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Substitution Invariant Inhomogeneous Beatty Sequences

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1. Introduction.

Given a real irrational θ and an arbitrary real ϕ we get

$$f_n=f(n;\theta,\phi)=[(n+1)\theta+\phi]-[n\theta+\phi]-[\theta]\,.$$

For brevity we write the infinite sequence (f_n) as $f_{\theta,\phi}=f_1f_2f_3\cdots$. Here our purpose will be to find those substitutions W leaving $f_{\theta,\phi}$ invariant, that is, so that $W(f_{\theta,\phi})=f_{\theta,\phi}$; and the θ and ϕ admitting such a substitution. We recall that a substitution is a pair of maps

$$W:0\longrightarrow W_0\,,\qquad 1\longrightarrow W_1\,,$$

where W_0 and W_1 are finite strings of 0's and 1's. We considered the homogeneous case $\phi=0$ in [5]. In this paper we shall give some solutions appropriate to the inhomogeneous case $\phi\neq 0$. The similar problem has been discussed by Ito and Yasutomi [3] and by Crisp [1], but we provide rather different argument.

Let $\theta=[a_0,a_1,a_2,\cdots]$ denote the continued fraction expansion of θ, where

$$\theta=a_0+\theta_0\,,\qquad a_0=[\theta]\,,$$
$$1/\theta_{n-1}=a_n+\theta_n\,,\qquad a_n=[1/\theta_{n-1}]\quad(n=1,2,\cdots)\,.$$

The n-th convergent $p_n/q_n=[a_0,a_1,\cdots,a_n]$ of θ is given by the recurrence relations

$$p_n=a_np_{n-1}+p_{n-2}\quad(n=0,1,\cdots)\,,\quad p_{-2}=0\,,\quad p_{-1}=1\,,$$
$$q_n=a_nq_{n-1}+q_{n-2}\quad(n=0,1,\cdots)\,,\quad q_{-2}=1\,,\quad q_{-1}=0\,.$$

Further, let $\phi={}_\theta[b_0,b_1,b_2,\cdots,]$ be the expansion of ϕ in terms of the sequence $\{\theta_0,\theta_1,\cdots\}$, where

$$\phi=b_0-\phi_0\,,\qquad b_0=\lceil\phi\rceil\,,$$
$$\phi_{n-1}/\theta_{n-1}=b_n-\phi_n\,,\qquad b_n=\lceil\phi_{n-1}/\theta_{n-1}\rceil\quad(n=1,2,\cdots)\,.$$

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2. Main results.

We define the sequence Γ_n $(n \geq 3)$ by

$$\Gamma_n = \{a_3 - b_3, a_4 - b_4, \cdots, a_n - b_n\},$$

and write $\pi_n = a_n - b_n$ if $a_n > b_n$, $\varpi_n = a_n - b_n$ if $a_n \geq b_n$ —so, if the entry 0 is permitted. Let $\beta_n = (b_n - 1)q_{n-1} + b_{n-1}q_{n-2} + \cdots + b_2 q_1 + 1$.

LEMMA 1 [4]. *Let θ be irrational and let ϕ be real with $0 < \theta$, $\phi < 1$. If $k\theta + \phi \notin \mathbf{Z}$ for any integer $k = 0, 1, 2, \cdots$ then we have*

$$\{[(k+1)\theta + \phi] - [k\theta + \phi]\}_{k=1}^{\infty} = \lim_{n \to \infty} \underbrace{0 \cdots 0}_{b_1 - 1} 1\, w_n .$$

Here w_n is the word of length q_n whose symbols are defined inductively by

$$w_1 = \underbrace{0 \cdots 0}_{a_1 - 1} 1 , \quad w_2 = w_1^{b_2 - 1} 0 w_1^{a_2 - b_2 + 1} , \quad w_n = w_{n-1}^{c_n} w_{n-2} w_{n-1}^{a_n - c_n} ,$$

where for some positive integer κ and a non-negative integer l

$$c_n = \begin{cases} b_n + 1 , & \text{if } \Gamma_n \text{ ends } (-1)0^{2\kappa - 1}\pi_n ; \\ 0 , & \text{if } \Gamma_{n-1} \text{ ends } (-1)0^{2\kappa - 2} \quad (\kappa \geq 2) , \\ & \qquad \pi_{n-l-2}\underbrace{\varpi_{n-l-1} \cdots \varpi_{n-2}}_{l}(-1) , \\ & \qquad \text{or} \quad 0^{n-4}(-1) ; \\ 1 , & \text{if } \Gamma_{n-1} \text{ ends } (-1)0^{2\kappa - 2}(-1) ; \\ \min(a_n, b_n) , & \text{otherwise} . \end{cases}$$

REMARK. Γ_n does end neither $(-1)0^{2k-1}(-1)$ nor $(-1)0^{2k-2}\pi_n$ as seen in [4].

LEMMA 2. *Suppose θ, ϕ satisfy the assumptions of Lemma 1 with $b_1 = 1$. Let W be a substitution. If there exists a positive integer m such that $W(w_i) = w_{i+m}$ for all large integers $i = l, l+1, \cdots$, then $W(f_{\theta,\phi}) = f_{\theta,\phi}$.*

PROOF. From Lemma 1, if $w_l \to w_{l+m}$, $w_{l+1} \to w_{l+m+1}$, $\cdots$, then

$$W(f_{\theta,\phi}) = \lim_{l \to \infty} w_{l+m} = \lim_{l \to \infty} w_l = f_{\theta,\phi} .$$

We suppose throughout that $k\theta + \phi$ is never integral. As usual $\overline{a_2, \cdots, a_n}$ is the pure periodic sequence with period $a_2, \cdots, a_n$. We interpret $\overline{b_2, \cdots, b_n}$ similarly. We may assume that $b_1 = 1$ without loss of generality, for $b_1 = 1$ if and only if $\theta + \phi > 1$. Otherwise, by $(1 - \theta) + (1 - \phi) > 1$ and $f(n; 1 - \theta, 1 - \phi) = 1 - f(n; \theta, \phi)$, we simply interchange the roles of 0 and 1 below.

THEOREM 1. *Set*

$$\theta=[0, a_1, \overline{a_2, \cdots, a_n}] \quad and \quad \phi={}_\theta[1, 1, \overline{b_2, \cdots, b_n}]\,.$$

Then $f_{\theta,\phi}$ is invariant under the block-to-block substitution

$$W\colon w_3 \longrightarrow w_{n+2}\,, \qquad w_4 \longrightarrow w_{n+3}\,,$$

except when Γ_{n+3} ends $(-1)0^{j+1}$ or $(-1)0^{2j}(-1)$ $(j\geq 1)$.

Moreover, if $c_4=c_{n+3}$, then $f_{\theta,\phi}$ is invariant under

$$W\colon w_2 \longrightarrow w_{n+1}\,, \qquad w_3 \longrightarrow w_{n+2}\,;$$

whilst if $c_3=c_{n+2}$, then $f_{\theta,\phi}$ is invariant under

$$W\colon w_1 \longrightarrow w_n\,, \qquad w_2 \longrightarrow w_{n+1}\,.$$

PROOF. The point is the different scenarios defining c_n. When Γ_{n+3} ends $(-1)0^{j+1}$ or $(-1)0^{2j}(-1)$ $(j\geq 1)$, Γ_4 and Γ_{n+3} arise from a different scenario. Otherwise, from $a_5=a_{n+4}$ and $c_5=c_{n+4}$ we have

$$w_5=w_4^{c_5}w_3w_4^{a_5-c_5} \longrightarrow w_{n+3}^{c_5}w_{n+2}w_{n+3}^{a_5-c_5}=w_{n+4}\,.$$

These may belong to the same scenario when Γ_{n+3} ends in an odd number of 0's, but then Γ_5 and Γ_{n+4} belong to different scenarios. Similarly, except for these exclusions we have $w_6\to w_{n+5}$, $w_7\to w_{n+6}$, $\cdots$, $w_i\to w_{n+i-1}$.

We need extra conditions to perform a descent on the substitutions. We can obtain

$$w_4=w_3^{c_4}w_2w_3^{a_4-c_4} \longrightarrow w_{n+2}^{c_4}w_{n+1}w_{n+2}^{a_4-c_4}=w_{n+3}$$

if and only if $c_4=c_{n+3}$; and

$$w_3=w_2^{c_3}w_1w_2^{a_3-c_3} \longrightarrow w_{n+1}^{c_3}w_nw_{n+1}^{a_3-c_3}=w_{n+2}$$

if and only if $c_3=c_{n+2}$.

We also have $w_2\to w_{n+1}$ if and only if $b_2-1=c_{n+1}$. But there is no substitution satisfying all the conditions above given the form of the expansion of ϕ.

We note that $c_{n+1}=0$, 1 or a_{n+1}. But if $c_{n+1}=0$, then from $a_2\leq b_2=1$ we have $a_2=b_2$; since $a_3>b_3$ to avoid the excluded scenario, we have $b_{n+2}+1=c_{n+2}\neq c_3=b_3$. The other cases are checked in similar ways.

But, if we allow 'words' of the shape w^{-1}, then there is a case that can be expressed by substitutions in the usual form $0\to W_0$, $1\to W_1$.

COROLLARY 1. *Let*

$$\theta=[0, a_1, \overline{a_2, \cdots, a_n}] \quad and \quad \phi={}_\theta[1, 1, \overline{b_2, \cdots, b_n}]\,.$$

If in addition to the conditions of Theorem 1 we have $b_2=c_{n+1}$ then $f_{\theta,\phi}$ is invariant under the substitution

$$W: 0 \longrightarrow w_n w_{n-1} w_n^{-1}, \qquad 1 \longrightarrow w_n w_{n-1}^{1-a_1}.$$

PROOF. We note that

$$w_1 = 0^{a_1 - 1}1 \qquad \longrightarrow w_n w_{n-1}^{a_1 - 1} w_n^{-1} w_n w_{n-1}^{1-a_n} = w_n,$$
$$w_2 = w_1^{b_2 - 1} 0 w_1^{a_2 - b_2 + 1} \longrightarrow w_n^{b_2 - 1} w_n w_{n-1} w_n^{-1} w_n^{a_2 - b_2 + 1} = w_{n+1},$$

and that the rest is readily checked by similar computations.

EXAMPLE 1. Set

$$\theta = \sqrt{3} - 1 = [0, 1, \overline{2, 1}] \quad \text{and} \quad \phi = \frac{\sqrt{3} - 1}{2} = {}_\theta[1, 1, \overline{1, 1}],$$

so that $k\theta + \phi \in \mathbf{Z}$ for any $k \in \mathbf{Z}$. Since $a_i \geq b_i$ $(i = 1, 2, \cdots)$, the conditions of Corollary 1 are satisfied. Notice that

$$\theta_0 = \theta_2 = \theta_4 = \cdots = \theta, \qquad \theta_1 = \theta_3 = \theta_5 = \cdots = \frac{1-\theta}{\theta} = \frac{\sqrt{3} - 1}{2},$$

$$\phi_0 = \phi_2 = \phi_4 = \cdots = 1 - \phi = \frac{3 - \sqrt{3}}{2}, \quad \phi_1 = \phi_3 = \phi_5 = \cdots = \frac{2 - \sqrt{3}}{2}.$$

Since $w_1 = 1$, $w_2 = 011$ and $w_3 = 0111$, the substitution W is given by

$$0 \longrightarrow 0111011(0111)^{-1}, \quad 1 \longrightarrow 0111$$

or

$$01 \longrightarrow 0111011, \quad 1 \longrightarrow 0111.$$

Therefore, we obtain

01	1	1	01	1	01	1	1	01	1	1	$\cdots$,
0111011	0111	0111	0111011	0111	0111011	0111	0111	0111011	0111	0111	$\cdots$.

We shall make a small change to the form of the inhomogeneous continued fraction expansion in Theorem 1 to obtain a substitution not involving extraordinary words like w^{-1}.

THEOREM 2. *Set*

$$\theta = [0, a_1, \overline{a_2, \cdots, a_n}] \quad \text{and} \quad \phi = {}_\theta[1, 1, b_2, \overline{b_3, \cdots, b_n, b_2 - 1}].$$

If in addition to the conditions of Theorem 1, $b_2 - 1 = c_{n+1}$ *and* Γ_n *ends in* $\varpi_3 \cdots \varpi_n$ *or* $\pi_{n-l} \varpi_{n-l+1} \cdots \varpi_n$, *then* $f_{\theta,\phi}$ *is invariant under the substitution*

$$W: 0 \longrightarrow w_{n-1}, \qquad 1 \longrightarrow w_{n-1}^{c_n + 1 - a_1} w_{n-2} w_{n-1}^{a_n - c_n},$$

where $c_n + 1 \geq a_1$ *and* $b_2 \geq 2$.

PROOF. Some of the conditions of Theorem 1 are not satisfied if Γ_n does not end in $\varpi_3 \cdots \varpi_n$ or $\pi_{n-l}\varpi_{n-l+1} \cdots \varpi_n$. Otherwise, since $b_2 - 1 = b_{n+1} = c_{n+1}$, we obtain

$$\begin{aligned} w_1 &= 0^{a_1-1}1 && \longrightarrow w_{n-1}^{a_1-1} w_{n-1}^{c_n+1-a_1} w_{n-2} w_{n-1}^{a_n-c_n} = w_n \,, \\ w_2 &= w_1^{b_2-1} 0 w_1^{a_2-b_2+1} && \longrightarrow w_n^{b_2-1} w_{n-1} w_n^{a_2-b_2+1} = w_{n+1} \,; \end{aligned}$$

the rest is readily checked by similar computations.

EXAMPLE 2. Let

$$\theta = \sqrt{2} - 1 = [0, 2, \overline{2, 2}] \quad \text{and} \quad \phi = \frac{3\sqrt{2}}{2}(\sqrt{2} - 1) = {}_\theta[1, 1, 2, \overline{1, 1}] \,,$$

so that $k\theta + \phi \in \mathbf{Z}$ for any $k \in \mathbf{Z}$. Since $a_i \geq b_i$ $(i = 3, 4, \cdots)$, the conditions of Theorem 2 are satisfied. Notice that

$$\theta_0 = \theta_1 = \theta_2 = \cdots = \theta \,, \qquad \phi_0 = 1 - \phi = \frac{3\sqrt{2} - 4}{2} \,,$$

$$\phi_1 = \frac{\sqrt{2}}{2} \,, \qquad \phi_2 = \phi_3 = \phi_4 = \cdots = \frac{2 - \sqrt{2}}{2} \,.$$

Since $w_1 = 01$ and $w_2 = 01001$, the substitution W is given by

$$0 \longrightarrow w_2 = 01001 \,, \quad 1 \longrightarrow w_2^{b_3+1-a_1} w_1 w_2^{a_3-b_3} = w_1 w_2 = 0101001 \,.$$

Therefore, we obtain

0	1	0	0	1	0	1	0	1	0	$\cdots$,
01001	0101001	01001	01001	0101001	01001	0101001	01001	0101001	01001	$\cdots$.

COROLLARY 2. *If* $\phi = {}_\theta[1, 1, b_2, \overline{b_3, \cdots, b_n, b_2 - 2}]$ *with* $b_2 \geq 3$ *but* Γ_{n+1} *ends* $(-1)0^{2\kappa-1}\pi_{n+1}$, *then* $f_{\theta,\phi}$ *is invariant under the same substitution as that of Theorem* 2.

From Lemma 3 below we know that for every pair of θ and ϕ above, θ is a quadratic irrational and $\phi \in \mathbf{Q}(\theta)$, $\phi \notin \theta\mathbf{Z}$.

If we allow the appearance of the 'partial quotient' 0 in the continued fraction expansion, then the different expressions in Theorem 1 and Theorem 2 become the same. We use the following Proposition:

PROPOSITION. *Let* s, t, u *and* v *be integers with* $t \geq v \geq 1$. *Then*

$$\theta = [\cdots, a_{i-1}, s + t, a_{i+1}, \cdots] = [\cdots, a_{i-1}, s, 0, t, a_{i+1} \cdots] \,,$$

$$\phi = {}_\theta[\cdots, b_{i-1}, u + v, b_{i+1}, \cdots] = {}_\theta[\cdots, b_{i-1}, u, 0, v, b_{i+1}, \cdots]$$

PROOF. The first assertion is a trivial consequence of

$$\begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} s+t & 1 \\ 1 & 0 \end{pmatrix}.$$

Indeed, if we set $a_i'=s$ instead of $a_i=s+t$, noting that $0<\theta_{i-1}<1/(s+t)$, we have

$$\theta_i'=\frac{1}{\theta_{i-1}}-s\,, \qquad a_{i+1}'=\left\lfloor\frac{1}{\theta_i'}\right\rfloor=\left\lfloor\frac{\theta_{i-1}}{1-s\theta_{i-1}}\right\rfloor=0\,,$$

$$\theta_{i+1}'=\frac{\theta_{i-1}}{1-s\theta_{i-1}}\,, \qquad a_{i+2}'=\left\lfloor\frac{1}{\theta_{i+1}'}\right\rfloor=\left\lfloor\frac{1}{\theta_{i-1}}-s\right\rfloor=t\,,$$

$$\theta_{i+2}'=\frac{1}{\theta_{i-1}}-s-t=\theta_2\,, \qquad a_{i+3}'=\left\lfloor\frac{1}{\theta_{i+2}'}\right\rfloor=a_{i+1}\,.$$

If we put $b_i'=u$ instead of $b_i=u+v$, noting that $-t\le -v<u-\phi_{i-1}/\theta_{i-1}<0$, we obtain

$$\phi_i'=u-\frac{\phi_{i-1}}{\theta_{i-1}}\,, \qquad b_{i+1}'=\left\lceil\frac{\phi_i'}{\theta_i'}\right\rceil=\left\lceil\left(u-\frac{\phi_{i-1}}{\theta_{i-1}}\right)\left(\frac{\theta_{i-1}}{1-s\theta_{i-1}}\right)\right\rceil=0\,,$$

$$\phi_{i+1}'=-\frac{\phi_i'}{\theta_i'}\,, \qquad b_{i+2}'=\left\lceil\frac{\phi_{i+1}'}{\theta_{i+1}'}\right\rceil=\left\lceil\frac{\phi_{i-1}}{\theta_{i-1}}-u\right\rceil=v\,,$$

$$\phi_{i+2}'=u+v-\frac{\phi_{i-1}}{\theta_{i-1}}=\phi_i\,, \qquad b_{i+3}'=\left\lceil\frac{\phi_{i+2}'}{\theta_{i+2}'}\right\rceil=b_{i+1}\,.$$

This provides the expansion of ϕ in terms of the sequence because

$$b_i'-\theta_i'b_{i+1}'+\theta_i'\theta_{i+1}'b_{i+2}'=b_i$$

(compare the proof of Lemma 3 below).

Therefore, the expansions of θ and ϕ in Theorem 2 become

$$\theta=[0, a_1, 0, 0, \overline{a_2, \cdots, a_n}] \quad\text{and}\quad \phi={}_\theta[1, 1, 1, 0, \overline{b_2-1, b_3, \cdots, b_n}]\,.$$

On the other hand those in Theorem 1 become

$$\theta=[0, a_1, 0, 0, \overline{a_2, \cdots, a_n}] \quad\text{and}\quad \phi={}_\theta[1, 1, 0, 0, \overline{b_2, b_3, \cdots, b_n}]\,.$$

3. Inhomogeneous continued fraction expansion.

We investigate the case where θ is a quadratic irrational and $\phi\in\mathbf{Q}(\theta)$. Similar results about a different type of inhomogeneous continued fraction expansion have already been obtained by Hara-Mimachi and Ito [2].

LEMMA 3. *If for some integers l, m, l', m' with $l, l'\ge 0$ and $m, m'\ge 1$*

$$\theta=[a_0, a_1, \cdots, a_l, \overline{a_{l+1}, \cdots, a_{l+m}}]\,,$$

$$\phi={}_\theta[b_0, b_1, \cdots, b_{l'}, \overline{b_{l'+1}, \cdots, b_{l'+m'}}]\,,$$

then we have $\phi \in \mathbf{Q}(\theta)$.

PROOF. We recall that θ is a quadratic irrational if and only if its continued fraction expansion is periodic, say $\theta=[a_0, a_1, \cdots, a_l, \overline{a_{l+1}, \cdots, a_{l+m}}]$. Thus, $\theta_i \in \mathbf{Q}(\theta)$ for every $i \geq 0$, So, clearly $\theta_0\theta_1\cdots\theta_i \in \mathbf{Q}(\theta)$ for every $i \geq 0$. By its definition ϕ is given by

$$\begin{aligned}\phi&=b_0+\sum_{i=1}^{n}(-1)^i\theta_0\theta_1\cdots\theta_{i-1}b_i+(-1)^{n-1}\theta_0\theta_1\cdots\theta_{n-1}\phi_n \quad (n\geq 0)\\ &=b_0+\sum_{i=1}^{\infty}(-1)^i\theta_0\theta_1\cdots\theta_{i-1}b_i\end{aligned}$$

because $0<\theta_i<1$ $(i\geq 0)$ and the cardinality of the set $\{\theta_0, \theta_1, \theta_2, \cdots\}$ is finite. So $\phi \in \mathbf{Q}(\theta)$. Similarly,

$$\begin{aligned}\phi_l&=\sum_{i=1}^{n}(-1)^{i-1}\theta_l\theta_{l+1}\cdots\theta_{l+i-1}b_{l+i}+(-1)^n\theta_l\theta_{l+1}\cdots\theta_{l+n-1}\phi_{l+n} \quad (n\geq 0)\\ &=\sum_{i=1}^{\infty}(-1)^{i-1}\theta_l\theta_{l+1}\cdots\theta_{l+i-1}b_{l+i}\end{aligned}$$

and

$$\begin{aligned}\phi_{l+m}&=\sum_{i=1}^{n}(-1)^{i-1}\theta_{l+m}\theta_{l+m+1}\cdots\theta_{l+m+i-1}b_{l+m+i}\\ &\quad+(-1)^n\theta_{l+m}\theta_{l+m+1}\cdots\theta_{l+m+n-1}\phi_{l+m+n} \quad (n\geq 0)\\ &=\sum_{i=1}^{\infty}(-1)^{i-1}\theta_{l+m}\theta_{l+m+1}\cdots\theta_{l+m+i-1}b_{l+m+i}\end{aligned}$$

yield $\phi_l=\phi_{l+m}$ because $\theta_j=\theta_{j+m}$ and $b_{j+1}=b_{j+m+1}$ $(j=l, l+1, l+2, \cdots)$.

LEMMA 4. *If θ is a quadratic irrational and $\phi \in \mathbf{Q}(\theta)$, then the cardinality of the set $\{\phi_0, \phi_1, \phi_2, \cdots\}$ is finite.*

PROOF. Firstly we shall show that the ϕ_i can be expressed by

$$\phi_i=\frac{u_i-u_{i-1}\theta_i}{t},$$

where $t \in \mathbf{N}$, and where u_i $(\in \mathbf{Z})$ is defined recursively by $u_i=b_it-a_iu_{i-1}+u_{i-2}$ $(i=1, 2, \cdots)$. If $\phi \in \mathbf{Q}(\theta)$, then there exist $u_0, u_{-1} \in \mathbf{Z}$ and $t \in \mathbf{N}$ such that $\phi_0=(u_0-u_{-1}\theta_0)/t$. For $i=0, 1, 2, \cdots$ we have

$$\phi_{i+1}=b_{i+1}-\frac{\phi_i}{\theta_i}=b_{i+1}+\frac{u_{i-1}}{t}-\frac{u_i}{t}(a_{i+1}+\theta_{i+1})=\frac{u_{i+1}-u_i\theta_{i+1}}{t}.$$

From $0\leq\phi_i<1$ $(i=0, 1, 2,)$ we have

$$u_i\theta_{i+1}\le u_{i+1}<u_i\theta_{i+1}+t\,.$$

If $u_{-1}\ge 0$, then $u_0, u_1, u_2, \cdots>0$. If $u_{-1}<0$, then there exists some non-negative integer k such that $u_{-1}<u_0<\cdots<u_{k-1}\le 0$ and $u_k, u_{k+1}, \cdots>0$. Even if $u_{i+1}\ge u_i$ for some integer i, we obtain $u_{i+1}<u_i\theta_{i+1}+t\le u_{i+1}\theta_{i+1}+t$ or $u_{i+1}\le t/(1-\theta_{i+1})$. Since θ is a quadratic irrational, the cardinality of the set $\{\theta_0, \theta_1, \theta_2, \cdots\}$ is finite. Thus, the cardinality of the set $\{u_{-1}, u_0, u_1, u_2, \cdots\}$ is also finite, yielding the claim.

By Lemma 4 there are integers l and m such that $\phi_l=\phi_{l+m}$. Together with Lemma 3 we obtain the following.

THEOREM 3. *θ is a quadratic irrational, and $\phi\in\mathbf{Q}(\theta)$, if and only if there exist l, $m\in\mathbf{Z}$ with $l\ge 0$ and $m\ge 1$ such that*

$$\theta=[a_0, a_1, \cdots, a_l, \overline{a_{l+1}, \cdots, a_{l+m}}]\,,$$

$$\phi={}_\theta[b_0, b_1, \cdots, b_l, \overline{b_{l+1}, \cdots, b_{l+m}}]\,.$$

Similarly to Theorem 1, we obtain the following.

THEOREM 4. *For $l\ge 2$ set*

$$\theta=[0, a_1, \cdots, a_l, \overline{a_{l+1}, \cdots, a_{l+m}}] \quad \textit{and}$$

$$\phi={}_\theta[1, 1, b_2, \cdots, b_l, \overline{b_{l+1}, \cdots, b_{l+m}}]\,.$$

Then $f_{\theta,\phi}$ is invariant under the block-to-block substitution

$$W:\ w_{l+1}\longrightarrow w_{l+m+1}\,, \qquad w_{l+2}\longrightarrow w_{l+m+2}\,,$$

except the following cases.

(1) *Γ_{l+m+2} ends $(-1)0^{j+1}$ $(j\ge 1)$—and if $l\ge 3$,*

(1a) *$a_3\ge b_3, \cdots, a_l\ge b_l$,*

(1b) *$a_i>b_i, a_{i+1}\ge b_{i+1}, \cdots, a_l\ge b_l$ for an integer i with $3\le i\le l$,*

(1c) *$a_i=b_i-1, a_{i+1}=b_{i+1}, \cdots, a_l=b_l$ for an integer i with $3\le i\le l$ and $i+j\equiv l$ (mod 2), or*

(2) *Γ_{l+m+2} ends $(-1)0^{2j}(-1)$ $(j\ge 1)$—and if $l\ge 3$, (1a) or (1b) is satisfied.*

REMARK. Furthermore, if $c_{l+1}=c_{l+m+1}$ and $c_{l+2}=c_{l+m+2}$, then $f_{\theta,\phi}$ is invariant under

$$W:\ w_{l-1}\longrightarrow w_{l+m-1}\,, \qquad w_l\longrightarrow w_{l+m}\,.$$

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