end{pmatrix}\cdot\overline{\begin{pmatrix}z+iw\\ z-iw\end{pmatrix}}\right\}^m\overline{\left\{\begin{pmatrix}b_1\\ b_2\end{pmatrix}\cdot\overline{\begin{pmatrix}z+iw\\ z-iw\end{pmatrix}}\right\}^n}dzdw\\
&=\int_{S^{2p-1}}\left\{\begin{pmatrix}a_1+a_2\\ -i(a_1-a_2)\end{pmatrix}\cdot\begin{pmatrix}z\\ w\end{pmatrix}\right\}^m\overline{\left\{\begin{pmatrix}b_1+b_2\\ -i(b_1-b_2)\end{pmatrix}\cdot\begin{pmatrix}z\\ w\end{pmatrix}\right\}^n}dzdw\\
&=\frac{2^{-n}n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}\{(a_1+a_2)\cdot(\bar{b}_1+\bar{b}_2)+(a_1-a_2)\cdot(\bar{b}_1-\bar{b}_2)\}^n\\
&=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}\langle X,Y\rangle^n
\end{aligned}$$

because $(a_1+a_2)\cdot(a_1+a_2)-(a_1-a_2)\cdot(a_1-a_2)=4a_1\cdot a_2=0$ and $(b_1+b_2)\cdot(b_1+b_2)-(b_1-b_2)\cdot(b_1-b_2)=4b_1\cdot b_2=0$. Q.E.D.

LEMMA 2.4. *Let $L_n \in \mathcal{H}_n$. If $L_n(kX)=L_n(X)$ for any $k \in K_0$, then $L_n(X)$ is expressed as follows*:

$$L_n(X)=\sum_{(l,m)\in\Lambda} C_{l,m}x_1^l y_1^m (x'\cdot y')^{(n-l-m)/2}, \tag{2.12}$$

where $\Lambda=\{(l,m)\in \mathbf{Z}_+^2\ ;\ n\equiv l+m\ (\mathrm{mod}\,2)\}$, $C_{l,m}\in\mathbf{C}$, $X=\begin{pmatrix} 0 & x \\ {}^t y & 0\end{pmatrix}\in\mathfrak{p}$, $x=\begin{pmatrix} x_1 \\ x'\end{pmatrix}$, $y=\begin{pmatrix} y_1 \\ y'\end{pmatrix}$, x_1, $y_1\in\mathbf{C}$ *and* $x', y'\in\mathbf{C}^{p-1}$.

PROOF. Since L_n is a homogeneous polynomial of degree n, L_n can be expressed as follows:

$$L_n(X)=\sum_{\substack{0\le l,m\le n\\ 0\le l+m\le n}} x_1^l y_1^m A_{l,m}({}^t(x'\ \ y')), \tag{2.13}$$

where $A_{l,m}$ is a homogeneous polynomial of $\begin{pmatrix} x' \\ y'\end{pmatrix}$ of degree $n-l-m$. For any $k=\mathrm{Ad}\begin{pmatrix} 1 & & \\ & B & \\ & & 1\end{pmatrix}\in K_0$ $(B\in U(p-1))$

$$kX=\begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & Bx' \\ y_1 & {}^t(\bar{B}y') & 0\end{pmatrix}.$$

From (2.13) we have

$$L_n(kX)=\sum_{\substack{0\le l,m\le n\\ 0\le l+m\le n}} x_1^l y_1^m A_{l,m}({}^t(Bx'\ \ \bar{B}y')).$$

Since $L_n(X)=L_n(kX)$ for any $B\in U(p-1)$,

$$A_{l,m}({}^t(Bx'\ \ \bar{B}y'))=A_{l,m}({}^t(x'\ \ y')). \tag{2.14}$$

In general it is known that $f\in J'_+$ if and only if f is fixed under $K_{\mathbf{R}}$ (cf. [3] p. 800). Let $\mathfrak{g}'=\mathfrak{sl}(p,\mathbf{C})$ and let $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ be the complexification of a Cartan decomposition of $\mathfrak{g}'_{\mathbf{R}}=\mathfrak{su}(p-1,1)$. Then $\begin{pmatrix} 0 & x' \\ {}^t y' & 0\end{pmatrix}$ belongs to $\mathfrak{p}'$ and $A_{l,m}({}^t(x'\ \ y'))$ is $\exp\mathrm{ad}\,\mathfrak{k}'_{\mathbf{R}}$-invariant from (2.14). Therefore, we have

$$A_{l,m}({}^t(x'\ \ y'))=C_{l,m}(x'\cdot y')^{(n-l-m)/2}, \tag{2.15}$$

where $C_{l,m}$ is some constant. Since $x'\cdot y'$ is a homogeneous polynomial of degree 2, (2.12) follows from (2.15). Q.E.D.

LEMMA 2.5. *For any $X\in\mathfrak{p}$ we have*

$$\int_0^1 \rho(r)\left(\int_{N(r)} \langle Y, E_0\rangle^m \overline{\langle Y, X\rangle}^n d\mu_r(Y)\right)dr = \frac{n!\Gamma(p)}{\Gamma(n+p)}\,\delta_{m,n} Q_{n,p}(X, E_0)\,. \tag{2.16}$$

PROOF. For any $X \in \mathfrak{p}$ we put

$$F(X) = \int_0^1 \rho(r)\left(\int_{N(r)} \langle Y, E_0\rangle^m \overline{\langle Y, X\rangle}^n d\mu_r(Y)\right)dr\,. \tag{2.17}$$

Then $F \in \mathscr{H}_n$ because $N(r) \subset \mathfrak{N}$. Furthermore, F is K_0-invariant because the inner product $\langle\ ,\ \rangle$ and $d\mu_r$ are $K_{\mathbf{R}}$-invariant. Hence, by Lemma 2.4 $F(X)$ can be expressed as follows:

$$F(X) = \sum_{(l,m)\in\Lambda} C_{l,m} x_1^l y_1^m (x'\cdot y')^{(n-l-m)/2}\,.$$

We put $X_\theta = \begin{pmatrix} 0 & e^{i\theta}e_1 \\ {}^t(e^{-i\theta}e_1) & 0 \end{pmatrix} \in \Sigma$, $h_\theta = \mathrm{Ad}\begin{pmatrix} e^{-i\theta}I_p & 0 \\ 0 & 1 \end{pmatrix} \in K_{\mathbf{R}}$ $(\theta \in \mathbf{R})$. So we have $X_\theta = h_\theta^{-1} E_0$ and $gh_\theta = h_\theta g$ for any $g \in K_{\mathbf{R}}$. We put

$$G_r(X_\theta) = \int_{N(r)} \langle Y, E_0\rangle^m \overline{\langle Y, X_\theta\rangle}^n d\mu_r(Y)\,.$$

Then it is valid that for any $g \in K_{\mathbf{R}}$

$$\begin{aligned} G_r(X_\theta) &= \int_{N(r)} \langle Y, E_0\rangle^m \overline{\langle Y, h_\theta^{-1}E_0\rangle}^n d\mu_r(Y) \\ &= \int_{N(r)} \langle Y, E_0\rangle^m \overline{\langle h_\theta Y, E_0\rangle}^n d\mu_r(Y) \\ &= \int_{N(r)} \langle g^{-1}Y, E_0\rangle^m \overline{\langle h_\theta g^{-1} Y, E_0\rangle}^n d\mu_r(Y) \\ &= \int_{N(r)} \langle g^{-1}Y, E_0\rangle^m \overline{\langle g^{-1} h_\theta Y, E_0\rangle}^n d\mu_r(Y) \\ &= \int_{N(r)} \langle Y, gE_0\rangle^m \overline{\langle h_\theta Y, gE_0\rangle}^n d\mu_r(Y)\,. \end{aligned} \tag{2.18}$$

Let dg be the Haar measure on $K_{\mathbf{R}}$ such that $\int_{K_{\mathbf{R}}} 1\, dg = 1$. (2.18) gives

$$\begin{aligned} G_r(X_\theta) &= \int_{K_{\mathbf{R}}} G_r(X_\theta)dg \\ &= \int_{K_{\mathbf{R}}}\left(\int_{N(r)} \langle Y, gE_0\rangle^m \overline{\langle h_\theta Y, gE_0\rangle}^n d\mu_r(Y)\right)dg \\ &= \int_{N(r)}\left(\int_{K_{\mathbf{R}}} \langle Y, gE_0\rangle^m \overline{\langle h_\theta Y, gE_0\rangle}^n dg\right)d\mu_r(Y)\,. \end{aligned}$$

Since $K_{\mathbf{R}}E_0=\Sigma$, and Y and $h_\theta Y$ belong to $N(r)\subset\mathfrak{N}$, we have from (2.11)

$$(2.19)\qquad \int_{K_{\mathbf{R}}}\langle Y, gE_0\rangle^m\overline{\langle h_\theta Y, gE_0\rangle}^n dg=\int_\Sigma \langle Y, Z\rangle^m\overline{\langle h_\theta Y, Z\rangle}^n d\mu_\Sigma(Z)$$

$$=\frac{n!\Gamma(p)}{\Gamma(n+p)}\,\delta_{m,n}\langle Y, h_\theta Y\rangle^n .$$

For any $Y=\begin{pmatrix}0 & a\\ {}^t b & 0\end{pmatrix}\in N(r)$ we have $h_\theta Y=\begin{pmatrix}0 & e^{-i\theta}a\\ {}^t(e^{i\theta}b) & 0\end{pmatrix}$, $a\cdot\bar{a}=r^2$, and $b\cdot\bar{b}=1-r^2$. Therefore, we get

$$(2.20)\qquad \langle Y, h_\theta Y\rangle=e^{i\theta}a\cdot\bar{a}+e^{-i\theta}b\cdot\bar{b}=e^{i\theta}r^2+e^{-i\theta}(1-r^2)$$

$$=\cos\theta+(2r^2-1)i\sin\theta .$$

If $n=m$, we have from (2.19) and (2.20)

$$(2.21)\qquad G_r(X_\theta)=\frac{n!\Gamma(p)}{\Gamma(n+p)}\int_{N(r)}\{\cos\theta+(2r^2-1)i\sin\theta\}^n d\mu_r(Y)$$

$$=\frac{n!\Gamma(p)}{\Gamma(n+p)}\{\cos\theta+(2r^2-1)i\sin\theta\}^n .$$

From (2.17), (2.21), and (2.9) we have

$$F(X_\theta)=\frac{n!\Gamma(p)}{\Gamma(n+p)}\int_0^1 \rho(r)\{\cos\theta+(2r^2-1)i\sin\theta\}^n dr$$

$$=\frac{n!\Gamma(p)}{\Gamma(n+p)}P_{n,2p}(\cos\theta)=\frac{n!\Gamma(p)}{\Gamma(n+p)}Q_{n,p}(X_\theta, E_0) .$$

On the other hand, we have

$$(2.22)\qquad F(X_\theta)=\sum_{l=0}^n C_{l,n-l}e^{i(2l-n)\theta} .$$

Since $F\in\mathscr{H}_n$, we have

$$P(D)F(X)=4\sum_{\substack{(l,m)\in\Lambda\\ 0\le l+m\le n-2}}\left(\left\{C_{l+1,m+1}(l+1)(m+1)+C_{l,m}\frac{(n-l-m)(n-l-m+2p-4)}{4}\right\}\cdot x_1^l y_1^m(x'\cdot y')^{(n-l-m-2)/2}\right)=0 .$$

If $(l, m)\in\Lambda$ and $0\le l+m\le n-2$, we have from this equality

$$(2.23)\qquad C_{l,m}=\frac{-4(l+1)(m+1)}{(n-l-m)(n-l-m+2p-4)}\,C_{l+1,m+1} .$$

(2.23) shows that we can determine all coefficients of $F(X)$ uniquely by $C_{l,n-l}$ $(l=0, 1, \cdots, n)$. If we put $H(X)=(n!\Gamma(p)/\Gamma(n+p))Q_{n,p}(X, E_0)$, H belongs to $\mathscr{H}_n$ and $H(kX)=H(X)$ for any $k\in K_0$. Hence we can express

$$H(X)=\sum_{(l,m)\in\Lambda} D_{l,m}x_1^l y_1^m(x'\cdot y')^{(n-l-m)/2}, \tag{2.24}$$

where $D_{l,m}\in\mathbf{C}$. In addition, $D_{l,m}$ also satisfies (2.23). Furthermore, since $H(X_\theta)=F(X_\theta)$, we have $D_{l,n-l}=C_{l,n-l}$ by (2.22) and (2.24). Therefore, for any $(l, m)\in\Lambda$ we obtain $D_{l,m}=C_{l,m}$, which implies (2.16).

When $n\neq m$, we have $C_{l,n-l}=0$ $(l=0, 1, \cdots, n)$ because $F(X_\theta)=0$. Therefore, we get $F(X)=0$ by (2.23). Q.E.D.

PROOF OF THEOREM 2.2. (i) From Lemma 2.5 we have for any $X\in\Sigma$

$$\int_0^1 \rho(r)\left(\int_{N(r)}\langle Z, E_0\rangle^m\overline{\langle Z, X\rangle}^n d\mu_r(Z)\right)dr$$
$$=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}P_{n,2p}\left(\frac{1}{2}\langle X, E_o\rangle\right).$$

For any $Y\in\Sigma$ there exists some $g\in K_{\mathbf{R}}$ such that $Y=gE_0$. Hence we have

$$\begin{aligned}
&\int_0^1 \rho(r)\left(\int_{N(r)}\langle Z, Y\rangle^m\overline{\langle Z, X\rangle}^n d\mu_r(Z)\right)dr \\
&=\int_0^1 \rho(r)\left(\int_{N(r)}\langle Z, gE_0\rangle^m\overline{\langle Z, X\rangle}^n d\mu_r(Z)\right)dr \\
&=\int_0^1 \rho(r)\left(\int_{N(r)}\langle g^{-1}Z, E_0\rangle^m\overline{\langle g^{-1}Z, g^{-1}X\rangle}^n d\mu_r(Z)\right)dr \\
&=\int_0^1 \rho(r)\left(\int_{N(r)}\langle Z, E_0\rangle^m\overline{\langle Z, g^{-1}X\rangle}^n d\mu_r(Z)\right)dr \\
&=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}P_{n,2p}\left(\frac{1}{2}\langle g^{-1}X, E_0\rangle\right) \\
&=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}P_{n,2p}\left(\frac{1}{2}\langle X, Y\rangle\right)=\frac{n!\Gamma(p)}{\Gamma(n+p)}\delta_{m,n}Q_{n,p}(X, Y).
\end{aligned} \tag{2.25}$$

It is clear that the restriction mapping $f\to f|_\Sigma$ is bijective. Since the left-hand side of (2.25) and $(n!\Gamma(p)/\Gamma(n+p))\delta_{m,n}Q_{n,p}(X, Y)$ belong to $\mathscr{H}_n$, we obtain (2.6) from (2.25).

(ii) For any $f\in\mathscr{H}_m$ there exist some positive integer M, $a_k\in\mathbf{C}$ and $Y_k\in\Sigma$ $(k=1, 2, \cdots, M)$ such that

$$f(Z)=\sum_{k=1}^{M} a_k Q_{m,p}(Z, Y_k). \tag{2.26}$$

(2.26) implies

$$\int_0^1 \rho(r)\left(\int_{N(r)} f(Z)\overline{\langle Z, X\rangle}^n d\mu_r(Z)\right)dr \tag{2.27}$$

$$= \sum_{k=1}^{M} a_k \int_0^1 \rho(r)\left(\int_{N(r)} Q_{m,p}(Z, Y_k)\overline{\langle Z, X\rangle}^n d\mu_r(Z)\right)dr .$$

Let C_n be the coefficient of the highest power in $P_{n,2p}(t)$. Then it is known that

$$C_n = \frac{2^n \Gamma(n+p)}{n!\Gamma(p)\dim \mathscr{H}_n} \tag{2.28}$$

(cf. [5], [7]) and $Q_{m,p}(Z, Y_k) = 2^{-m}C_m\langle Z, Y_k\rangle^m$ for $Z \in N(r) \subset \mathfrak{N}$. Hence the right-hand side of (2.27) equals

$$2^{-m}C_m \sum_{k=1}^{M} a_k \int_0^1 \rho(r)\left(\int_{N(r)} \langle Z, Y_k\rangle^m \overline{\langle Z, X\rangle}^n d\mu_r(Z)\right)dr$$

$$= 2^{-n}C_n \sum_{k=1}^{M} a_k \frac{n!\Gamma(p)}{\Gamma(n+p)} \delta_{m,n} Q_{n,p}(X, Y_k) = (\dim \mathscr{H}_n)^{-1}\delta_{m,n} f(X)$$

by (2.6), (2.26) and (2.28).

(iii) For any $g \in \mathscr{H}_n$ there exist some positive integer M, $X_k \in \Sigma$ and $a_k \in \mathbf{C}$ $(k=1, 2, \cdots, M)$ such that $g(Z) = \sum_{k=1}^{M} a_k Q_{n,p}(Z, X_k) = 2^{-n}C_n \sum_{k=1}^{M} a_k \langle Z, X_k\rangle^n$ for $Z \in \mathfrak{N}$. Hence, we get from (2.7) and (2.4)

$$\int_0^1 \rho(r)\left(\int_{N(r)} f(Z)\overline{g(Z)}d\mu_r(Z)\right)dr = 2^{-n}C_n(\dim \mathscr{H}_n)^{-1}\delta_{m,n} \sum_{k=1}^{M} \bar{a}_k f(X_k) \tag{2.29}$$

$$= 2^{-n}C_n \int_{\Sigma} f(Z)\overline{g(Z)}d\mu_\Sigma(Z) .$$

Finally (2.8) follows from (2.28) and (2.29). Q.E.D.

We put $\mathcal{O}_\lambda(\mathfrak{p}) = \{f \in \mathcal{O}(\mathfrak{p}) ; (P(D)+\lambda^2)f = 0\}$, where $\mathcal{O}(\mathfrak{p})$ denotes the space of holomorphic functions on $\mathfrak{p}$ and $\lambda \in \mathbf{C}$. From Theorem 2.2 and [15] Theorem 2.4 we have

COROLLARY 2.6 (cf. [15] Theorem 2.4). *For any $\lambda \in \mathbf{C}$ the restriction mapping $\alpha_\lambda : F \to F|_{\mathfrak{N}}$ is a bijection from $\mathcal{O}_\lambda(\mathfrak{p})$ onto $\mathcal{O}(\mathfrak{N})$ and $\alpha_\lambda^{-1} f$ is expressed as follows:*

$$\alpha_\lambda^{-1} f(X) = \int_0^1 \rho(r)\left(\int_{N(r)} f(Z)K_\lambda(X, Z)d\mu_r(Z)\right)dr \qquad (X \in \mathfrak{p}),$$

where

$$K_\lambda(X, Z) = \sum_{n=0}^{\infty} \dim \mathscr{H}_n\, \Gamma(n+p)\left(\frac{\lambda\sqrt{P(Z)}}{4}\right)^{-n-p+1} J_{n+p-1}(\lambda\sqrt{P(Z)}/2)\langle X, Z\rangle^n ,$$

and $J_\nu(t)$ is the Bessel function of order ν.

In particular if $\lambda=0$ and $\langle X, X\rangle < t$ $(t>0)$,

$$\alpha_0^{-1} f(X)=\int_0^1 \rho(r)\left(\int_{N(r)} f(Z)\frac{1+\langle X/t, Z\rangle}{(1-\langle X/t, Z\rangle)^{2p-1}}\, d\mu_r(Z)\right)dr\,. \tag{2.30}$$

Appendix.

In this appendix, we state the integral representations of harmonic polynomials in the case $\mathfrak{g}_{\mathbf{R}}=\mathfrak{so}(p,1)$ which are reformulation of Lemma 1.2. When $\mathfrak{g}_{\mathbf{R}}=\mathfrak{so}(p,1)$ $(p\in\mathbf{Z}_+, p\geq 3)$, we have

$$\mathfrak{g}=\left\{\begin{pmatrix} A & z \\ {}^t z & 0\end{pmatrix};\ A\in\mathfrak{so}(p,\mathbf{C}),\ z\in\mathbf{C}^p\right\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}\in M(p+1,\mathbf{C});\ A\in\mathfrak{so}(p,\mathbf{C})\right\},$$

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & z \\ {}^t z & 0\end{pmatrix}\in M(p+1,\mathbf{C}):\ z\in\mathbf{C}^p\right\}.$$

Furthermore, we have

$$K_{\mathbf{R}}=\left\{\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1\end{pmatrix};\ A\in SO(p)\right\}.$$

For $g=\mathrm{Ad}\begin{pmatrix} A & 0 \\ 0 & 1\end{pmatrix}\in K_{\mathbf{R}}$ and $Z=\begin{pmatrix} 0 & z \\ {}^t z & 0\end{pmatrix}\in\mathfrak{p}$ we have

$$gz=\begin{pmatrix} 0 & Az \\ {}^t(Az) & 0\end{pmatrix}.$$

For $Z\in\mathfrak{p}$ we put $P(Z)=z\cdot z$. Then the generator of J'_+ is $P(Z)$ and $\mathscr{H}_n$ can be identified with $H_n(\mathbf{C}^p)$. We put $\Sigma=\{Z\in\mathfrak{p}\,;\ z\cdot\bar{z}=1\}$, $N=\{Z\in\mathfrak{p}\,;\ P(Z)=0,\ z\cdot\bar{z}=2\}$. Then Σ and N are simple $K_{\mathbf{R}}$-orbits. For $Z\in\mathfrak{p}$ we define $\varphi(Z)=z$ and $Q_n(Z,X)=P_{n,p}(\varphi(Z),\varphi(X))$ $(X\in\Sigma)$. μ_Σ and μ_N denote the unique $K_{\mathbf{R}}$-invariant measures on Σ and N such that $\int_\Sigma 1 d\mu_\Sigma=\int_N 1 d\mu_N=1$, respectively. From (1.1)–(1.4) we have the following formulas which are similar to (2.4), (2.7), (2.8) and (2.30).

$$\dim\mathscr{H}_n\int_\Sigma F(Z)Q_n(Z,X)d\mu_\Sigma(Z)=\delta_{m,n}F(X)\,, \tag{A.1}$$

$$\dim\mathscr{H}_n\int_N F(Z)\langle Z,\overline{Y}\rangle^n d\mu_N(Z)=2^n\delta_{m,n}F(Y)\,, \tag{A.2}$$

(A.3) $$\int_{\Sigma} F(Z)\overline{G(Z)}d\mu_{\Sigma}(Z)=2^{-2n}\frac{n!\Gamma(p/2)}{\Gamma(n+p/2)}\dim\mathscr{H}_n\int_N F(z)\overline{G(z)}d\mu_N(z)\,,$$

(A.4) $$H(X)=\int_N H(Z)(1+\langle X/2, Z\rangle)(1-\langle X/2, Z\rangle)^{-p+1}d\mu_N(Z)\,,$$

where $F\in\mathscr{H}_m$, $G\in\mathscr{H}_n$, $H\in\mathscr{O}_0(\mathfrak{p})$, $X\in\Sigma$ and $Y\in\mathfrak{p}$.

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