TOKYO J. MATH.
VOL. 22, NO. 1, 1999

A Characterization of Invertible Trace Maps Associated with a Substitution

Zhi-Xiong WEN and Zhi-Ying WEN

Wuhan University

(Communicated by Y. Ito)

Abstract. Let $F=\langle a, b\rangle$ be the free group generated by a, b. Let $\phi\in\mathrm{Hom}(F, SL(2, \mathbf{C}))$ be a homomorphism from F to $SL(2, \mathbf{C})$. Define $T(\phi)=(\mathrm{tr}\,\phi(a), \mathrm{tr}\,\phi(b), \mathrm{tr}\,\phi(ab))$, where $\mathrm{tr}\,A$ stands for the trace of the matrix A. Let $\sigma\in\mathrm{Aut}\,F$. Then from [2, 12, 4], there exists a unique polynomial map $\Phi_\sigma\in(\mathbf{Z}[x, y, x])^3$, such that

$$(\mathrm{tr}\,\phi(\sigma(a)), \mathrm{tr}\,\phi(\sigma(b)), \mathrm{tr}\,\phi(\sigma(ab)))=\Phi_\sigma(\mathrm{tr}\,\phi(a), \mathrm{tr}\,\phi(b), \mathrm{tr}\,\phi(ab))$$

with $x=\mathrm{tr}\,\phi(a)$, $y=\mathrm{tr}\,\phi(b)$, $z=\mathrm{tr}\,\phi(ab)$, and there exists a unique polynomial Q_σ, such that $\lambda\circ\Phi_\sigma=\lambda\cdot Q_\sigma$, where $\lambda(x, y, z)=x^2+y^2+z^2-xyz-4$. In this paper, we will show that $\sigma\in\mathrm{Aut}\,F$ if and only if $Q_\sigma(2, 2, z)\equiv 1$, and that this result cannot be improved.

Since the discovery of quasicrystals by Shechtman et al. [14] many authors have investigated nonperiodic ordered chains of atoms generated by a substitution acting on a finite alphabet, with each letter representing an atom between two neighbouring atoms (see [3] and the references therein). Various physical properties of such systems have been obtained in a dynamical map approach leading to a trace map [1, 10, 11]. For these trace maps, the invertible trace maps (and the invertible substitutions) possess some very interesting properties which play an important role in the studies mentioned above [11, 12, 16, 19, 20, 21, 23], therefore it is important to characterize an invertible trace map.

We recall first some preliminaries.

Let $\mathscr{A}=\{a, b\}$ be an alphabet of two letters, let $\mathscr{A}^*$ and F be the free monoid and the free group generated by $\mathscr{A}$ respectively. The elements of $\mathscr{A}^*$ and F are called words. The neutral element of $\mathscr{A}^*$ is called the empty word which we denote by ε. We denote by $\mathrm{Aut}\,F$ and $\mathrm{End}\,F$ the group of automorphisms and the group of endomorphisms of F respectively. A morphism σ from $\mathscr{A}^*$ to $\mathscr{A}^*$ is called a substitution over $\mathscr{A}$. Such a morphism can naturally be extended to be a morphism of F. If σ is also in $\mathrm{Aut}\,F$, it is

Received April 25, 1997
Revised November 5, 1998
Research support by the NSFC of China.

called an invertible substitution.

Let $\sigma\in\mathrm{End}F$, then σ is determined uniquely by the couple $(\sigma(a), \sigma(b))\in F\times F$, and we denote by $\sigma=(u, v)$ the homomorphism $\sigma(a)=u$, $\sigma(b)=v$. Let $U=\{u_i\}$ be a finite set of freely reduced words ($\neq\varepsilon$) of F, an initial word of U (i.e., of either u_i or u_i^{-1}) is called isolated if it does not occur as an initial word of any other words of U. Similarly, we define an isolated terminal words of U.

Let w be a freely reduced word ($\neq\varepsilon$). The initial word v of w is called the major initial word of w if $|w|/2<|v|\leq|w|/2+1$, and the minor initial word v' of w is that initial word satisfying $|w|/2-1\leq|v|<|w|/2$, where by $|w|$ we mean the length of the word w, that is, the number of letters appearing in w. If the length of w is even we define the right half and the left half of w in an obvious manner.

Let $U=\{u_i\}$ be a set of freely reduced words ($\neq\varepsilon$.) Then U is called a Nielson reduced form if the following conditions are satisfied:

(i) Both the major initial and major terminal subwords of each u_i are isolated,

(ii) for each u_i of even length, either its left half or its right half is isolated.

One of the following three transformations is called an elementary Nielson transformation: (i) exchange a and b, (ii) replace a (resp. b) by a^{-1} (reps. b^{-1}), (iii) replace a (or b) by ab or ba.

THEOREM (Nielsen [5]). *Let $W=(w_1, \cdots, w_m)$ be a finite m-tuple of freely reduced words in F. Then we can find a sequence $\tau_1, \cdots, \tau_k$ of elementary Nielsen transformations such that*:

$$\tau_k\cdots\tau_1 W=(v_1, \cdots, v_m),$$

where $(v_1, \cdots, v_t)$ is a Nielsen reduced form and $v_{t+1}=\cdots=v_m=\varepsilon$.

We call also that $(v_1, \cdots, v_t)$ is the Nielson reduced form of W.

Now let $\phi\in\mathrm{Hom}(F, SL(2, \mathbf{C}))$ (the group of homomorphisms from F to $SL(2, \mathbf{C})$). Define $T(\phi)=(\mathrm{tr}\,\phi(a), \mathrm{tr}\,\phi(b), \mathrm{tr}\,\phi(ab))$, where $\mathrm{tr}A$ stands for the trace of the matrix A. Let $w\in F$, from [12] (see also [13, 15, 17, 18]), there exists a unique polynomial $P_w(x, y, z)\in\mathbf{Z}[x, y, z]$, such that

$$\mathrm{tr}\,\phi(w)=P_w(T(\phi)) \tag{1}$$

with $x=\mathrm{tr}\,\phi(a)$, $y=\mathrm{tr}\,\phi(b)$, $z=\mathrm{tr}\,\phi(ab)$.

Moreover, if $\sigma\in\mathrm{End}F$, then there exists a unique polynomial map $\Phi_\sigma(x, y, z)=(\Phi_1, \Phi_2, \Phi_3)\in(\mathbf{Z}[x, y, z])^3$, such that

$$\Phi_\sigma(T\phi)=T(\phi\circ\sigma), \tag{2}$$

where $\phi\in\mathrm{Hom}(F, SL(2, \mathbf{C}))$. Φ_σ is called the trace map associated with σ. By (1) and (2),

$$\Phi_1=P_{\sigma(a)}, \quad \Phi_2=P_{\sigma(b)}, \quad \Phi_3=P_{\sigma(ab)}.$$

To state the main result of this note, we recall some definitions and terminology. Let $\sigma, \tau \in \mathrm{End}F$, we define $\sigma\tau = \tau = \tau \circ \sigma$.

The abelianization of F is homomorphic to $\mathbf{Z}^2$. For $w \in F$, let $\tilde{w}$ be the image of w, that is $\tilde{w} = (|w|_a - |w|_{a^{-1}}, |w|_b - |w|_{b^{-1}})$, where $|w|_s$ denotes the number of the letter s in w. For $\sigma \in \mathrm{End}F$, let $\tilde{\sigma}$ be the matrix $(\widetilde{\sigma(a)}^t, \widetilde{\sigma(b)}^t)$, where A^t is the transpose of A. Notice that $\tilde{\sigma} \in M_2(\mathbf{Z})$ and $\widetilde{(\sigma\tau)} = \tilde{\sigma}\tilde{\tau}$, we have $\mathrm{End}F \sim GL(2, \mathbf{Z})$.

Let $\lambda(x, y, z) = x^2 + y^2 + z^2 - xyz - 4$ be the Markov polynomial. Peyrière [10] proved that for any $\sigma \in \mathrm{End}F$, there exists $Q_\sigma \in \mathbf{Z}[x, y, z]$, such that

$$\lambda \circ \Phi_\sigma = \lambda \cdot Q_\sigma . \tag{3}$$

The following known results will be used in the sequel which may be found in [12].

THEOREM A. *Let* $\alpha = (b, a)$, $\beta = (a, b^{-1})$, $\gamma = (ab, b^{-1})$.

1. *Let* $\sigma, \tau \in \mathrm{End}F$, *then*

$$\Phi_{\sigma\tau} = \Phi_\sigma \circ \Phi_\tau ;$$

2. *For any* $\sigma \in \mathrm{End}F$, *we have* $\Phi_\sigma(2, 2, 2) = (2, 2, 2)$;

3. *The trace maps associated with* α, β, γ (*which are the elements of* $\mathrm{Aut}F$) *are respectively* $\Phi_\alpha(x, y, z) = (y, x, z)$, $\Phi_\beta(x, y, z) = (x, y, xy - z)$, $\Phi_\gamma(x, y, z) = (z, y, x)$.

Now we can characterize $\mathrm{Aut}F$ by the Nielsen reduced forms, Φ_σ and Q_σ. In fact, we have

THEOREM B. *Let* $\sigma \in \mathrm{End}F$. *The following assertions are equivalent*:

1. σ *is invertible* (*that is*, $\sigma \in \mathrm{Aut}F$);
2. $\sigma = \tau_1 \cdots \tau_n$, *where* $\tau_i \in \{\alpha, \beta, \gamma\}$, $1 \le i \le n$ (*that is*, $\mathrm{Aut}F$ *is generated by* α, β *and* γ) [8, 9, 7];
3. *The Nielsen reduced form of* $(\sigma(a), \sigma(b))$ *is* (a, b) [8, 9, 7];
4. Φ_σ *is invertible* [12];
5. $Q_\sigma \equiv 1$ ([12]), *that is*

$$P^2_{\sigma(a)}(x, y, z) + P^2_{\sigma(b)}(x, y, z) + P^2_{\sigma(ab)}(x, y, z) - P_{\sigma(a)}(x, y, z) P_{\sigma(b)}(x, y, z) P_{\sigma(ab)}(x, y, z) \equiv x^2 + y^2 + z^2 - xyz . \tag{4}$$

6. *The system of equations* $\Phi_\sigma(x, y, z) = (2, 2, 2)$ *has a unique solution* $(2, 2, 2)$ [22].

In particular, if we only consider the invertible substitution, we can replace the condition 2 of Theorem B by

2′. $\sigma = \tau_1 \cdots \tau_n$, *where* $\tau_i \in \{\alpha, (ab, a), (ba, a)\}$, $1 \le i \le n$ (*that is*, $\mathrm{IS}(\mathscr{A})$ *is generated by* α *and Fibonacci substitutions* (ab, a), (ba, a)) [19].

Moreover, by [6], this condition is equivalent to the fact that σ is a Sturmian substitution. For the definition of the Sturmian substitution, we refer to [6].

A natural question is posed: can we weaken the condition $Q_\sigma \equiv 1$ and what is the best condition? The main aim of this note consists of answering this question: we will prove that

$$\sigma \in \mathrm{Aut}F \Longleftrightarrow Q_\sigma(2, 2, z) \equiv 1 .$$

Moreover, this condition cannot be improved.

Now let $\mathbf{A} = \{\Phi_\sigma; \sigma \in \mathrm{End}F, \lambda \circ \Phi_\sigma = \lambda\}$, then if $\Phi_\sigma \in \mathbf{A}$, then $\sigma \in \mathrm{Aut}F$, and $\mathbf{A} = \langle \Phi_\alpha, \Phi_\beta, \Phi_\gamma \rangle$, that is, $\mathbf{A}$ is the group generated by Φ_α, Φ_β, Φ_γ. We then have

THEOREM C. *Let $\sigma \in \mathrm{End}F$, then $\sigma \in \mathrm{Aut}F$ if and only if $Q_\sigma(2, 2, z) \equiv 1$, that is*

$$P^2_{\sigma(a)}(2, 2, z) + P^2_{\sigma(b)}(2, 2, z) + P^2_{\sigma(ab)}(2, 2, z) - P_{\sigma(a)}(2, 2, z)P_{\sigma(b)}(2, 2, z)P_{\sigma(ab)}(2, 2, z) \equiv (z-2)^2 .$$

REMARK 1. From Theorem A.3, we obtain that the condition $Q_\sigma(2, 2, z) \equiv 1$ of Theorem B is equivalent to $Q_\sigma(x, 2, 2) \equiv 1$ or $Q_\sigma(2, y, 2) \equiv 1$.

From the equation (4), $\sigma \in \mathrm{Aut}F \Rightarrow Q_\sigma(x, y, z) \equiv 1 \Rightarrow Q_\sigma(2, 2, z) \equiv 1$, thus we only need to prove that $Q_\sigma(2, 2, z) \equiv 1 \Rightarrow \sigma \in \mathrm{Aut}F$.

To prove the following lemmas, we recall first some facts about Chebyshev polynomials which can be found in [12].

Let $u_0(x) = 0$, $u_1(x) = 1$, and by the following recurrence relations we define two polynomial sequences $\{u_n(x)\}_{n \in \mathbf{Z}}$, $\{t_n(x)\}_{n \in \mathbf{Z}}$, which are called respectively the first and the second class of Chebyshev polynomials:

$$u_{n+1}(x) := xu_n(x) - u_{n-1}(x) ,$$
$$t_n(x) := xu_n(x) - 2u_{n-1}(x) .$$

It is easy to verify:

$$u_{-n}(x) = -u_n(x) , \quad \deg u_n(x) = n-1 , \quad u_n(2) = n , \qquad n \geq 1 , \tag{5}$$

$$t_{-n}(x) = t_n(x) , \quad \deg t_n(x) = |n| , \quad t_n(2) = 2 . \tag{6}$$

Using these two classes of Chebyshev polynomials, we can obtain the following result [12]:

Let $A \in SL(2, \mathbf{C})$, and let $x = \mathrm{tr}A$, $y = \mathrm{tr}B$, $x = \mathrm{tr}AB$, then for any $n \in \mathbf{Z}$, we have

$$A^n = u_n(x)A - u_{n-1}(x) , \quad \mathrm{tr}A^n = t_n(x) . \tag{7}$$

LEMMA 1. *Let $w = a^{m_1}b^{n_1}a^{m_2}b^{n_2} \cdots a^{m_k}b^{n_k} \in F$, $m_i, n_i \in \mathbf{Z}$, $m_1m_2 \cdots m_kn_1n_2 \cdots n_k \neq 0$, $k > 0$ (if $k = 0$, we take $w = c^n$ by convention, where $c \in \{a, b\}$, $n \in \mathbf{Z}$). Then*

$$\deg P_w(2, 2, z) = k .$$

PROOF. We prove the lemma by induction. Since $P_\varepsilon = 2$, the conclusion of the lemma is true for $k = 0$. Now suppose that the conclusion is true for the positive integers smaller than $k-1$. Let

$$w_1 = a^{m_1}b^{n_1}\cdots a^{m_{k-1}}b^{n_{k-1}},$$

then from (7), for $\phi \in \mathrm{Hom}(F, SL(2, \mathbf{C}))$, we have

$$\begin{aligned}\phi_w &= \phi(w_1)(u_{m_k}(x)\phi(a) - u_{m_k-1}(x))(u_{n_k}(y)\phi(b) - u_{n_k-1}(y)) \\ &= u_{m_k}(x)u_{n_k}(y)\phi(w_1ab) - u_{m_k-1}(x)u_{n_k}(y)\phi(w_1b) \\ &\quad - u_{m_k}(x)u_{n_k-1}(y)\phi(w_1a) + u_{m_k-1}(x)u_{n_k-1}(y).\end{aligned}$$

By $u_{m_k}(2)u_{n_k}(2) = m_k \cdot n_k \neq 0$, and by the induction hypothesis

$$\deg P_{w_1a}(2, 2, z),\ \deg P_{w_1b}(2, 2, z) \leq k-1,$$

thus we only need to prove that $\deg P_{w_1ab}(2, 2, z) = k$.

By repeating the process above, we are led to compute the degree of $P_{(ab)^k}(2, 2, z)$. From (7), we get immediately

$$\deg P_{(ab)^k}(2, 2, z) = \deg(t_k(z)) = k.$$

We thus finish the proof of the lemma.

By this lemma, we obtain immediately the following

COROLLARY 1. *With the notations as above. If* $\deg P_w(2, 2, z) = k$, *then there exists* $u \in F$, *such that* $w = uvu^{-1}$, *where* $v = a^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}$, $m_1m_2\cdots m_kn_1n_2\cdots n_k \neq 0$, $m_i, n_i \in \mathbf{Z}$. *In particular, if* $\deg P_w(2, 2, z) = 0$, *then* $v = d^n$, *where* $d \in \{a, b\}$, $n \in \mathbf{Z}$.

Let $\psi \in \mathbf{Z}[x, y, z]$, we denote by $\deg\psi$ the degree of ψ. (For $\Psi = (\psi_1, \psi_2, \psi_3) \in (\mathbf{Z}[x, y, z])^3$, we set $\deg\Psi := \deg\psi_1 + \deg\psi_2 + \deg\psi_3$.)

From Theorem A.4, for any permutation Φ of (x, y, z), there exists $\pi \in \langle\alpha, \gamma\rangle$, such that $\Phi = \Phi_\pi$, thus, without loss of generality, we can suppose that

$$\deg\Phi_1 \leq \deg\Phi_2 \leq \deg\Phi_3.$$

LEMMA 2. *Suppose that* $\sigma \in \mathrm{End}\,F$ *and suppose that* $Q_\sigma(2, 2, z) \equiv 1$. *If* $\deg\phi \geq 3$, *then*

$$\deg(\phi_3 - \phi_1\phi_2) < \deg\phi_3, \qquad (*)$$

where $\phi_i(z) = \Phi_i(2, 2, z)$, $i = 1, 2, 3$, $\phi(z) = \Phi_\sigma(2, 2, z)$.

PROOF. By the hypotheses $Q_\sigma(2, 2, z) \equiv 1$ and the formula (3), we have

$$\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_1\phi_2\phi_3 - 4 = (z-2)^2 \qquad (8)$$

that is

$$\phi_1^2 + \phi_2^2 + \phi_3(\phi_3 - \phi_1\phi_2) - 4 = (z-2)^2. \qquad (9)$$

Note first that ϕ_1 and ϕ_2 cannot be constants simultaneously, otherwise the degree of the left hand side of (8) ≥ 6, and that of the right hand side is 2.

Now we consider two different cases:

Case 1): $\phi_1(z)\equiv c$. Since $\deg\phi_\sigma\geq 3$, we have $\deg\phi_3\geq 2$. If $\deg\phi_3\neq\deg\phi_2$, the degree of the left hand of (8) $\geq 4>$ the degree of the right hand of (8), therefore we must have $\deg\phi_3=\deg\phi_2=k\geq 2$.

Now by corollary 1, we have

$$\sigma(a)=ud^m u^{-1}, \qquad u\in F, \quad d\in\{a,b\}, \quad m\in\mathbf{Z},$$

$$\sigma(b)=va^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}v^{-1}, \qquad v\in F,$$

where $m_1m_2\cdots m_kn_1n_2\cdots n_k\neq 0$, $m_i, n_i\in\mathbf{Z}$. In particular, by the equations (7) and (6), $\phi_1(z)=t_m(2)=2$, thus by the equation (8), we obtain that

$$(\phi_3-\phi_2)^2=(z-2)^2$$

i.e., $\phi_3-\phi_2=\pm(z-2)$.

Now, we consider the case of $d=a$, the case of $d=b$ can be discussed in the same way. If $u^{-1}v\neq b^{n_k}$ and $u^{-1}v\neq a^l$ for any $l\in\mathbf{Z}$, then $\sigma(ab)=ua^mu^{-1}va^{m_1}b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}v^{-1}$, so by Lemma 1,

$$\deg\phi_3=\deg P_{\sigma(ab)}(2,2,z)>k,$$

which is contradictory to the condition $\deg\phi_2=\deg\phi_3$.

Putting now

$$P_k(x,y,z)=P_{a^1b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}}(x,y,z),$$

$$P_{k-1}(x,y,z)=P_{b^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}}(x,y,z),$$

then by Lemma 1, $\deg p_k(2,2,z)=k$, $\deg p_{k-1}(2,2,z)=k-1$.

From the equations (7), (5) and (6), we have

$$\begin{aligned}P_{a^nb^{n_1}a^{m_2}b^{n_2}\cdots a^{m_k}b^{n_k}}&=\operatorname{tr}(u_n(x)A-u_{n-1}(x))B^{n_1}A^{m_2}B^{n_2}\cdots A^{m_k}B^{n_k}\\&=u_n(x)p_k(x,y,z)-u_{n-1}(x)p_{k-1}(x,y,z)\end{aligned}$$

so that

$$\begin{aligned}\Phi_3(x,y,z)-\Phi_2(x,y,z)=&(u_{m+m_1}(x)-u_{m_1}(x))p_k(x,y,z)\\&+(u_{m+m_1-1}(x)-u_{m_1-1}(x))p_{k-1}(x,y,z),\end{aligned}$$

if $u^{-1}v=a^l$.

Thus by the equation again, we have

$$\begin{aligned}\phi_3-\phi_2&=\Phi_3(2,2,z)-\Phi_2(2,2,z)\\&=m(p_k(2,2,z)-p_{k-1}(2,2,z))\end{aligned}$$

which contradicts that $\phi_3-\phi_2=\pm(z-2)$. We have a contradiction for the case $u^{-1}v=b^{n_k}$, similarly.

Case 2): None of ϕ_1, ϕ_2, ϕ_3 is constant. If $\deg\phi_3\neq\deg\phi_1\phi_2$, the degree of the left hand of (9) $=\deg\phi_3+\max(\deg\phi_3, \deg\phi_1\phi_2)\geq 2+2=4>$ the degree of the right hand of (9), this contradiction shows that

$$\deg\phi_3=\deg\phi_1\phi_2\geq 2\ .$$

This implies that $\deg(\phi_3-\phi_1\phi_2)\leq\deg\phi_3$.

If $\deg(\phi_3-\phi_1\phi_2)=\deg\phi_3$, the same analysis as above will give also a contradiction, so we still have $(*)$.

Notice that α, γ are the permutations of (x, y, z) and the role of β is to change the third component to $xy-z$ (in the discussions above, it is $\phi_1\phi_2-\phi_3$), we get therefore

COROLLARY 2. *With the notations above, let* $\sigma\in\mathrm{End}F$. *If* $\deg\phi_\sigma\geq 3$, *then there exists* $\pi\in\mathrm{Aut}F$, *such that*

$$\deg\phi_{\pi\circ\sigma}<\deg\phi_\sigma\ .$$

Moreover, there exists $\tau\in\mathrm{Aut}F$, *such that*

$$\deg\phi_{\tau\circ\sigma}\leq 2\ .$$

By corollary 2, we can suppose that $\deg\phi_\sigma\leq 2$.

LEMMA 3. *Suppose that* $\sigma\in\mathrm{End}F$ *and* $\deg\phi_\sigma\leq 2$, *then there exists* $\tau\in\mathrm{Aut}F$, *such that*

$$\Phi_\tau\circ\Phi_\sigma(2, 2, z)=(2, 2, z)\ .$$

PROOF. From Theorem A.2, we have $\phi_i(2)=2$, $i=1, 2, 3$.

1) ϕ_1, ϕ_2 are constants. From the discussion above we have $\phi_1=\phi_2\equiv 2$. By (8) we get either $\phi_3=z$ or $\phi_3=4-z$. In the latter case, notice that

$$\Phi_\beta\circ\Phi_\sigma(2, 2, z)=\Phi_\beta(2, 2, 4-z)=(2, 2, z)\ .$$

Thus for both of these two cases, the conclusion of the lemma is true.

2) ϕ_1 is constant, $\deg\phi_2=\deg\phi_3=1$. From $\phi_1\equiv 2$ and (8), we get

$$(\phi_3-\phi_2)^2=(z-2)^2\ ,$$

so $\pm(\phi_3-\phi_2)=z-2$. Using α and γ we can always exchange ϕ_2 and ϕ_3, thus we only need to consider the case $\phi_3-\phi_2=z-2$. Since $\deg\phi_3=1$, we can suppose that $\phi_3=nz+m$, $n, m\in\mathbf{Z}$. By $\phi_3(2)=2$, we see that $m\in 2\mathbf{Z}$ and $\phi_2=\phi_3-(z-2)=(n-1)z+(m+2)$.

Let $\pi=\alpha\gamma\alpha$, then $\Phi_\pi(x, y, z)=(x, z, y)$, so

$$\Phi_{\pi\circ\beta\circ\pi\circ\sigma}(2, 2, z)=(2, 2\phi_3-\phi_2, \phi_3)=:(2, \phi_2', \phi_3)\ .$$

Notice that $\phi_2'=2\phi_3-\phi_2=(n+1)z+(m-2)$, by exchanging ϕ_2' and ϕ_3, we can suppose that $\phi_3=(n+1)z+(m-2)$. Repeat this process and notice that $m\in 2\mathbf{Z}$, after a finite number of steps, we can suppose that $\phi_3=nz$. From $\phi_3(2)=2$ again, we obtain finally:

after a finite number of applications of elements of $\mathrm{Aut}F$, $\phi_3=z$. For the case $m<0$, we can treat in the same way. From the discussions avove, we have proved that there exists $\tau\in\mathrm{Aut}F$, such that $\Phi_\tau\circ\Phi_\sigma(2,2,z)=\Phi_{\tau\sigma}(2,2,z)=(2,2,z)$.

PROOF OF THEOREM C. Assume that $Q_\sigma(2,2,z)\equiv 1$. By Lemma 3, there exists $\tau\in\mathrm{Aut}F$, such that $\Phi_{\tau\sigma}(2,2,z)=(2,2,z)$. For convenience, we write simply σ instead of $\tau\sigma$. From the equality above, it follows that

$$P_{\sigma(a)}(2,2,z)=2\,,\quad P_{\sigma(b)}(2,2,z)=2\,,\quad P_{\sigma(ab)}(2,2,z)=z\,.$$

Thus

$$\deg P_{\sigma(a)}(2,2,z)=\deg P_{\sigma(b)}(2,2,z)=0\,, \tag{10}$$

$$\deg P_{\sigma(ab)}(2,2,z)=1\,. \tag{11}$$

By the equality (10) and Corollary 1, there exist $u, v\in F$, $m, n\in\mathbf{Z}$, such that

$$\sigma=(u^{-1}a^m u, v^{-1}b^n v)\quad\text{or}\quad(u^{-1}b^n u, v^{-1}a^m v)\,.$$

Thus

$$\sigma(ab)=u^{-1}a^m uv^{-1}b^n v\quad\text{or}\quad u^{-1}b^n uv^{-1}a^m v\,. \tag{12}$$

Without loss of generality, it suffices to consider the case of $|u|\geq|v|$. Assume that $u=b^k u'$ and that the first letter of u' is a, then by the equality (12),

$$\sigma(ab)=u'^{-1}b^{-k}a^m b^k u'v^{-1}b^n v\,.$$

From (8), $\deg P_{\sigma(ab)}(2,2,z)=1$, we have thus $u'v^{-1}=b^l$ by Lemma 1, so $u=b^k v$. It follows that $\sigma(ab)=u^{-1}a^m b^n u$, $u\in F$, i.e. $P_{\sigma(ab)}=P_{a^m b^n}$. By (7),

$$\begin{aligned} z&=P_{\sigma(ab)}(2,2,z)=P_{a^m b^n}(2,2,z)\\ &=u_m(2)u_n(2)z-xu_{n-1}-yu_{m-1}(2)+2u_{m-1}(2)u_{n-1}(2)\,. \end{aligned} \tag{13}$$

By comparing the coefficients of z in the two sides of (13), we have $u_n(2)u_m(2)=1$, thus m and n have the same sign. Suppose that $m, n>0$, by comparing the coefficients of x and y in (13), we see that $m=n=1$ are the unique solution of (13). For $m, n<0$, we obtain $m=n=-1$ in the same way.

So, we have

$$\sigma=(u^{-1}a^\varepsilon u, u^{-1}b^\varepsilon u)\,,\quad\text{or}\quad(u^{-1}b^\varepsilon u, u^{-1}a^\varepsilon u)\,,$$

where $\varepsilon=\pm 1$, $u\in F$. That is, σ is either an inner automorphism or a product of an inner automorphism and an involution of F.

From the discussions above, we see that, if $Q_\sigma(2,2,z)\equiv 2$, then there exists $\tau\in\mathrm{Aut}F$, such that $\tau\sigma\in\mathrm{Aut}F$, i.e. $\sigma\in\mathrm{Aut}F$. This completes the proof of Theorem C.

REMARK 2. As $Q_\sigma(2, 2, z)$ is a polynomial of z, if there exists an infinite number of z, such that $Q_\sigma(2, 2, z)=0$, then $Q_\sigma(2, 2, z)\equiv 0$; on the other hand, for any $n\in\mathbf{N}$, there exists σ, such that $\deg Q_\sigma(2, 2, z)>n$, that is, the condition of Theorem C cannot be weakened.

REMARK 3. By Theorem C, to determine that $\sigma\in\operatorname{End}F$ is invertible, we only need to verify $\lambda\circ\Phi_\sigma(2, 2, z)=\lambda(2, 2, z)$.

By using the same method as the one used in the proof of Theorem C and the proof of Theorem 5 of [12], we have

THEOREM D. *Let* $\sigma\in\operatorname{End}F$, *then the following assertions are equivalent*:

i) σ *is not an injection*;

ii) *there exists* $m, n\in\mathbf{Z}$, $w\in F$, *such that* $\sigma=(w^m, w^n)$;

iii) $Q_\sigma(2, 2, z)\equiv 0$ (*or* $Q_\sigma(x, 2, 2)\equiv 0$, *or* $Q_\sigma(2, y, 2)\equiv 0$).

ACKNOWLEDGEMENT. Z. Y. Wen thanks Prof. Yuji Ito for his hospitality when he visited Keio University, and the authors thank the referee for his suggestion.

References

[1] J.-P. ALLOUCHE et J. PEYRIÈRE, Sur une formule récurrence sur les traces de produits de matrices associés à certaines substitutions, C. R. Acad. Sci. Paris Sér. II **302** (1986), 1135.

[2] R. D. HOROWITZ, Induced automorphisms on Fricke characters of free groups, Trans. Amer. Math. Soc. **208** (1975), 41–50.

[3] J. M. LUCK, C. GODRÈCHE, A. JANNER and T. JANSSEN, The nature of the atomic surfaces of quasiperiodic self-similar structures, J. Phys. A **26** (1993), 1951–1999.

[4] M. KOHMOTO, L. KADANOFF and C. TANG, Phys. Rev. Lett. **50** (1983), 1870.

[5] W. MAGNUS, A. KARRAS and D. SOLITAR, *Combinatorial Group Theory*, Interscience (1966)/Dover (1976).

[6] F. MIGNOSI and P. SÉÉBOLD, Morphismes sturmiens et règles de Rauzy, J. Théor. Nombres Bordeaux **5** (1993), 221–233.

[7] B. H. NEUMANN, Die Automorphismengruppe der freien Gruppen, Math. Ann. **107** (1933), 367–386.

[8] J. NIELSEN, Die Isomorphismen der allgemeinen unendlichen Grupper mit zwei Erzen-genden, Math. Ann. **78** (1918), 385–397.

[9] J. NIELSEN, Die Isomorphismen der freien Gruppen, Math. Ann. **91** (1924), 169–209.

[10] J. PEYRIÈRE, On the trace map for products of matrices associated with substitutive sequences, J. Stat. Phys. **62** (1991), 411–414.

[11] J. PEYRIÈRE, Z.-X. WEN and Z.-Y. WEN, On the dynamic behaviours of the iterations of the trace map associated with substitutive sequences, *Nonlinear Problems in Engineering and Science* (eds. S. Xiao and X.-C. Hu), Science Press, Beijing (1992), 259–266.

[12] J. PEYRIÈRE, Z.-Y. WEN et Z.-X. WEN, Polynômes associés aux endomorphismes de groupes libres, Enseign. Math. **39** (1993), 153–175.

[13] C. PROCESI, The invariant theory of $n\times n$ matrices, Adv. in Math. **19** (1976), 306–381.

[14] D. SHECHTMAN, I. BLECH, D. GRATIAS and J. W. CAHN, Metallic phase with long-range orientational order and no translational symmetry, Phys. Rev. Lett. **53** (1984), 1951.

[15] JU. P. RAZMYSLOV, Trace identities of full matrix algebras over a field of characteristic zero, Izv.

Akad. Nauk SSSR **38** (1974); English transl. *Math. USSR Izv.* **8** (1974), 727–760.

[16] A. SÜTO, The spectrum of a quasiperiodic Schrödinger operator, Comm. Math. Phys. **111** (1987), 409–415.

[17] C. TRAINA, Representation of the trace polynomial of cyclically reduced words in a free group on two generators. Ph. D. Thesis, Polytech. Inst. of New York, 1978.

[18] C. TRAINA, Trace polynomial for two generators subgroups of $SL(2, \mathbf{C})$. Proc. Amer. Math. Soc. **79** (1980), 369–372.

[19] WEN ZHI-XIONG and WEN ZHI-YING, Local isomorphism of the invertible substitutions, C. R. Acad. Sci. Paris Sér. I **318** (1994), 299–304.

[20] WEN ZHI-XIONG and WEN ZHI-YING, Some properties of the singular words of the Fibonacci word, Europ. J. Combin. **15** (1994), 1–12.

[21] WEN, ZHI-XIONG and WEN ZHI-YING, Some studies of factors of infinite words generated by invertible substitution, *Formal Power Series and Algebraic Combinatorics*, Proc. of 5th Conf. (eds. A. Barlotti, M. Delest and R. Pinzani), (1993), 455–466.

[22] WEN ZHI-XIONG and WEN ZHI-YING, A classification of the trace mappings associated with substitutive sequences, in preparation.

[23] Z. Y. WEN, F. WIJNANDS and J. LAMB, A nature class of generalized Fibonacci chains, J. Phys. A **27** (1994), 3689–3706.

Present Addresses:
ZHI-XIONG WEN
DEPARTMENT OF MATHEMATICS, WUHAN UNIVERSITY,
WUHAN 430072, P. R. OF CHINA.
e-mail: zhxwen@whu.edu.cn

ZHI-YING WEN
DEPARTMENT OF MATHEMATICS, STATE KEY LAB. OF SOFT ENGINEER,
WUHAN UNIVERSITY, WUHAN 430072,
and
INSTITUTE OF APPLIED MATHEMATICS, Academic SINICA, BEIJING 100080,
P. R. OF CHINA.
e-mail: wenzy@tsinghua.edu.cn