TOKYO J. MATH.
VOL. 22, NO. 1, 1999

# On Log Canonical Thresholds of Surfaces in $\mathbf{C}^3$

Takayasu KUWATA

*University of Utah*

(Communicated by K. Akao)

## 1. Introduction.

There are some invariants to measure how singular a hypersurface is. The log canonical threshold is useful for classifying non log canonical pairs. In this paper, we will study hypersurface singularities in $\mathbf{C}^n$. The sigularities are not necessarily isolated.

Let $f$ be a holomorphic function near $0 \in \mathbf{C}^n$, and let $D = div(f)$. Then the log canonical threshold of $f$ at 0 is defined by

$$c_0(\mathbf{C}^n, f) := sup\{c : (\mathbf{C}^n, cD) \text{ is log canonical near } 0\} .$$

We frequently write $c_0(f)$ instead of $c_0(\mathbf{C}^n, f)$, if there is no confusion. If $f \neq 0$, then we see that $0 < c_0(f) \leq 1$ and $c_0(f) \in \mathbf{Q}$. The log canonical threshold $c_0(f)$ is an interesting number, because it has some equivalent definitions (See Chapter 8, 9, 10 in [4] for a detailed explanation):

1. $c_0(f) = sup\{c : |f|^{-c} \text{ is locally } L^2 \text{ near } 0\}$
2. $c_0(f) = -$(the largest root of the Bernstein-Sato polynomial of $f$).

Shokurov proposed the following conjecture in [8].

CONJECTURE. *Let $\mathscr{T}_n = \{c_0(\mathbf{C}^n, f) : f \neq 0$ is holomorphic near $0 \in \mathbf{C}^n\}$. For every $n \in \mathbf{N}$, the set $\mathscr{T}_n$ satisfies the ascending chain condition.*

Shokurov proved it for the case $n = 2$ in the paper. And this was proved for the case $n = 3$ by Alexeev [2].

It is an interesting problem to describe $\mathscr{T}_n$ explicitly. By Shokurov's method, we can find $\mathscr{T}_2$ (Lemma 3.1), but Alexeev's method is ineffective and gives us little information about $\mathscr{T}_3$. So the case $n \geq 3$ is unknown. We do not even know the accumulation points of $\mathscr{T}_3$.

The aim of this paper is to show the following theorem.

THEOREM. *Let $f$ be a nonzero holomorphic function near $0 \in \mathbf{C}^3$. Then all the list of $\{c_0(\mathbf{C}^3, f) \geq 5/6\}$ are as follows:*

Received September 25, 1997

$$\frac{5}{6}+\frac{1}{m}\quad (m\geq 6)\,,\quad \frac{5}{6}+\frac{2}{3m}\quad (m\geq 4)\,,\quad \frac{5}{6}+\frac{4}{9m+6}\quad (m\geq 2)\,,$$

$$\frac{19}{20}\,,\quad \frac{15}{16}\,,\quad \frac{12}{13}\,,\quad \frac{25}{28}\,,\quad \frac{15}{17}\,,\quad \frac{5}{6}\,.$$

*As a consequence, the largest accumulation point of* $\mathscr{T}_3$ *is* 5/6.

NOTATION. *lc* expresses log canonical.

$f \sim g$ means that $f$ is holomorphically equivalent to $g$.

Numerical equivalence is denoted by $\equiv$, and the convergent power series ring by $\mathbf{C}\{x_1, \cdots, x_n\}$.

ACKNOWLEDGEMENTS. I am very grateful to Prof. János Kollár for giving me this problem and for advising me.

## 2. Preliminaries.

All propositions in this section are in Chapter 8 of [4]. The first proposition implies a lower semicontinuity for the log canonical threshold. The other propositions are basic techniques to find $c_0(f)$.

PROPOSITION 2.1. *Given nonzero holomorphic functions* $f$, $g$ *near* $0 \in \mathbf{C}^n$, *let* $f_t = f + tg$, *where* $t \in \mathbf{C}$. *Pick a point* $t_0 \in \mathbf{C}$. *Then*

$$c_0(f_{t_0}) \leq c_0(f_t) \qquad \text{for } t \text{ near } t_0\,.$$

EXAMPLE 2.2. Let $f_t = x^2 + ty^3$. Then $c_0(f_0) \leq c_0(f_t)$ for any $t \in \mathbf{C}$. Indeed, $c_0(f_0) = 1/2$ and $c_0(f_t) = 5/6$ for $t \neq 0$.

PROPOSITION 2.3. *Let* $f$ *be a nonzero holomorphic function near* $0 \in \mathbf{C}^n$. *Let* $d = \mathrm{mult}_0(f)$. *Let* $f_d$ *denote the degree* $d$ *homogeneous part of the Taylor series of* $f$. *Then*

(1) $1/d \leq c_0(f) \leq n/d$.

(2) $c_0(f) = n/d$ *iff* $(\mathbf{P}^n, \frac{n}{d}(f_d = 0))$ *is lc*.

(3) *If* $(f_d = 0) \subset \mathbf{P}^{n-1}$ *is lc, then* $c_0(f) = \min\{1, n/d\}$.

PROPOSITION 2.4. *Let* $f$ *be a holomorphic function near* $0 \in \mathbf{C}^n$. *Assign rational weights* $w(x_i)$ *to the variables and let* $w(f)$ *be the weighted multiplicity of* $f$. *Let* $f_w$ *denote the weighted homogeneous leading term of* $f$. *Then*

$$c_0(f) \leq \frac{\sum w(x_i)}{w(f)}\,.$$

*If* $(f_w = 0) \subset \mathbf{C}^n$ *is lc outside the origin* 0, *then equality holds.*

EXAMPLE 2.5. Let $f = x^3 + yz^2 + x^2y^2 + x^5z$. Let $w = (4, 2, 5)$. Then $f_w = x^3 + yz^2 + x^2y^2$. Since $Sing(f = 0) = (x = z = 0)$ and $(f_w = 0) \subset \mathbf{C}^3$ is *lc* outside $0 \in \mathbf{C}^3$, we get

$$c_0(f)=\frac{4+2+5}{12}=\frac{11}{12}\,.$$

PROPOSITION 2.6. *Let $f(x_1, \cdots, x_m, y_1, \cdots, y_n)=g(x_1, \cdots, x_m)+h(y_1, \cdots, y_n)$ be a sum of nonzero holomorphic functions near $0\in\mathbf{C}^m$, $0\in\mathbf{C}^n$. Then $c_0(\mathbf{C}^{m+n}, f)=min\{1, c_0(\mathbf{C}^m, g)+c_0(\mathbf{C}^n, h)\}$.*

EXAMPLE 2.7. Let $f=\sum x_i^{n_i}$ ($n_i\in\mathbf{N}$ for every $i$). Then $c_0(f)=min\{1, \sum 1/n_i\}$.

## 3. Proof of the main theorem.

LEMMA 3.1. *Let $f$ be a nonzero holomorphic function near $0\in\mathbf{C}^2$. Then $\{c_0(f)\}$ is one of the following forms:*

1) $\dfrac{2}{n}$ $\quad(n\in\mathbf{Z}_{\geq 2})$

2) $\dfrac{m_1+m_2}{km_1m_2+n_1m_2+n_2m_1}$ $\quad(-km_1<n_1-n_2<km_2, gcd(m_1, m_2)=1)$

*where $k$, $m_1$, $m_2$, $n$, $n_1$ and $n_2$ are all nonnegative integers.*

*In particular, the set of $\{c_0(f)\}$ satisfies the ascending chain condition and the accummulation points are $1/n$ $(n=2, 3, \cdots)$ and $0$.*

PROOF. This result comes from Shokurov's method in [8]. See also Chapter 16 in [6] for an explanation, and [7] for the proof itself. □

For ease of reference, we state the following result.

LEMMA 3.2. *Let $f$ be a nonzero holomorphic function near $0\in\mathbf{C}^n$, with a not necessarily isolated critical point at 0. Assign rational weights $w(x_i)$ to the variables. Suppose that $f=f_w+g$, where $w(g)>w(f)$. Let $\{u_i\}$ be a basis for $\mathbf{C}\{x_1, \cdots, x_n\}/(\partial f_w/\partial x_1, \cdots, \partial f_w/\partial x_n)$. Then for any $N>w(f)$, we can write*

$$f\sim f_w+\sum_{w(u_i)\leq N} a_iu_i+\varphi$$

*for some $a_i\in\mathbf{C}$ and for some $\varphi$ which satisfies $w(\varphi)>N$.*

REMARK. The basis $\{u_i\}$ is an infinite set unless the singularity of $(f_w=0)$ at 0 is isolated.

PROOF. Let $f=f_w+g$, where $w(f)=d_0<d_1=w(g)$. Then $g_w$ can be expressed as

$$g_w=\sum_{w(u_i)=d_1} a_iu_i+\sum_{j=1}^{n} v_j\frac{\partial f_w}{\partial x_j}\,.$$

Note that $w(v_i)=w(x_i)+(d_1-d_0)$ for any $i$, and that

$$\psi(x_1+v_1, \cdots, x_n+v_n)=\psi(x_1, \cdots, x_n)+\sum_{j=1}^{n} v_j \frac{\partial\psi}{\partial x_j}+r$$

for any $\psi \in \mathbf{C}\{x_1, \cdots, x_n\}$, where $w(r)>w(\psi)+(d_1-d_0)$. So

$$\begin{aligned} f(x_1-v_1, \cdots, x_n-v_n)&=f(x_1, \cdots, x_n)-\sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}+r \quad (w(r)>d_1) \\ &=f_w+g_w+h-\sum_{j=1}^{n} v_j \frac{\partial f_w}{\partial x_j}-\sum_{j=1}^{n} v_j \frac{\partial g}{\partial x_j}+r \quad (g=g_w+h) \\ &=f_w+\Bigl(\sum_{w(u_i)=d_1} a_i u_i+\sum_{j=1}^{n} v_j \frac{\partial f_w}{\partial x_j}\Bigr)+h-\sum_{j=1}^{n} v_j \frac{\partial f_w}{\partial x_j}-\sum_{j=1}^{n} v_j \frac{\partial g}{\partial x_j}+r \\ &=f_w+\sum_{w(u_i)=d_1} a_i u_i+\Bigl(h-\sum v_j \frac{\partial g}{\partial x_j}+r\Bigr) \\ &=f_w+\sum_{w(u_i)=d_1} a_i u_i+\varphi_1 \quad (w(\varphi_1)>d_1)\,. \end{aligned}$$

We then take the coordinate change. We find that

$$f \sim f_w+\sum_{w(u_i)=d_1} a_i u_i+\varphi_1 \qquad \text{where} \quad d_2=w(\varphi_1)>d_1\,.$$

Next applying the similar process we get

$$f \sim f_w+\sum_{w(u_i)\le d_2} a_i u_i+\varphi_2 \qquad \text{where} \quad d_3=w(\varphi_2)>d_2\,.$$

We repeat this process until we get $w(\varphi_k)>N$. □

We now start to prove the main theorem.

PROOF OF THE MAIN THEOREM. Let $d=mult_0(f)$. Let $f_d$ denote the degree $d$ homogeneous part of the Taylor series of $f$.

(1) Case $d=1$: Then $(f=0)\subset \mathbf{C}^3$ is smooth, so $c_0(f)=1$.

(2) Case $d=2$: Then $f_2 \sim x^2+y^2+z^2$, $x^2+y^2$, or $x^2$. If $(f_2=0)\subset \mathbf{P}^2$ is *lc*, then $c_0(f)=1$, by Proposition 2.3. Otherwise, $f\sim x^2+g(y, z)$ $(mult_0(g)\ge 3)$ and $c_0(\mathbf{C}^3, f)=min\{1, 1/2+c_0(\mathbf{C}^2, g)\}$, by Proposition 2.6. Applying Lemma 3.1, we get the following list of $c_0(f)$ between 5/6 and 1:

$$\frac{5}{6}+\frac{1}{m}\ (m\ge 7)\,, \quad \frac{5}{6}+\frac{2}{3m}\ (m\ge 5)\,, \quad \frac{5}{6}+\frac{4}{9m+6}\ (m\ge 3)\,, \quad \frac{15}{16}, \frac{19}{20}, \frac{25}{28}\,.$$

(3) Case $d=3$: If $(f_3=0)\subset \mathbf{P}^2$ is *lc*, then $c_0(f)=1$. Otherwise, the curve $(f_3=0)\subset \mathbf{P}^2$ is one of five cases (a cuspidal cubic, a line tangent to a conic, three lines through a point (two or three may coincide)). We use the rotating ruler method in

Newton polyhedra to get an inequality in Proposition 2.4 and we then apply the latter part of Proposition 2.4. The method basically comes from [3]. Arnol'd researched isolated hypersurface singularities and their hierarchies. In this paper, singularities are not necessarily isolated, and we have to classify them further in a part of (3.4) than in [3].

(3.1) Case $f_3 \sim x^3+yz^2$: Let $w_k=(2k, 2, 3k-1)$ be the weight. If $f_{w_k} \sim x^3+yz^2$, then $f \sim x^3+yz^2+g(y)+xh(y)+\varphi$ ($w_k(g)>6k$, $w_k(h)>4k$, and $w_k(\varphi)$ is large enough). So $f$ can be expressed as

$$f \sim x^3+yz^2+ay^{3k+1}+bxy^{2k+1}+cy^{3k+2}+dxy^{2k+2}+ey^{3k+3}+\cdots .$$

(3.1.k;1) Case $a \neq 0$: Let $w_{k_1}=(6k+2, 6, 9k)$. Then $f_{w_{k_1}}=x^3+yz^2+ay^{3k+1}$. $(f_{w_{k_1}}=0) \subset \mathbf{C}^3$ has an isolated singularity at $0 \in \mathbf{C}^3$. By Proposition 2.4,

$$c_0(f)=\frac{15k+8}{18k+6}=\frac{5}{6}+\frac{1}{6k+2} \qquad (k=1, 2, 3, \cdots).$$

(3.1.k;2) Case $a=0$, $b \neq 0$: Let $w_{k_2}=(4k+2, 4, 6k+1)$. Then $f_{w_{k_2}}=x^3+yz^2+bxy^{2k+1}$. $(f_{w_{k_2}}=0) \subset \mathbf{C}^3$ has an isolated singularity at $0 \in \mathbf{C}^3$. By Proposition 2.4,

$$c_0(f)=\frac{10k+7}{12k+6}=\frac{5}{6}+\frac{1}{6k+3} \qquad (k=1, 2, 3, \cdots).$$

(3.1.k;3) Case $a=b=0$, $c \neq 0$: Let $w_{k_3}=(6k+4, 6, 9k+3)$. Then $f_{w_{k_3}}=x^3+yz^2+cy^{3k+2}$, and $(f_{w_{k_3}}=x^3+yz^2+cy^{3k+2}=0) \subset \mathbf{C}^3$ has an isolated singularity at $0 \in \mathbf{C}^3$. By Proposition 2.4,

$$c_0(f)=\frac{15k+13}{18k+12}=\frac{5}{6}+\frac{1}{6k+4} \qquad (k=1, 2, 3, \cdots).$$

(3.1.k;4) Case $a=b=c=0$, $(d, e) \neq (0, 0)$: Let $w_{k_4}=(2k+2, 2, 3k+2)$. Then $f_{w_{k_4}}=x^3+yz^2+dxy^{2k+2}+ey^{3k+3}$, and $(f_{w_{k_4}}=0) \subset \mathbf{C}^3$ is *lc* outside $0 \in \mathbf{C}^3$. By Proposition 2.4,

$$c_0(f)=\frac{5k+6}{6k+6}=\frac{5}{6}+\frac{1}{6k+6} \qquad (k=1, 2, 3, \cdots).$$

(3.1.k;5) Case $a=b=c=d=e=0$: Let $w_{k_4}=(2k+2, 2, 3k+2)$. Then $f_{w_{k_4}}=x^3+yz^2$. Since $w_{k_4}=w_{k+1}$, this case is (3.1.k+1) and so we can repeat the process. Hence

$$c_0(f)=\frac{5}{6}+\frac{1}{6k+i} \qquad (i=2, 3, 4, 6,\ k \in \mathbf{N}).$$

By similar methods, we will get the rest of these cases.

(3.2) Case $f_3 \sim x^2z+yz^2$:

$$c_0(f)=\frac{3}{4}+\frac{9}{48k+i} \qquad (i=0, 4, 12, 20, 24,\ k \in N).$$

In particular, if $5/6<c_0(f)<1$, then

$$c_0(f)=\frac{15}{16},\frac{12}{13},\frac{9}{10},\frac{15}{17},\frac{7}{8},\frac{6}{7},\frac{27}{32},\frac{21}{25}.$$

(3.3) Case $f_3\sim x^3+xz^2$:

$$c_0(f)=\frac{2}{3}+\frac{2}{6k+i} \qquad (i=2,3,4,6,\ k\in N).$$

In particular, if $5/6<c_0(f)<1$, then

$$c_0(f)=\frac{11}{12},\frac{8}{9},\frac{13}{15}.$$

(3.4) Case $f_3\sim x^2y$: Since the ideal generated by partial derivatives of $f_3$ is $(xy, x^2)$, applying Lemma 3.2, $f$ can be expressed as $f\sim x^2y+g(y,z)+xh(z)+\varphi$, where $mult_0(\varphi)$ is sufficiently large. Let $w=(3,2,2)$. Then

$$f_w\sim x^2y+az^4+byz^3+cy^2z^2+dy^3z+ey^4.$$

If $(f_w=0)\subset \mathbf{C}^3$ is *lc* outside 0, then $c_0(f)=7/8$, by Proposition 2.4. Otherwise, one of the following five cases occurs.

(3.4.1) Case $f_w\sim x^2y+z^4+z^3y$, or $x^2y+z^3y$: Then one of the following three cases occurs:

$f_{w_1}\sim x^2y+yz^3+ay^p$ $(p\geq 5,\ a\neq 0)$ with respect to $w_1=(3p-3,6,2p-2)$,

$f_{w_2}\sim x^2y+yz^3+by^qz$ $(q\geq 4,\ b\neq 0)$ with respect to $w_2=(3q-3,4,2q-2)$,

or $c_0(f)\leq 5/6$.

Applying Proposition 2.4, we get

$$c_0(f)=\frac{5}{6}+\frac{1}{6p}\ (p\geq 5),\quad \frac{5q-1}{6q-2}\ (q\geq 4),\quad \text{or}\quad c_0(f)\leq\frac{5}{6}.$$

By a similar method, we get the rest of the cases.

(3.4.2) Case $f_w\sim x^2y+z^4$: Then $c_0(f)=17/20$, $27/32$, or $c_0(f)\leq 5/6$.

(3.4.3) Case $f_w\sim x^2y+z^2y^2+zy^3$, or $x^2y+z^2y^2$: Then $c_0(f)=6/7$, $17/20$, or $c_0(f)\leq 5/6$.

(3.4.4) Case $f_w\sim x^2y+zy^3$: Then $c_0(f)=16/19$, or $c_0(f)\leq 5/6$.

(3.4.5) Case $f_w\sim x^2y+y^4$, or $x^2y$: Then $c_0(f)\leq 5/6$.

(3.5) Case $f_3\sim x^3$: Then $f\sim x^3+xg(y,z)+h(y,z)+\varphi$, where $mult_0(g)\geq 3$, $mult_0(h)\geq 4$ and $mult_0(\varphi)$ is large enough. Let $w=(4,3,3)$. Applying Proposition 2.4, we get

$$c_0(f) \leq \frac{4+3+3}{12} = \frac{5}{6}.$$

(4) Case $d \geq 4$: Then $c_0(f) \leq 3/4$, by Proposition 2.3. □

As a consequence of the main theorem, we get the following result.

COROLLARY 3.3. *The following conjecture is true for the case* $n=3$.

*The largest accumulation point of* $\mathcal{T}_n$ *is* $\max \mathcal{T}_{n-1} \cap (0, 1)$.

PROOF. We can get $\max \mathcal{T}_2 \cap (0, 1) = \frac{5}{6}$ from Lemma 3.1. Combine this with the main theorem. We obtain this corollary. □

## References

[1] V. I. ARNOL'D, S. M. GUSEIN-ZADE and A. N. VARCHENKO, *Singularities of Differentiable Maps*, Birkhäuser (1985).

[2] V. A. ALEXEEV, Two two-dimensional terminations, Duke, Math. J **69** (1993), 527–545.

[3] V. I. ARNOL'D, Critical points of smooth functions and their normal forms, Russian Math. Surveys **35:5** (1975), 1–75.

[4] J. KOLLÁR, *Singularities of Pairs*, Proc. Sympos. Pure Math. 62 (1997), 221–287.

[5] J. KOLLÁR, Log surfaces of general type; *some conjectures*, Contemp, Math. **162** (1994), Amer. Math. Soc., 261–275.

[6] J. KOLLÁR (with 14 coauthors), *Flips and Abundance for Algebraic Threefolds*, Astérisque (1992).

[7] T. KUWATA, On log canonical thresholds of reducible plane curves, to appear in Amer. J. Math.

[8] V. SHOKUROV, 3-fold log flips, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), 105–203.

*Present Address*:
TOKYO DENKI UNIVERSITY,
INZAI-SHI, CHIBA, 270–1382 JAPAN.
*e-mail*: kuwata@chiba.dendai.ac.jp