

Strongly q -Additive Functions and Algebraic Independence

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1. Introduction.

Let $q \geq 2$ be a fixed integer. A complex-valued function $a(n)$ is said to be q -additive or q -multiplicative if

$$(1) \quad a(kq^t + r) = a(kq^t) + a(r), \quad a(0) = 0,$$

or

$$(2) \quad a(kq^t + r) = a(kq^t)a(r), \quad a(0) = 1$$

for any integer $k \geq 0$, $t \geq 0$, and $0 \leq r < q^t$, respectively. Furthermore, if

$$(3) \quad a(kq) = a(k),$$

$a(n)$ is said to be *strongly q -additive* or *strongly q -multiplicative*, respectively. We note that the strongly q -additive or q -multiplicative function $a(n)$ is determined completely by the initial values $a(1), \dots, a(q-1)$. This paper concerns mainly with q -additive functions.

Let $a_1(n), \dots, a_m(n)$ be m strongly q -additive functions. For each $a_k(n)$, we define a power series $f_k(z)$ by

$$(4) \quad f_k(z) := \sum_{n \geq 0} a_k(n)z^n \in \mathbf{C}[[z]] \quad (1 \leq k \leq m).$$

It follows from (1) and (3) that each $f_k(z)$ converges in $|z| < 1$ and satisfies the functional equation

$$(5) \quad f_k(z) = \frac{1-z^q}{1-z} f_k(z^q) + \frac{1}{1-z^q} \sum_{r=0}^{q-1} a_k(r)z^r \quad (1 \leq k \leq m),$$

since

$$\begin{aligned}
f_k(z) &= \sum_{n \geq 0} a_k(n)z^n = \sum_{t \geq 0} \sum_{r=0}^{q-1} \{a_k(t) + a_k(r)\} z^{tq+r} \\
&= \sum_{t \geq 0} a_k(t)z^{tq} \sum_{r=0}^{q-1} z^r + \sum_{t \geq 0} z^{tq} \sum_{r=0}^{q-1} a_k(r)z^r \\
&= \frac{1-z^q}{1-z} f_k(z^q) + \frac{1}{1-z^q} \sum_{r=0}^{q-1} a_k(r)z^r.
\end{aligned}$$

Putting

$$g_k(z) = (1-z)f_k(z) \quad (1 \leq k \leq m),$$

we have

$$(6) \quad g_k(z) = g_k(z^q) + \frac{1}{\varphi(z)} \sum_{r=0}^{q-1} a_k(r)z^r,$$

where $\varphi(z) = (1-z^q)/(1-z) \in \mathbf{C}[z]$. Because of the functional equations (6), we see by a theorem of Mahler [4] that if the functions $a_1(n), \dots, a_m(n)$ are algebraic-valued and $g_1(z), \dots, g_m(z)$ are algebraically independent over $\mathbf{C}(z)$, then the values $g_1(\alpha), \dots, g_m(\alpha)$ are algebraically independent, where α is an algebraic number with $0 < |\alpha| < 1$.

An example of a strongly q -additive function is the *sum of digits functions* $s^{(q)}(n)$, that is,

$$s^{(q)}(n) = \sum_{j=0}^t b_j,$$

where

$$(7) \quad n = b_0 + b_1q + \dots + b_tq^t,$$

$$b_t \neq 0, \quad b_j \in \{0, 1, \dots, q-1\} \quad (0 \leq j \leq t)$$

is the q -adic expansion of an integer $n \geq 0$. Another example is the function $e_i^{(q)}(n)$ defined for each $i \in \{1, 2, \dots, q-1\}$ by the number of the b_j 's ($0 \leq j \leq t$) that are equal to i .

We state our results.

THEOREM 1. *Let $a_k(n)$ ($1 \leq k \leq m$) be m strongly q -additive functions and $f_k(z)$ be defined by (4). The following statements are equivalent:*

- (i) *The functions $f_1(z), \dots, f_m(z)$ are algebraically dependent over $\mathbf{C}(z)$.*
- (ii) *The functions $a_1(n), \dots, a_m(n)$ are linearly dependent over \mathbf{C} .*
- (iii) *The $(q-1)$ -dimensional vectors $(a_1(1), \dots, a_1(q-1)), \dots, (a_m(1), \dots, a_m(q-1))$ are linearly dependent over \mathbf{C} .*

Theorem 1 implies that $\{e_i^{(q)}(n), \dots, e_{q-1}^{(q)}(n)\}$ is a base of the vector space of strongly q -additive functions over \mathbf{C} , since $e_i^{(q)}(j) = \delta_{ij}$ ($1 \leq i, j \leq q-1$).

THEOREM 2. *Let α be an algebraic number with $0 < |\alpha| < 1$. Then the values $\{\sum_{n \geq 0} s^{(q)}(n)\alpha^n\}_{q \geq 2}$ are algebraically independent. For any fixed positive integer i , the values $\{\sum_{n \geq 0} e_i^{(q)}(n)\alpha^n\}_{q > i}$ are algebraically independent.*

COROLLARY. *The functions $\{\sum_{n \geq 0} s^{(q)}(n)z^n\}_{q \geq 2}$ are algebraically independent over $\mathbf{C}(z)$. For any fixed positive integer i , the functions $\{\sum_{n \geq 0} e_i^{(q)}(n)z^n\}_{q > i}$ are algebraically independent over $\mathbf{C}(z)$.*

In Theorem 2 and Corollary, if i is not fixed, then the functions $\{\sum_{n \geq 0} e_i^{(q)}(n)z^n\}_{q > i}$ are algebraically dependent over $\mathbf{C}(z)$. In fact, the functions

$$\left\{ \sum_{n \geq 0} e_i^{(q)}(n)z^n \right\}_{(q,i) = (2,1), (4,1), (4,2), (4,3)}$$

are algebraically dependent over $\mathbf{C}(z)$, since these functions are strongly 4-additive and the dimension of the vector space of the strongly 4-additive functions is 3.

THEOREM 3. *If a function $a(n)$ is strongly q_1 -additive and also strongly q_2 -additive, where $\log q_1 / \log q_2$ is not a rational number, then $a(n)$ must be identically zero.*

THEOREM 4. *If a function $a(n)$ is strongly q_1 -multiplicative and also strongly q_2 -multiplicative, where $\log q_1 / \log q_2$ is not a rational number, then $a(n) = 0$ ($n \geq 1$) or $a(n) = \gamma^n$ ($n \geq 1$), where $\gamma^{q_1 - 1} = \gamma^{q_2 - 1} = 1$.*

2. Lemmas.

LEMMA 1 (Kubota [1], Loxton and van der Poorten [2], see Nishioka [6]). *Suppose $f_{i,j}(z) \in \mathbf{C}[[z]]$ ($1 \leq i \leq n, 1 \leq j \leq M_i$) satisfy the functional equations*

$$f_{i,j}(z^q) = a_i f_{i,j}(z) + b_{i,j}(z) \quad (1 \leq i \leq n, 1 \leq j \leq m_i),$$

where $a_i \in \mathbf{C}, a_i \neq a_j$ ($i \neq j$), and $b_{i,j}(z) \in \mathbf{C}(z)$. If $f_{i,j}(z)$ ($1 \leq i \leq n, 1 \leq j \leq M_i$) are algebraically dependent over $\mathbf{C}(z)$, then there exists an integer i_0 ($1 \leq i_0 \leq n$) and $c_1, \dots, c_{M_{i_0}} \in \mathbf{C}$, not all zero, such that

$$c_1 f_{i_0,1}(z) + \dots + c_{M_{i_0}} f_{i_0,M_{i_0}}(z) \in \mathbf{C}(z).$$

LEMMA 2 (Toshimitsu [7]). *Let N be a positive integer and put $\varphi(z) = (1 - z^q)/(1 - z)$. If a polynomial $P(z) \in \mathbf{C}[z]$ with $\deg P(z) \leq N(q - 1)$ and a rational function $c(z) \in \mathbf{C}(z)$ satisfy*

$$c(z) + \frac{P(z)}{\varphi(z)^N} = c(z^q),$$

then $c(z) \in \mathbf{C}$ and $P(z)$ is identically zero.

LEMMA 3 (Nishioka [5]). *Let d_1, \dots, d_t be integers greater than 1 and suppose that $\log d_i / \log d_j \notin \mathbf{Q}$ if $i \neq j$. Let K be an algebraic number field. Assume that $f_{i,j}(z) \in K[[z]]$ ($1 \leq i \leq t, 1 \leq j \leq M_i$) satisfy the functional equations*

$$f_{i,j}(z^{d_i}) = a_{i,j}(z)f_{i,j}(z) + b_{i,j}(z) \quad (1 \leq i \leq t, 1 \leq j \leq M_i),$$

where $a_{i,j}(z), b_{i,j}(z) \in K(z)$, $a_{i,j}(0) = 1$, and $f_{i,1}(z), \dots, f_{i,M_i}(z)$ are algebraically independent over $K(z)$ for each i ($1 \leq i \leq t$). If α is an algebraic number with $0 < |\alpha| < 1$, $a_{i,j}(\alpha^{d^k}) \neq 0$ ($k \geq 0$) and all $f_{i,j}(z)$ converge at α , then the values

$$f_{i,j}(\alpha) \quad (1 \leq i \leq t, 1 \leq j \leq M_i)$$

are algebraically independent.

3. Proof of the theorems.

PROOF OF THEOREM 1. (ii) \rightarrow (i) is trivial. We prove (i) \rightarrow (iii). Assume that $f_1(z), \dots, f_m(z)$ are algebraically dependent over $\mathbf{C}(z)$. Then $g_1(z), \dots, g_m(z)$ are algebraically dependent over $\mathbf{C}(z)$. By the functional equation (6) and Lemma 1 there exist complex numbers c_j ($1 \leq j \leq m$), not all zero, such that

$$(8) \quad c_1 g_1(z) + \dots + c_m g_m(z) = c(z) \in \mathbf{C}(z).$$

We have by (6) and (8)

$$c(z) = c(z^q) + \frac{1}{\varphi(z)} P(z),$$

where $P(z) = \sum_{j=1}^m c_j \sum_{r=0}^{q-1} a_j(r) z^r$ is a polynomial of $\deg P(z) \leq q-1$. Using Lemma 2, we see that $P(z)$ is identically zero. Hence for any r with $1 \leq r \leq q-1$, $\sum_{j=1}^m c_j a_j(r) = 0$, which implies the third statement.

Finally we prove (iii) \rightarrow (ii). By (1), (3) and (7) we get

$$(9) \quad a_j(n) = a_j(b_0) + a_j(b_1) + \dots + a_j(b_t), \quad b_j \in \{0, 1, \dots, q-1\}$$

for each function $a_j(n)$ ($1 \leq j \leq m$). By the assumption,

$$(10) \quad \sum_{j=1}^m c_j a_j(r) = 0 \quad (1 \leq r \leq q-1),$$

where $c_j \in \mathbf{C}$ ($1 \leq j \leq m$) are not all zero. The second statement follows from (9) and (10).

PROOF OF THEOREM 2. Define a subset of the positive integers by

$$D = \{d \in \mathbf{N} \mid d \neq a^n \text{ for any } a, n \in \mathbf{N}, n \geq 2\}.$$

Then we have

$$\mathbf{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \dots\} = \{d^j \in \mathbf{N} \mid d \in D, j \geq 1\}.$$

Here we note that if $d, d' \in D$ are distinct, then $\log d / \log d' \notin \mathbf{Q}$. We prove the first assertion of the theorem. By Lemma 3, it is enough to prove the algebraic independence

of the functions $\{f_{d^j}(z) = \sum_{n \geq 0} s^{(d^j)}(n)z^n\}_{1 \leq j \leq h}$ over $\mathbf{C}(z)$ for a fixed $d \in D$. Putting $H = h!$, we see that all the functions $s^{(d^j)}(n)$ ($1 \leq j \leq h$) are strongly d^H -additive. Assume that $f_d(z), \dots, f_{d^h}(z)$ are algebraically dependent over $\mathbf{C}(z)$. Then by Theorem 1, there exist complex numbers c_i ($1 \leq i \leq h$), not all zero, such that

$$(11) \quad c_1 s^{(d)}(n) + c_2 s^{(d^2)}(n) + \dots + c_h s^{(d^h)}(n) = 0 \quad (n = 1, 2, \dots).$$

Let h_0 be the smallest index with $c_{h_0} \neq 0$. Substituting $n=1$ in (11), we have $c_{h_0} + \dots + c_h = 0$, since $s^{(d^{h_0})}(1) = \dots = s^{(d^h)}(1) = 1$. Putting $n = d^{h_0}$ in (11), we get

$$c_{h_0} + d^{h_0} c_{h_0+1} + \dots + d^{h_0} c_h = 0.$$

Hence $c_{h_0} = 0$, a contradiction.

We prove the second assertion of the theorem. We may prove the algebraic independence of the functions $\{g_{d^i}(z) = \sum_{n \geq 0} e_i^{(d^i)}(n)z^n\}_{\log d^i < j \leq h}$. By the same argument as above, we have

$$(12) \quad c_{h_0} e_i^{(d^{h_0})}(n) + \dots + c_h e_i^{(d^h)}(n) = 0 \quad (n = 1, 2, \dots),$$

where i ($1 \leq i < d^{h_0}$) is fixed and $c_{h_0} \neq 0$. Putting $n = id^{h_0}$ in (12), we get $c_{h_0} = 0$, since $e_i^{(d^{h_0})}(id^{h_0}) = 1, e_i^{(d^{h_0+1})}(id^{h_0}) = \dots = e_i^{(d^h)}(id^{h_0}) = 0$; which is a contradiction and the proof is completed.

PROOF OF THEOREM 3. First we assume that $a(n)$ takes algebraic-values and $a(n)$ is not identically zero. Since $a(n)$ is determined completely by $a(1), \dots, a(q_1 - 1)$, we have $\{a(n)\}_{n \geq 0} \subset K$ for some algebraic number field K . Putting

$$f(z) = (1 - z) \sum_{n \geq 0} a(n)z^n,$$

we see that $f(z)$ is transcendental over $\mathbf{C}(z)$ by Theorem 1. Since $a(n)$ is strongly q_1 -additive and strongly q_2 -additive, we have

$$f(z) = f_i(z) = f_i(z^{q_i}) + \frac{1 - z}{1 - z^{q_i}} \sum_{r=0}^{q_i-1} a(r)z^r \quad (i = 1, 2)$$

by (5). Applying Lemma 3, we see that the values $f_1(\alpha)$ and $f_2(\alpha)$ are algebraically independent for an algebraic number α with $0 < |\alpha| < 1$, which contradicts $f_1(z) = f_2(z)$.

Next we treat general case. Suppose $a(n)$ is not identically zero. Then we have $a(r_0) \neq 0$ for some r_0 ($1 \leq r_0 \leq q_1 - 1$). Let $s_1, \dots, s_{q_1-1}, t_1, \dots, t_{q_2-1}, u$ be variables. Let $P(s_1, \dots, s_{q_1-1}; n)$ be a strongly q_1 -additive function satisfying $P(s_1, \dots, s_{q_1-1}; r) = s_r$ for $1 \leq r \leq q_1 - 1$ and $Q(t_1, \dots, t_{q_2-1}; n)$ be a strongly q_2 -additive function satisfying $Q(t_1, \dots, t_{q_2-1}; r) = t_r$ for $1 \leq r \leq q_2 - 1$. For each non-negative integer n , $P(s_1, \dots, s_{q_1-1}; n)$ and $Q(t_1, \dots, t_{q_2-1}; n)$ are polynomials in $\mathbf{Q}[s_1, \dots, s_{q_1-1}, t_1, \dots, t_{q_2-1}]$. Let I be an ideal of $\mathbf{Q}[s_1, \dots, s_{q_1-1}, t_1, \dots, t_{q_2-1}, u]$ which is generated by

$$(13) \quad \{P(s_1, \dots, s_{q_1-1}; n) - Q(t_1, \dots, t_{q_2-1}; n)\}_{n \geq 1} \cup \{s_{r_0}u - 1\}.$$

Since $(a(1), \dots, a(q_1-1), a(1), \dots, a(q_2-1), a(r_0)^{-1}) \in \mathbf{C}^{q_1+q_2-1}$ is a zero of the ideal I , there exists an algebraic zero $(b_1, \dots, b_{q_1-1}, c_1, \dots, c_{q_2-1}, d) \in \overline{\mathbf{Q}}^{q_1+q_2-1}$ of I . We note that $b_{r_0} \neq 0$ by $b_{r_0}d = 1$. Let $b(n)$ be a strongly q_1 -additive function satisfying $b(r) = b_r$ ($1 \leq r \leq q_1 - 1$) and $c(n)$ be a strongly q_2 -additive function satisfying $c(r) = c_r$ ($1 \leq r \leq q_2 - 1$). Noting

$$P(b_1, \dots, b_{q_1-1}; n) = b(n), \quad Q(c_1, \dots, c_{q_2-1}; n) = c(n),$$

and (13), we have $b(n) = c(n)$ ($n \geq 1$). Hence $b(n)$ is an algebraic-valued q_1 -additive and q_2 -additive function and so $b(n)$ must be identically zero. This contradicts $b_{r_0} \neq 0$ and the proof is completed.

PROOF OF THEOREM 4. Putting $f(z) = \sum_{n \geq 0} a(n)z^n$, we have

$$f(z) = \prod_{k \geq 0} (1 + a(1)z^{q_1^k} + \dots + a(q_i - 1)z^{(q_i - 1)q_1^k}) \quad (i = 1, 2),$$

$$f(z^{q_i}) = \frac{f(z)}{1 + a(1)z + \dots + a(q_i - 1)z^{q_i - 1}} \quad (i = 1, 2)$$

by (2) and (3). First we assume that $a(n)$ takes algebraic-values. If $f(z)$ is transcendental over $\mathbf{C}(z)$, we get a contradiction by the same argument as in the proof of Theorem 3. Hence $f(z)$ is algebraic over $\mathbf{C}(z)$. By Example 1.3.1 in Nishioka [6], we have $a(n) = 0$ ($n \geq 1$) or $a(n) = \gamma^n$ ($n \geq 1$), where $\gamma^{q_1-1} = \gamma^{q_2-1} = 1$.

Next we treat general case. Define polynomials $P(s_1, \dots, s_{q_1-1}; n)$, $Q(t_1, \dots, t_{q_2-1}; n)$ in the same way as in the proof of Theorem 3. Assume that

$$1 + a(1)z + \dots + a(q_1 - 1)z^{q_1 - 1} \neq 1 + \gamma z + \dots + z^{q_1 - 1},$$

$$1 + a(1)z + \dots + a(q_1 - 1)z^{q_1 - 1} \neq 1.$$

Then $(a(1), \dots, a(q_1 - 1))$ is not a zero of at least one of the polynomials $s_1^{q_1} - s_1, s_r - s_1^r$ ($r = 2, \dots, q_1 - 1$), say $R(s_1, \dots, s_{q_1-1})$. Considering $R(s_1, \dots, s_{q_1-1})u - 1$ in place of $s_{r_0}u - 1$ in the proof of Theorem 3, we can prove the theorem in the similar way.

As for the algebraic case, a similar argument is found in Loxton [3]. The deduction of the general case from the algebraic case is suggested by Kumiko Nishioka.

REMARK (by Kumiko Nishioka). By considering the rational function field $\mathbf{C}(t)$ in place of an algebraic number field K , the similar argument as in [5] leads to the following: if $f(z) \in \mathbf{C}[[z]]$ satisfies the functional equations

$$f(z^{q_i}) = a_i(z)f(z) + b_i(z) \quad (i = 1, 2),$$

where $\log q_1 / \log q_2$ is not a rational number and $a_i(z), b_i(z) \in \mathbf{C}(z)$, $a_i(0) = 1$, then $f(z) \in \mathbf{C}(t)(z)$, and so $f(z) \in \mathbf{C}(z)$.

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