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Solvability of Nonstationary Problems for Nonhomogeneous Incompressible Fluids and the Convergence with Vanishing Viscosity

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1. Introduction.

Let Ω be a bounded or unbounded domain in $\mathbf{R}^3$ with a smooth boundary S. We consider the system of equations

$$(1.1)\qquad \begin{cases} \rho_t + v\cdot\nabla\rho = 0\,, \\ \rho[v_t + (v\cdot\nabla)v] + \nabla p = \mu\Delta v + \rho f\,, \\ \operatorname{div} v = 0 \end{cases}$$

in $Q_T = \Omega\times[0, T]$, $T>0$, where $f(x, t)$ is a given vector field of external forces, while the density $\rho(x, t)$, the velocity vector $v(x, t)$ and the pressure $p(x, t)$ are the unknowns. The viscosity coefficient μ is assumed to be a nonnegative constant.

This paper consists of two parts. In the first part, Part 1, we solve (1.1) under the following initial-boundary conditions:

If $\mu>0$,

$$(1.2)\qquad \begin{cases} v|_{S_T} = 0\,, \\ \rho|_{t=0} = \rho_0(x)\,, \\ v|_{t=0} = v_0(x)\,, \end{cases}$$

and if $\mu=0$,

$$(1.3)\qquad \begin{cases} v\cdot n|_{S_T} = 0\,, \\ \rho|_{t=0} = \rho_0(x)\,, \\ v|_{t=0} = v_0(x)\,, \end{cases}$$

where n is the unit outward normal to S, and $S_T = S\times[0, T]$.

In the second part, Part 2, when $\Omega = \mathbf{R}^3$, we consider the Cauchy problem (1.1) and

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$$(1.4)\qquad \begin{cases} \rho|_{t=0}=\rho_0(x)\,, \\ v|_{t=0}=v_0(x)\,, \end{cases}$$

and establish the uniform convergence of the solution of (1.1) and (1.4) with $\mu>0$ to the one with $\mu=0$ as $\mu\to 0$.

We will use the classical notations and results of the Sobolev spaces. For $k=0, 1, 2, \cdots$ and $1\le p\le\infty$,

$$W_p^k(\Omega)=\left\{ u\in L_p(\Omega);\ \sum_{|\alpha|\le k} \|D_x^\alpha u\|_{L_p(\Omega)}<\infty \right\},$$

$$W_p^{2,1}(Q_T)=\left\{ u\in L_p(Q_T);\ \|u\|_{W_p^{2,1}(Q_T)}=\|u_t\|_{L_p(Q_T)}+\sum_{|\alpha|\le 2} \|D_x^\alpha u\|_{L_p(Q_T)}<\infty \right\},$$

where $D_x^\alpha=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3}$.

If $u(x, t)\in W_p^{2,1}(Q_T)$ and $p>3$, then for any fixed $t\in[0, T]$, the value of $u(x, t)$ belongs to the Slobodetskii-Besov space $W_p^{2-2/p}(\Omega)$ in which the norm is given by

$$\|u\|_{W_p^{2-2/p}(\Omega)}=\left(\sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{L_p(\Omega)}^p+\sum_{|\alpha|=1} \int_\Omega\int_\Omega \frac{|D_x^\alpha u(x)-D_y^\alpha u(y)|^p}{|x-y|^{1+p}}\,dxdy \right)^{1/p}.$$

Moreover, we have the inequality

$$\|u(\cdot, t)\|_{W_p^{2-2/p}(\Omega)}\le \|u(\cdot, 0)\|_{W_p^{2-2/p}(\Omega)}+\hat{c}\|u\|_{W_p^{2,1}(Q_t)}\,,$$

where the constant $\hat{c}$ does not depend on t (cf. Ladyzhenskaya-Solonnikov-Ural'ceva [7]).

Our theorems related to the unique solvability are the following.

THEOREM 1.1. *Let $p>3$ and $\mu>0$. Assume that*

$$(1.5)\qquad \rho_0(x)\in C^0(\bar{\Omega})\,,\quad \nabla\rho_0(x)\in W_p^1(\Omega)\,,\quad 0<m\le\rho_0(x)\le M<\infty\,,$$

$$(1.6)\qquad v_0(x)\in W_p^{2-2/p}(\Omega)\,,\quad v_0|_S=0\,,\quad \operatorname{div} v_0=0\,,$$

$$(1.7)\qquad f(x, t)\in L_p(Q_T)\,.$$

Then there exists $T_1\in(0, T]$ such that problem (1.1), (1.2) has a unique solution $(\rho, v, p)(x, t)$ which satisfies

$$(1.8)\qquad \begin{gathered} \rho(x, t)\in C^0(\bar{Q}_{T_1})\,,\quad \nabla\rho(x, t)\in C^0([0, T_1];\ W_p^1(\Omega))\,, \\ 0<m\le\rho(x, t)\le M<\infty\,, \end{gathered}$$

$$(1.9)\qquad v(x, t)\in W_p^{2,1}(Q_{T_1})\,,$$

$$(1.10)\qquad \nabla p(x, t)\in L_p(Q_{T_1})\,.$$

THEOREM 1.2. *Let $p>3$ and $\mu=0$. Assume that*

$$(1.5)\qquad \rho_0(x)\in C^0(\bar{\Omega})\,,\quad \nabla\rho_0(x)\in W_p^1(\Omega)\,,\quad 0<m\le\rho_0(x)\le M<\infty\,,$$

(1.11) $$v_0(x) \in W_p^2(\Omega), \quad v_0 \cdot n|_S = 0, \quad \operatorname{div} v_0 = 0,$$

(1.12) $$f(x, t) \in C^0([0, T]; W_p^2(\Omega)).$$

Then there exists $T_2 \in (0, T]$ *such that problem* (1.1), (1.3) *has a unique solution* $(\rho, v, p)(x, t)$ *which satisfies*

(1.13) $$\rho(x, t) \in C^0(\bar{Q}_{T_2}), \quad \nabla\rho(x, t) \in C^0([0, T_2]; W_p^1(\Omega)), \quad 0 < m \le \rho(x, t) \le M < \infty,$$

(1.14) $$v(x, t) \in C^0([0, T_2]; W_p^2(\Omega)),$$

(1.15) $$\nabla p(x, t) \in C^0([0, T_2]; W_p^2(\Omega)).$$

REMARK. In the case that Ω is bounded, Theorems 1.1 and 1.2 were proved by Ladyzhenskaya-Solonnikov [6] and Valli-Zajaczkowski [13], respectively. See also [2–4, 8, 9]. However, in the case that Ω is unbounded, it seems to the authors that the rigorous proofs for these theorems have not been given yet.

The next theorem is concerned with the vanishing viscosity. Analogous result was obtained in [5] in the Sobolev spaces of Hilbert type.

THEOREM 1.3. *Let* $p \ge 4$ *and* $0 \le \mu \le 1$, *and assume that*

(1.16) $$\rho_0(x) \in C^0(\mathbf{R}^3), \quad \nabla\rho_0(x) \in W_p^1(\mathbf{R}^3), \quad 0 < m \le \rho_0(x) \le M < \infty,$$

(1.17) $$v_0(x) \in W_p^2(\mathbf{R}^3), \quad \operatorname{div} v_0 = 0,$$

(1.18) $$f(x, t) \in C^0([0, T]; W_p^2(\mathbf{R}^3)).$$

Then there exists $T_0 \in (0, T]$ *independent of* μ *such that problem* (1.1), (1.4) *has a unique solution* $(\rho, v, p)(x, t)$ *which satisfies*

(1.19) $$\rho(x, t) \in C^0(\mathbf{R}^3 \times [0, T_0]), \quad \nabla\rho(x, t) \in C^0([0, T_0]; W_p^1(\mathbf{R}^3)), \quad 0 < m \le \rho(x, t) \le M < \infty,$$

(1.20) $$v(x, t) \in C^0([0, T_0]; W_p^2(\mathbf{R}^3)),$$

(1.21) $$\nabla p(x, t) \in C^0([0, T_0]; W_p^1(\mathbf{R}^3)).$$

Furthermore, let (ρ^0, v^0, p^0) *be the solution of problem* (1.1), (1.4) *with* $\mu = 0$ *and* (ρ^μ, v^μ, p^μ) *the one with* $\mu > 0$, *then we have*

(1.22) $$\sup_{0 \le t \le T_0} [\|(\rho^0 - \rho^\mu)(t)\|_{W_p^1(\mathbf{R}^3)} + \|(v^0 - v^\mu)(t)\|_{W_p^1(\mathbf{R}^3)} + \|\nabla(p^0 - p^\mu)(t)\|_{W_p^1(\mathbf{R}^3)}] \to 0 \quad \textit{as} \quad \mu \to 0.$$

Part 1 is divided into two sections: In sections 2 and 3, we shall prove Theorems 1.1 and 1.2, respectively. Finally, Theorem 1.3 will be established in Part 2, section 4.

Part I. Existence Theorems

2. The case $\mu>0$.

In this section, we prove Theorem 1.1 by dividing into three subsections.

2.1. Auxiliary problems. By $C^{\alpha,\beta}(\bar{Q}_T)$ $(0<\alpha<1,\ 0<\beta<1)$, we mean the space of functions which are defined in $\bar{Q}_T$ and Hölder continuous with exponent α with respect to x and with exponent β with respect to t. The norm is

$$\|u\|_{C^{\alpha,\beta}(\bar{Q}_T)}=\sup_{\bar{Q}_T}|u(x,t)|+[u]_{C^{\alpha,\beta}(\bar{Q}_T)},$$

where

$$[u]_{C^{\alpha,\beta}(\bar{Q}_T)}=\sup_{(x,t),(y,s)\in\bar{Q}_T,x\neq y,t\neq s}\frac{|u(x,t)-u(y,s)|}{|x-y|^\alpha+|t-s|^\beta}.$$

LEMMA 2.1. *Let* $\rho(x,t)\in C^{\alpha,\beta}(\bar{Q}_T)$, $\alpha,\beta\in(0,1)$ *such that* $0<m\le\rho(x,t)\le M<\infty$. *Then for any* $g(x,t)\in L_p(Q_T)$ *and* $v_0(x)\in W_p^{2-2/p}(\Omega)$ *with* $v_0|_S=0$ *and* $\operatorname{div}v_0=0$, *problem*

$$(2.1)\qquad\begin{cases}\rho v_t-\mu\Delta v+\nabla p=g,\\ \operatorname{div}v=0,\\ v|_{S_T}=0,\\ v|_{t=0}=v_0(x)\end{cases}$$

has a unique solution $v(x,t)\in W_p^{2,1}(Q_T)$ *and* $\nabla p(x,t)\in L_p(Q_T)$, *satisfying*

$$(2.2)\qquad\|v\|_{W_p^{2,1}(Q_T)}+\|\nabla p\|_{L_p(Q_T)}\le K_1(\|\rho\|_{C^{\alpha,\beta}(\bar{Q}_T)},T)(\|v_0\|_{W_p^{2-2/p}(\Omega)}+\|g\|_{L_p(Q_T)}),$$

where K_1 *is an increasing function of* $\|\rho\|_{C^{\alpha,\beta}(\bar{Q}_T)}$ *and* T, *depending on* m *and* M.

PROOF. Let us seek the solution of (2.1) in the form $v(x,t)=u(x,t)+w(x,t)$ and $p(x,t)=r(x,t)+s(x,t)$, where (u,r) and (w,s) satisfy the following systems, respectively:

$$(2.3)\qquad\begin{cases}u_t-\mu\Delta u+\nabla r=g,\\ \operatorname{div}u=0,\\ u|_{S_T}=0,\\ u|_{t=0}=v_0(x),\end{cases}$$

and

$$(2.4)\qquad\begin{cases}\rho w_t-\mu\Delta w+\nabla s=(1-\rho)u_t\equiv g',\\ \operatorname{div}w=0,\\ w|_{S_T}=0,\\ w|_{t=0}=0.\end{cases}$$

Problem (2.3) was solved by Solonnikov [11] so that it has a unique solution $u(x, t) \in W_p^{2,1}(Q_T)$ and $\nabla r(x, t) \in L_p(Q_T)$, satisfying

$$\|u\|_{W_p^{2,1}(Q_T)} + \|\nabla r\|_{L_p(Q_T)} \leq c_1(1 + e^{c_1 T})(\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|g\|_{L_p(Q_T)}) , \tag{2.5}$$

where c_1 is a constant independent of T. Therefore it is sufficient to establish the unique solvability of (2.4). It is proved by the method of regularizer. To this end, we introduce two systems of covering $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ such that

1. $\omega^{(k)} \subset \Omega^{(k)} \subset \bar{\Omega}$ and $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \bar{\Omega}$,
2. for any x, there exists $\omega^{(k)}$ such that $x \in \omega^{(k)}$ and $\mathrm{dist}(x, \bar{\Omega} - \omega^{(k)}) \geq \delta_1 > 0$,
3. for any $\lambda > 0$, there exists a number N_0 independent of λ such that $\bigcap_{k=1}^{N_0+1} \Omega^{(k)} = \varnothing$,
4. (a) if $\Omega^{(k)} \cap S = \varnothing$ (we denote the set of indices k by $\mathcal{M}$), then $\omega^{(k)}$ and $\Omega^{(k)}$ are the cubes with the same center and with the length of their edges equal to $\lambda/2$ and λ, respectively,
   (b) if $\omega^{(k)} \cap S \neq \varnothing$ (we denote the set of indices k by $\mathcal{N}$), then for a local rectangular coordinate system $\{y\}$ with the center $\xi^{(k)} \in S$,

$$\omega^{(k)} = \left\{ |y_i| \leq \frac{1}{2} \delta_2 \lambda \ (i=1, 2),\ 0 \leq y_3 - F(y'; \xi^{(k)}) \leq \delta_2 \lambda \right\} ,$$

$$\Omega^{(k)} = \{ |y_i| \leq \delta_2 \lambda \ (i=1, 2),\ 0 \leq y_3 - F(y'; \xi^{(k)}) \leq 2\delta_2 \lambda \} ,$$

where $F(y'; \xi^{(k)})$ $(y' = (y_1, y_2))$ is a function describing the boundary S in the neighborhood of $\xi^{(k)}$ and δ_2 is a positive constant independent of λ.

By changing the variables in such a way that $z_i = y_i$ $(i=1, 2)$ and $z_3 = y_3 - F(y')$, $\Omega^{(k)}$ $(k \in \mathcal{N})$ and the boundary in $\Omega^{(k)}$ are, respectively, transformed into a standard cube

$$K = \{ |z_i| \leq \delta_2 \lambda \ (i=1, 2),\ 0 \leq z_3 \leq 2\delta_2 \lambda \} ,$$

$$K' = \{ |z_i| \leq \delta_2 \lambda \ (i=1, 2),\ z_3 = 0 \} .$$

Furthermore, it is well known that there exist the smooth functions $\{\zeta^{(k)}(x)\}$ and $\{\eta^{(k)}(x)\}$ such that

$$\begin{cases} \zeta^{(k)}(x) = \begin{cases} 1 & \text{if } x \in \omega^{(k)} , \\ 0 & \text{if } x \in \bar{\Omega} - \Omega^{(k)} , \end{cases} & 0 \leq \zeta^{(k)}(x) \leq 1 , \\ \eta^{(k)}(x) = 0 \quad \text{if } x \in \bar{\Omega} - \Omega^{(k)} , & \sum_k \zeta^{(k)}(x) \eta^{(k)}(x) = 1 , \\ |D_x^\alpha \zeta^{(k)}(x)| \leq c_\alpha \lambda^{-|\alpha|} , & |D_x^\alpha \eta^{(k)}(x)| \leq c_\alpha \lambda^{-|\alpha|} . \end{cases}$$

Now, let us construct regularizer. For $k \in \mathcal{M}$, let $(\bar{w}^{(k)}, \bar{s}^{(k)})(x, t)$ be the solution of problem

$$\left\{\begin{array}{l}\rho(\xi^{(k)},0)\bar{w}_t^{(k)}-\mu\Delta\bar{w}^{(k)}+\nabla\bar{s}^{(k)}=\zeta^{(k)}(x)g'\,,\\ \operatorname{div}\bar{w}^{(k)}=0\,,\\ \bar{w}^{(k)}|_{t=0}=0\,.\end{array}\right. \tag{2.6}$$

Further, for $k\in\mathcal{N}$, let $(\bar{w}^{(k)},\bar{s}^{(k)})(z,t)$ be the solution of problem

$$\left\{\begin{array}{l}\rho(\xi^{(k)},0)\bar{w}_t^{(k)}-\mu\Delta\bar{w}^{(k)}+\nabla\bar{s}^{(k)}=\Pi_z^x\zeta^{(k)}(x)g'\,,\\ \operatorname{div}\bar{w}^{(k)}=0\,,\\ \bar{w}^{(k)}|_{z_n=0}=0\,,\\ \bar{w}^{(k)}|_{t=0}=0\,,\end{array}\right. \tag{2.7}$$

where Π_z^x is the transformation from x to z.

These problems were also solved in [11] to have a unique solution satisfying

$$\|\bar{w}_t^{(k)}\|_{L_p(\mathbf{R}_T^3)}+\sum_{|\alpha|=2}\|D_x^\alpha\bar{w}^{(k)}\|_{L_p(\mathbf{R}_T^3)}+\|\nabla\bar{s}^{(k)}\|_{L_p(\mathbf{R}_T^3)}\le c_2\|\zeta^{(k)}g'\|_{L_p(\mathbf{R}_T^3)}\,, \tag{2.8}$$

$$\|\nabla\bar{w}^{(k)}\|_{L_p(\mathbf{R}_T^3)}+\|\bar{s}^{(k)}\|_{L_p(\mathbf{R}_T^3)}\le c_2\sqrt{T}\,\|\zeta^{(k)}g'\|_{L_p(\mathbf{R}_T^3)}\,, \tag{2.9}$$

$$\|\bar{w}^{(k)}\|_{L_p(\mathbf{R}_T^3)}\le c_2T\|\zeta^{(k)}g'\|_{L_p(\mathbf{R}_T^3)}\,, \tag{2.10}$$

where $\mathbf{R}_T^3=\begin{cases}\mathbf{R}^3\times[0,T] & \text{for } k\in\mathcal{M}\\ \mathbf{R}_+^3\times[0,T] & \text{for } k\in\mathcal{N}\end{cases}$ and $\mathbf{R}_+^3=\{x\in\mathbf{R}^3;\,x_3>0\}$.

Defining the regularizer R by the formula

$$Rh=\sum_k\eta^{(k)}(x)(w^{(k)},s^{(k)})(x,t)=\sum_k\eta^{(k)}(x)\Pi_x^z(\bar{w}^{(k)},\bar{s}^{(k)})(z,t) \tag{2.11}$$

for $h=(g',0,0)$, and the operator A by

$$A(w,s)=(\rho w_t-\mu\Delta w+\nabla s,\ \operatorname{div}w,\ w|_{S_T})\,,$$

we obtain $ARh=h+Mh$. Here $Mh=(M_1h,M_2h,0)$,

$$\begin{aligned}M_1h=&\sum_k[\{\rho(x,t)(\eta^{(k)}w^{(k)})_t-\mu\Delta(\eta^{(k)}w^{(k)})+\nabla(\eta^{(k)}s^{(k)})\}\\ &\qquad-\eta^{(k)}\{\rho(x,t)w_t^{(k)}-\mu\Delta w^{(k)}+\nabla s^{(k)}\}]\\ &+\sum_k\eta^{(k)}[\{\rho(x,t)w_t^{(k)}-\mu\Delta w^{(k)}+\nabla s^{(k)}\}-\{\rho(\xi^{(k)},0)w_t^{(k)}-\mu\Delta w^{(k)}+\nabla s^{(k)}\}]\\ &+\sum_{k\in\mathcal{N}}\eta^{(k)}\Pi_x^z[\{\rho(\xi^{(k)},0)\bar{w}_t^{(k)}-\mu(\nabla-\nabla F\cdot\nabla_3)^2\bar{w}^{(k)}+(\nabla-\nabla F\cdot\nabla_3)\bar{s}^{(k)}\}\\ &\qquad-\{\rho(\xi^{(k)},0)\bar{w}_t^{(k)}-\mu\Delta\bar{w}^{(k)}+\nabla\bar{s}^{(k)}\}]\\ =&\sum_k[-2\mu\nabla\eta^{(k)}\nabla w^{(k)}-\mu\Delta\eta^{(k)}w^{(k)}+\nabla\eta^{(k)}s^{(k)}]\end{aligned}$$

$$+\sum_k \eta^{(k)}[\rho(x,t)-\rho(\xi^{(k)},0)]w_t^{(k)}$$

$$+\sum_{k\in\mathcal{N}} \eta^{(k)}\Pi_x^z[-\mu\{(\nabla F)^2\nabla_3^2\bar{w}^{(k)}-2\nabla F\nabla_3\nabla\bar{w}^{(k)}-\nabla^2F\nabla_3\bar{w}^{(k)}\}-\nabla F\nabla_3\nabla_3\bar{s}^{(k)}]\,,$$

$$M_2h=\sum_k[\nabla\cdot(\eta^{(k)}w^{(k)})-\eta^{(k)}(\nabla\cdot w^{(k)})]+\sum_{k\in\mathcal{N}}\eta^{(k)}\Pi_x^z[(\nabla-\nabla F\nabla_3)\cdot\bar{w}^{(k)}-\nabla\cdot\bar{w}^{(k)}]$$

$$=\sum_k\nabla\eta^{(k)}\cdot w^{(k)}-\sum_{k\in\mathcal{N}}\eta^{(k)}\Pi_x^z\nabla F\nabla_3\cdot\bar{w}^{(k)}\,.$$

It is easily seen that M is a bounded operator in the space

$$\mathcal{B}_{p,T}\equiv L_p(Q_T)\times L_p(0,T;W_p^1(\Omega))\times W_p^{2-1/p}(S_T)$$

with norm

$$\|h\|_{\mathcal{B}_{p,T}}\equiv\|h_1\|_{L_p(Q_T)}+\|h_2\|_{L_p(0,T;W_p^1(\Omega))}+\|h_3\|_{W_p^{2-1/p}(S_T)}$$

for $h=(h_1,h_2,h_3)\in\mathcal{B}_{p,T}$.

If $\tau=\kappa\lambda^2$ with $\kappa\leq 1$ and $\kappa\lambda^2\leq T$, then by (2.8), (2.9) and (2.10), we get

$$\|M_1h\|_{L_p(Q_\tau)}\leq c_3\left(\frac{\sqrt{\tau}}{\lambda}+\frac{\tau}{\lambda^2}+\lambda^\alpha+\lambda^\beta+\lambda^2+\lambda+\sqrt{\tau}\right)\|g'\|_{L_p(Q_\tau)}$$
$$\leq c_4(\sqrt{\kappa}+\lambda^\alpha+\lambda^\beta)\|g'\|_{L_p(Q_\tau)}\,,$$

$$\|M_2h\|_{L_p(0,\tau;W_p^1(\Omega))}\leq c_5\left(\frac{\sqrt{\tau}}{\lambda}+\frac{\tau}{\lambda}+\frac{\tau}{\lambda^2}+\lambda\sqrt{\tau}+\lambda+\sqrt{\tau}\right)\|g'\|_{L_p(Q_\tau)}$$
$$\leq c_6(\sqrt{\kappa}+\lambda)\|g'\|_{L_p(Q_\tau)}\,,$$

where c_4 and c_6 are independent of κ and λ. Therefore we have for sufficiently small κ and λ,

$$\|Mh\|_{\mathcal{B}_{p,\tau}}\leq\frac{1}{2}\|h\|_{\mathcal{B}_{p,\tau}}\,.$$

Whence the solution (w,s) on the interval $[0,\tau]$ can be found in the form

$$(w,s)=R(I+M+M^2+\cdots)h=R(I-M)^{-1}h\,.$$

Moreover, (2.8), (2.9) and (2.10) imply

$$\|w\|_{W_p^{2,1}(Q_\tau)}+\|\nabla s\|_{L_p(Q_\tau)}\leq c_7\|g'\|_{L_p(Q_\tau)}\,.$$

Next we prove the solvability on the interval $[\tau,2\tau]$. Put

$$w^*(x,t)=\begin{cases}w(x,t) & \text{for } 0\leq t\leq\tau\,,\\ w(x,2\tau-t) & \text{for } \tau\leq t\leq 2\tau\,,\end{cases}$$

$$s^*(x,t)=\begin{cases} s(x,t) & \text{for } 0\le t\le\tau\,, \\ s(x,2\tau-t) & \text{for } \tau\le t\le 2\tau\,. \end{cases}$$

Let $(\tilde{w},\tilde{s})(x,t)$ be a solution of problem

$$\begin{cases} \rho\tilde{w}_t-\mu\Delta\tilde{w}+\nabla\tilde{s}=g'(x,t)-g'(x,2\tau-t) \\ \qquad\qquad\qquad +\rho(x,t)w_t(x,2\tau-t)+\rho(x,2\tau-t)w_t(x,2\tau-t)\,, \\ \operatorname{div}\tilde{w}=0\,, \\ \tilde{w}|_{\tilde{S}_\tau}=0\,, \\ \tilde{w}|_{t=\tau}=0 \end{cases}$$

for $\tau\le t\le 2\tau$, where $\tilde{S}_\tau=S\times[\tau,2\tau]$, and $(\tilde{w},\tilde{s})=(0,0)$ for $0\le t\le\tau$. Then it is easy to verify that $w=w^*+\tilde{w}$ and $s=s^*+\tilde{s}$ is the solution of the problem

$$\begin{cases} \rho w_t-\mu\Delta w+\nabla s=g'\,, \\ \operatorname{div} w=0\,, \\ w|_{S_{2\tau}}=0\,, \\ w|_{t=0}=0\,. \end{cases}$$

Repeating this argument, we can obtain the solution on $[0,T]$ satisfying

$$\|w\|_{W_p^{2,1}(Q_T)}+\|\nabla s\|_{L_p(Q_T)}\le c_8\|g'\|_{L_p(Q_T)}\,. \tag{2.12}$$

From (2.5) and (2.12), we can derive the estimate (2.2). □

LEMMA 2.2. *If $v(x,t)$ satisfies $\operatorname{div} v=0$, $v|_{S_T}=0$ and*

$$\|v\|_{L_\infty(Q_T)}+\int_0^T\|\nabla v(t)\|_{L_\infty(\Omega)}dt<\infty\,, \tag{2.13}$$

then for any $\rho_0(x)\in C^1(\bar{\Omega})$ such that $0<m\le\rho_0(x)\le M<\infty$, problem

$$\begin{cases} \rho_t+v\cdot\nabla\rho=0\,, \\ \rho|_{t=0}=\rho_0(x)\,, \end{cases} \tag{2.14}$$

has a unique solution $\rho(x,t)\in C^{1,1}(\bar{Q}_T)$, which satisfies

$$m\le\rho(x,t)\le M\,, \tag{2.15}$$

$$\|\nabla\rho\|_{L_\infty(Q_T)}\le\sqrt{3}\,\|\nabla\rho_0\|_{L_\infty(\Omega)}\exp\left(\int_0^T\|\nabla v(t)\|_{L_\infty(\Omega)}dt\right), \tag{2.16}$$

$$\|\rho_t\|_{L_\infty(Q_T)}\le\sqrt{3}\,\|v\|_{L_\infty(Q_T)}\|\nabla\rho_0\|_{L_\infty(\Omega)}\exp\left(\int_0^T\|\nabla v(t)\|_{L_\infty(\Omega)}dt\right). \tag{2.17}$$

Moreover, if $\nabla\rho_0(x)\in W_p^1(\Omega)$ and $v(x,t)\in L_1(0,T;W_p^2(\Omega))$, then

$$\frac{d}{dt}\|\nabla\rho(t)\|_{W_p^1(\Omega)}\le c_9\|v(t)\|_{W_p^2(\Omega)}\|\nabla\rho(t)\|_{W_p^1(\Omega)}\,. \tag{2.18}$$

PROOF. It is well-known that, according to the classical method of characteristics, the solution of problem (2.14) is given by $\rho(x, t) = \rho_0(y(\tau, x, t)|_{\tau=0})$, where $y(\tau, x, t)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dy}{d\tau} = v(y, \tau), \\ y|_{\tau=t} = x. \end{cases}$$

From this the estimate (2.15) results. For the estimates (2.16) and (2.17), we refer to Lemma 1.3 in [6].

Next let us establish (2.18). Apply the operator D_x^α on each side of $(2.14)_1$. Multiplying the result by $D_x^\alpha \rho |D_x^\alpha \rho|^{p-2}$, integrating over Ω and summing over $|\alpha| = 1, 2$, we have the equality

$$\frac{1}{p}\frac{d}{dt}\|\nabla\rho(t)\|^p_{W_p^1(\Omega)} = -\sum_{|\alpha|=1}^{2}\int_\Omega (v\cdot\nabla D_x^\alpha\rho) D_x^\alpha\rho |D_x^\alpha\rho|^{p-2}dx$$
$$-\sum_{|\alpha|=1}^{2}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega (D_x^\beta v\cdot\nabla D_x^{\alpha-\beta}\rho) D_x^\alpha\rho |D_x^\alpha\rho|^{p-2}dx.$$

The first term of the right hand side is zero, by integration by parts, since $\mathrm{div}\, v = 0$ and $v|_{S_T} = 0$. The second term can be estimated as follows:

$$\sum_{|\alpha|=1}^{2}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\left|\int_\Omega (D_x^\beta v\cdot\nabla D_x^{\alpha-\beta}\rho) D_x^\alpha\rho |D_x^\alpha\rho|^{p-2}dx\right|$$
$$\le c_{10}\|\nabla v(t)\|_{W_p^1(\Omega)}\|\nabla\rho(t)\|^p_{W_p^1(\Omega)}.$$

Hence we get the estimate (2.18). □

The next lemma is directly proved by means of the method of characteristics.

LEMMA 2.3. *Let $v(x, t)$ be the same as in Lemma 2.2. If $\rho(x, t) \in C^{1,1}(\bar{Q}_T)$ satisfies*

$$\begin{cases} \rho_t + v\cdot\nabla\rho = \tilde{g} \in L_1(0, T; L_\infty(\Omega)), \\ \rho|_{t=0} = 0, \end{cases}$$

then we have

$$\|\rho(t)\|_{L_\infty(\Omega)} \le \int_0^t \|\tilde{g}(s)\|_{L_\infty(\Omega)}ds.$$

2.2. Successive approximations. We construct approximate solutions inductively:

(2.19) $$v^{(0)} = 0,$$

and for $k=1, 2, 3, \cdots, \rho^{(k)}$ and $(v^{(k)}, p^{(k)})$ are, respectively, the solutions of problems

$$\begin{cases} \rho_t^{(k)} + v^{(k-1)} \cdot \nabla \rho^{(k)} = 0 , \\ \rho^{(k)}|_{t=0} = \rho_0(x) , \end{cases} \tag{2.20}$$

and

$$\begin{cases} \rho^{(k)}[v_t^{(k)} + (v^{(k-1)} \cdot \nabla) v^{(k-1)}] + \nabla p^{(k)} = \mu \Delta v^{(k)} + \rho^{(k)} f , \\ \operatorname{div} v^{(k)} = 0 \\ v^{(k)}|_{S_T} = 0 , \\ v^{(k)}|_{t=0} = v_0(x) . \end{cases} \tag{2.21}$$

In this subsection, we show their boundedness.

LEMMA 2.4. *For sufficiently small $T_1 \in (0, T]$, the sequence $\{v^{(k)}, \nabla p^{(k)}\}_k$ is bounded in $W_p^{2,1}(Q_{T_1}) \times L_p(Q_{T_1})$.*

PROOF. Let

$$V^{(k)}(T) = \|v^{(k)}\|_{W_p^{2,1}(Q_T)} + \|\nabla p^{(k)}\|_{L_p(Q_T)} . \tag{2.22}$$

From the consequences in subsection 2.1, we have

$$m \le \rho^{(k)} \le M ,$$

$$\|\nabla \rho^{(k)}\|_{L_\infty(Q_T)} \le \sqrt{3}\, \|\nabla \rho_0\|_{L_\infty(\Omega)} \exp\left(\int_0^T \|\nabla v^{(k-1)}(t)\|_{L_\infty(\Omega)} dt \right) ,$$

$$\|\rho_t^{(k)}\|_{L_\infty(Q_T)} \le \sqrt{3}\, \|v^{(k-1)}\|_{L_\infty(Q_T)} \|\nabla \rho_0\|_{L_\infty(\Omega)} \exp\left(\int_0^T \|\nabla v^{(k-1)}(t)\|_{L_\infty(\Omega)} dt \right) ,$$

$$\begin{aligned} \|v^{(k)}\|_{W_p^{2,1}(Q_T)} + \|\nabla p^{(k)}\|_{L_p(Q_T)} &\le K_1(\|\rho^{(k)}\|_{C^{1,1}(\bar{Q}_T)}, T) \\ &\times (\|v_0\|_{W_p^{2-2/p}(\Omega)} + \|\rho^{(k)} f\|_{L_p(Q_T)} + \|\rho^{(k)} (v^{(k-1)} \cdot \nabla) v^{(k-1)}\|_{L_p(Q_T)}) . \end{aligned} \tag{2.23}$$

Let us estimate the right hand side of (2.23).

First, we can get the inequality

$$\|v^{(k-1)}\|_{L_\infty(Q_T)} \le c_{11} (\|v_0\|_{W_p^{2-2/p}(\Omega)} + T^{(1-1/p)(1-3/p)} V^{(k-1)}(T)) . \tag{2.24}$$

Indeed, since imbedding theorems imply the inequalities

$$\|v^{(k-1)}(t) - v_0\|_{L_\infty(\Omega)} \le c_{12} \|v^{(k-1)}(t) - v_0\|_{W_p^1(\Omega)}^{3/p} \|v^{(k-1)}(t) - v_0\|_{L_p(\Omega)}^{1-3/p} ,$$

$$\|v_0\|_{L_\infty(\Omega)} \le c_{12} \|v_0\|_{W_p^1(\Omega)} \le c_{13} \|v_0\|_{W_p^{2-2/p}(\Omega)} ,$$

$$\begin{aligned} \sup_{0 \le t \le T} \|v^{(k-1)}\|_{W_p^1(\Omega)} &\le \sup_{0 \le t \le T} \|v^{(k-1)}\|_{W_p^{2-2/p}(\Omega)} \\ &\le c_{14} \|v^{(k-1)}\|_{W_p^{2,1}(Q_T)} + \|v_0\|_{W_p^{2-2/p}(\Omega)} , \end{aligned}$$

$$\|v^{(k-1)}(t)-v_0\|^p_{L_p(\Omega)}=\int_\Omega |v^{(k-1)}(t)-v_0|^p dx=\int_\Omega\left|\int_0^t v_t^{(k-1)}(s)ds\right|^p dx$$
$$\leq\int_\Omega\left|\left(\int_0^t ds\right)^{1/q}\left(\int_0^t |v_t^{(k-1)}(s)|^p ds\right)^{1/p}\right|^p dx$$
$$\leq t^{p/q}\|v^{(k-1)}\|^p_{W_p^{2,1}(Q_T)}$$

for $p^{-1}+q^{-1}=1$, the estimate (2.24) is easily derived.

Secondly, we have

$$(2.25)\qquad \int_0^T \|\nabla v^{(k-1)}(t)\|_{L_\infty(\Omega)}dt\leq c_{15}\int_0^T \|v^{(k-1)}(t)\|_{W_p^2(\Omega)}dt$$
$$\leq c_{15}\left(\int_0^T dt\right)^{1/q}\left(\int_0^T \|v^{(k-1)}(t)\|^p_{W_p^2(\Omega)}dt\right)^{1/p}$$
$$\leq c_{15}T^{1/q}\|v^{(k-1)}\|_{W_p^{2,1}(Q_T)}\leq c_{15}T^{1/q}V^{(k-1)}(T)\,.$$

Thirdly, it is obvious that

$$(2.26)\qquad \|\rho^{(k)}f\|_{L_p(Q_T)}\leq M\|f\|_{L_p(Q_T)}\,.$$

Finally, we have the estimate

$$(2.27)\qquad \|\rho^{(k)}(v^{(k-1)}\cdot\nabla)v^{(k-1)}\|_{L_p(Q_T)}\leq c_{16}MT^{(2-3/p)/p}V^{(k-1)}(T)^{(p-3)/(2p-3)}$$
$$\times(\|v_0\|_{W_p^{2-2/p}(\Omega)}+T^{(1-1/p)(1-3/p)}V^{(k-1)}(T))^{3(p-1)/(2p-3)}$$
$$\leq c_{16}M[\|v_0\|^2_{W_p^{2-2/p}(\Omega)}+T^\delta V^{(k-1)}(T)^2]$$

with some positive constants δ and $c_{16}\geq M^{-1}+1$. Indeed, from the inequality

$$\|\nabla v^{(k-1)}(t)\|_{L_p(\Omega)}\leq\|v^{(k-1)}(t)\|_{W_p^1(\Omega)}\leq\|v^{(k-1)}(t)\|^a_{W_p^2(\Omega)}\|v^{(k-1)}(t)\|^{1-a}_{L_\infty(\Omega)}$$

with $a=(p-3)/(2p-3)$, it follows that

$$\|\rho^{(k)}(v^{(k-1)}\cdot\nabla)v^{(k-1)}\|^p_{L_p(Q_T)}\leq M^p\|v^{(k-1)}\|^p_{L_\infty(Q_T)}\int_0^T \|\nabla v^{(k-1)}(t)\|^p_{L_p(\Omega)}dt$$
$$\leq c_{17}M^p\|v^{(k-1)}\|^{p(2-a)}_{L_\infty(Q_T)}\int_0^T \|v^{(k-1)}(t)\|^{ap}_{W_p^2(\Omega)}dt$$
$$\leq c_{17}M^p\|v^{(k-1)}\|^{p(2-a)}_{L_\infty(Q_T)}T^{1-a}\|v^{(k-1)}\|^{ap}_{W_p^{2,1}(Q_T)}\,.$$

This inequality combined with (2.24) leads to (2.27). Moreover, we obtain

$$\|\rho^{(k)}\|_{C^{1,1}(\bar{Q}_T)}\leq M+\sqrt{3}\,(1+\|v^{(k-1)}\|_{L_\infty(Q_T)})$$
$$\times\|\nabla\rho_0\|_{L_\infty(\Omega)}\exp\left(\int_0^T \|\nabla v^{(k-1)}(t)\|_{L_\infty(\Omega)}dt\right)$$

$$\begin{aligned}&\le M+\sqrt{3}\,[1+c_{11}(\|v_0\|_{W_p^{2-2/p}(\Omega)}+T^{(1-1/p)(1-3/p)}V^{(k-1)}(T))]\\&\quad\times\|\nabla\rho_0\|_{L_\infty(\Omega)}\exp(c_{15}T^{(1-1/p)}V^{(k-1)}(T))\\&\equiv K_3(V^{(k-1)}(T),T)\,.\end{aligned}$$

Consequently, we have

$$\begin{aligned}V^{(k)}(T)\le{}&K_1(K_3(V^{(k-1)}(T),T),T)\\&\times[\|v_0\|_{W_p^{2-2/p}(\Omega)}+M\|f\|_{L_p(Q_T)}+c_{16}M(\|v_0\|^2_{W_p^{2-2/p}(\Omega)}+T^\delta V^{(k-1)}(T)^2)]\\&\le c_{18}MK_1(K_3(V^{(k-1)}(T),T),T)\\&\quad\times[\|v_0\|_{W_p^{2-2/p}(\Omega)}+\|v_0\|^2_{W_p^{2-2/p}(\Omega)}+\|f\|_{L_p(Q_T)}+T^\delta V^{(k-1)}(T)^2]\,.\end{aligned}\tag{2.28}$$

We choose

$$\begin{aligned}A_1\ge{}&K_1(M+\sqrt{3}\,e[1+c_{11}(1+\|v_0\|_{W_p^{2-2/p}(\Omega)})]\|\nabla\rho_0\|_{L_\infty(\Omega)},T)\\&\times c_{18}M[\|v_0\|_{W_p^{2-2/p}(\Omega)}+\|v_0\|^2_{W_p^{2-2/p}(\Omega)}+\|f\|_{L_p(Q_T)}+1]\,,\end{aligned}$$

and define

$$T_1=\min\{A_1^{-(1-1/p)^{-1}(1-3/p)^{-1}},\,A_1^{-2/\delta},\,(c_{15}A_1)^{-(1-1/p)^{-1}}\}\,.$$

Then it is easily seen that $V^{(k)}(T_1)\le A_1$ holds provided that $V^{(k-1)}(T_1)\le A_1$. Since

$$V^{(1)}(T_1)\le K_1(M+\|\nabla\rho_0\|_{L_\infty(\Omega)},T_1)(\|v_0\|_{W_p^{2-2/p}(\Omega)}+M\|f\|_{L_p(Q_{T_1})})\le A_1\,,$$

the assertion of the lemma comes out. □

Furthermore, we can immediately get

LEMMA 2.5. *For any* $k=1,2,3,\cdots$,

$$\|\nabla\rho^{(k)}\|_{L_\infty(Q_{T_1})}+\|\rho_t^{(k)}\|_{L_\infty(Q_{T_1})}\le K_3(A_1,T_1)\equiv A_2\,,\tag{2.29}$$

$$\sup_{0\le t\le T_1}\|\nabla\rho^{(k)}(t)\|_{W_p^1(\Omega)}\le\|\nabla\rho_0\|_{W_p^1(\Omega)}\exp(c_{19}A_1(T_1)^{(p-1)/p})\equiv A_3\,.\tag{2.30}$$

2.3. Proof of Theorem 1.1. Setting $\sigma^{(k)}=\rho^{(k)}-\rho^{(k-1)}$, $w^{(k)}=v^{(k)}-v^{(k-1)}$ and $q^{(k)}=p^{(k)}-p^{(k-1)}$, we have

$$\begin{cases}\sigma_t^{(k)}+v^{(k-1)}\cdot\nabla\sigma^{(k)}=-w^{(k-1)}\cdot\nabla\rho^{(k-1)}\,,\\ \sigma^{(k)}|_{t=0}=0\,,\end{cases}\tag{2.31}$$

(2.32)
$$\begin{cases} \rho^{(k)}w_t^{(k)} - \mu\Delta w^{(k)} + \nabla q^{(k)} = g^{(k)} , \\ \operatorname{div} w^{(k)} = 0 , \\ w^{(k)}|_{S_T} = 0 , \\ w^{(k)}|_{t=0} = 0 , \end{cases}$$

where

$$g^{(k)} = -\sigma^{(k)}[v_t^{(k-1)} + (v^{(k-1)} \cdot \nabla)v^{(k-1)} + f] \\ -\rho^{(k-1)}[(w^{(k-1)} \cdot \nabla)v^{(k-2)} + (v^{(k-1)} \cdot \nabla)w^{(k-1)}] .$$

Let

$$W^{(k)}(t) = \|w^{(k)}\|_{W_p^{2,1}(Q_t)} + \|\nabla q^{(k)}\|_{L_p(Q_t)} .$$

Then, from Lemma 2.3, it follows that for $t \in (0, T_1]$,

(2.33)
$$\|\sigma^{(k)}(t)\|_{L_\infty(\Omega)} \le A_2 \int_0^t \|w^{(k-1)}(s)\|_{L_\infty(\Omega)} ds \\ \le c_{20}A_2 \int_0^t \|w^{(k-1)}\|_{W_p^{2,1}(Q_s)} ds \\ \le c_{21} \int_0^t W^{(k-1)}(s) ds .$$

Furthermore, we have

$$\|\sigma^{(k)}(t)\|_{W_p^1(\Omega)} \le c_{22}A_3 \int_0^t \|w^{(k-1)}(s)\|_{W_p^1(\Omega)} ds .$$

Here we used the imbeddings

$$\|u\|_{L_\infty(\Omega)} \le c_{23}\|u\|_{W_p^1(\Omega)} \le c_{23}\|u\|_{W_p^{2-2/p}(\Omega)} \\ \le c_{24}(\|u(x, 0)\|_{W_p^{2-2/p}(\Omega)} + \|u\|_{W_p^{2,1}(Q_t)}) .$$

Next, using Lemma 2.4, we can estimate each term in $g^{(k)}$:

$$\|\sigma^{(k)}[v_t^{(k-1)} + (v^{(k-1)} \cdot \nabla)v^{(k-1)} + f]\|_{L_p(Q_t)}^p \\ \le \|\sigma^{(k)}\|_{L_\infty(Q_t)}^p (\|v_t^{(k-1)}\|_{L_p(Q_t)}^p + \|v^{(k-1)}\|_{L_\infty(Q_t)}^p \|\nabla v^{(k-1)}\|_{L_p(Q_t)}^p + \|f\|_{L_p(Q_t)}^p) \\ \le \|\sigma^{(k)}\|_{L_\infty(Q_t)}^p (A_1^p + c_{25}(1 + \|v_0\|_{W_p^{2-2/p}(\Omega)})^p A_1^p + \|f\|_{L_p(Q_t)}^p) ,$$

$$\|\rho^{(k-1)}[(w^{(k-1)} \cdot \nabla)v^{(k-2)} + (v^{(k-1)} \cdot \nabla)w^{(k-1)}]\|_{L_p(Q_t)}^p \\ \le M^p \int_0^t ds \int_\Omega (|\nabla v^{(k-2)}|^p |w^{(k-1)}|^p + |v^{(k-1)}|^p |\nabla w^{(k-1)}|^p) dx$$

$$\le M^p\Big(\sup_{0\le s\le t}\|\nabla v^{(k-2)}\|^p_{L_p(\Omega)}\int_0^t\|w^{(k-1)}\|^p_{L_\infty(\Omega)}ds$$

$$+\|v^{(k-1)}\|^p_{L_\infty(Q_t)}\int_0^t\|\nabla w^{(k-1)}\|^p_{L_p(\Omega)}ds\Big)$$

$$\le c_{26}M^p\Big(\|v^{(k-2)}\|^p_{W^{2,1}_p(Q_t)}\int_0^t\|w^{(k-1)}\|^p_{W^{2,1}_p(Q_s)}ds$$

$$+(1+\|v_0\|_{W^{2-2/p}_p(\Omega)})^p\int_0^t\|w^{(k-1)}\|^p_{W^{2,1}_p(Q_s)}ds\Big).$$

Hence, Lemma 2.1 yields

$$\begin{aligned}W^{(k)}(t)&\le c_{27}\Big[\int_0^t W^{(k-1)}(s)ds+\Big(\int_0^t W^{(k-1)}(s)^p ds\Big)^{1/p}\Big]\\ &\le c_{28}\Big(\int_0^t W^{(k-1)}(s)^p ds\Big)^{1/p},\end{aligned}\tag{2.34}$$

consequently,

$$W^{(k)}(T_1)\le c_{28}^{k-1}\frac{(T_1)^{(k-1)/p}}{\Gamma(k)^{1/p}}W^{(1)}(T_1).\tag{2.35}$$

Therefore we find that

$$\sum_{k=1}^{\infty}W^{(k)}(T_1)<\infty,$$

which implies that the sequence $(\rho^{(k)}, v^{(k)}, p^{(k)})(x, t)$ converges to the desired solution $(\rho, v, p)(x, t)$ as $k\to\infty$.

The uniqueness is proved by making use of the estimates analogous to (2.33) and (2.34).

3. The case $\mu=0$.

In this section, we prove Theorem 1.2.

3.1. Auxiliary problems. We assume that $v(x, t)\in C^0([0, T];\ W^2_p(\Omega))$ is a given function such that $\operatorname{div} v=0$ and $v\cdot n|_{S_T}=0$.

LEMMA 3.1. *Let* $\rho(x, t)\in C^{1,1}([0, T]\times\bar\Omega)$ *such that* $0<m\le\rho(x, t)\le M<\infty$, $\nabla\rho(x, t)\in C^0([0, T];\ W^1_p(\Omega))$ *and* $f(x, t)\in C^0([0, T];\ W^2_p(\Omega))$. *Then problem*

$$(3.1)\qquad \begin{cases} \operatorname{div}(\rho^{-1}\nabla p)=\operatorname{div} f-\sum_{i,j=1}^{3} v^i_{x_j}v^j_{x_i}\equiv F, \\ \rho^{-1}\dfrac{\partial p}{\partial n}\bigg|_S=f\cdot n+\sum_{i,j=1}^{3} v^i v^j \phi^{ij}\equiv G, \quad \phi^{ij}=n^j_{x_i} \end{cases}$$

has a unique solution $\nabla p(x,t)\in C^0([0,T];\ W^2_p(\Omega))$, *satisfying*

$$(3.2)\qquad \|\nabla p(t)\|_{W^2_p(\Omega)}\le K_4(\|\nabla\rho(t)\|_{W^1_p(\Omega)})(\|f(t)\|_{W^2_p(\Omega)}+\|v(t)\|^2_{W^2_p(\Omega)}),$$

where K_4 *is a nondecreasing function of* $\|\nabla\rho(t)\|_{W^1_p(\Omega)}$, *depending on* m *and* M. *Hereafter,* K_j*'s are functions, having the same properties as* K_4.

PROOF. We first note that $(3.1)_1$ comes from applying the divergence operator on both sides of $(1.1)_2$, and $(3.1)_2$ from taking the scalar product of each side of $(1.1)_2$ with n (cf. Temam [12]). It is well-known from the result of Agmon, Douglis and Nirenberg [1] that problem (3.1) is solvable in $W^2_p(\Omega)$ and the estimate

$$\|\nabla p(t)\|_{W^1_p(\Omega)}\le K_5(\|\rho^{-1}\|_{C^1(\bar\Omega)})(\|F\|_{L_p(\Omega)}+\|G\|_{W^{1-1/p}_p(S)})$$

is valid. Writing the problem in the form

$$\begin{cases} \Delta p=\rho F-\rho\nabla(\rho^{-1})\cdot\nabla p, \\ \dfrac{\partial p}{\partial n}\bigg|_S=\rho G, \end{cases}$$

we get

$$\begin{aligned}\|\nabla p(t)\|_{W^2_p(\Omega)}&\le c_{29}(\|\rho F-\rho\nabla(\rho^{-1})\cdot\nabla p\|_{W^1_p(\Omega)}+\|\rho G\|_{W^{2-1/p}_p(S)})\\ &\le K_6(\|\nabla\rho(t)\|_{W^1_p(\Omega)})(\|F\|_{W^1_p(\Omega)}+\|G\|_{W^{2-1/p}_p(S)})\\ &\le K_7(\|\nabla\rho(t)\|_{W^1_p(\Omega)})(\|f(t)\|_{W^2_p(\Omega)}+\|v(t)\|^2_{W^2_p(\Omega)}). \qquad \square\end{aligned}$$

LEMMA 3.2. *Let* $\rho(x,t)$ *and* $f(x,t)$ *be the same as in Lemma* 3.1, *and* $\nabla p(x,t)\in C^0([0,T];\ W^2_p(\Omega))$ *be the unique solution of* (3.1) *guaranteed in Lemma* 3.1. *Then problem*

$$(3.3)\qquad \begin{cases} u_t+v\cdot\nabla u=-\rho^{-1}\nabla p+f, \\ u|_{t=0}=v_0(x), \end{cases}$$

has a unique solution $u(x,t)\in C^0([0,T];\ W^2_p(\Omega))$. *Moreover,* $u(x,t)$ *satisfies*

$$(3.4)\qquad \frac{d}{dt}\|u(t)\|_{W^2_p(\Omega)}\le c_{30}\|v(t)\|_{W^2_p(\Omega)}\|u(t)\|_{W^2_p(\Omega)} + K_8(\|\nabla\rho(t)\|_{W^1_p(\Omega)})(\|f(t)\|_{W^2_p(\Omega)}+\|v(t)\|^2_{W^2_p(\Omega)}).$$

PROOF. Referring to the proof of Lemma 2.2, we should only estimate the term

$$\sum_{|\alpha|=0}^{2} \int_{\Omega} D_x^\alpha(f-\rho^{-1}\nabla p)\cdot D_x^\alpha u|D_x^\alpha u|^{p-2}dx\,.$$

Since

$$\sum_{|\alpha|=0}^{2} \int_{\Omega} |D_x^\alpha(\rho^{-1}\nabla p)||D_x^\alpha u|^{p-1}dx$$

$$\le m^{-1}\|\nabla p\|_{W_p^2(\Omega)}\|u\|_{W_p^2(\Omega)}^{p-1}+\|\nabla\rho^{-1}\|_{W_p^1(\Omega)}\|\nabla p\|_{W_p^1(\Omega)}\|\nabla u\|_{W_p^1(\Omega)}^{p-1}$$

$$\le K_9(\|\nabla\rho\|_{W_p^1(\Omega)})(\|f\|_{W_p^2(\Omega)}+\|v\|_{W_p^2(\Omega)}^2)\|u\|_{W_p^2(\Omega)}^{p-1}\,,$$

the desired estimate is obtained. □

3.2. Successive approximations. In order to prove Theorem 1.2, we use the method of successive approximations in the following form:

$$v^{(0)}=0\,, \tag{3.5}$$

and for $k=1, 2, 3, \cdots, \rho^{(k)}$, $p^{(k)}$ and $u^{(k)}$ are, respectively, the solutions of problems

$$\begin{cases} \rho_t^{(k)}+v^{(k-1)}\cdot\nabla\rho^{(k)}=0\,, \\ \rho^{(k)}|_{t=0}=\rho_0(x)\,, \end{cases} \tag{3.6}$$

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{\rho^{(k)}}\nabla p^{(k)}\right)=\operatorname{div}f-\displaystyle\sum_{i,j=1}^{3} v_{x_j}^{(k-1),i}v_{x_i}^{(k-1),j}\,, \\ \dfrac{1}{\rho^{(k)}}\dfrac{\partial p^{(k)}}{\partial n}\bigg|_S=f\cdot n+\displaystyle\sum_{i,j=1}^{3} v^{(k-1),i}v^{(k-1),j}\phi^{ij}\,, \end{cases} \tag{3.7}$$

and

$$\begin{cases} u_t^{(k)}+v^{(k-1)}\cdot\nabla u^{(k)}=-\dfrac{1}{\rho^{(k)}}\nabla p^{(k)}+f\,, \\ u^{(k)}|_{t=0}=v_0(x)\,. \end{cases} \tag{3.8}$$

Finally, let

$$v^{(k)}=u^{(k)}-\nabla\psi^{(k)}\,, \tag{3.9}$$

where $\psi^{(k)}$ is the solution of problem

$$\begin{cases} \Delta\psi^{(k)}=\operatorname{div}u^{(k)}\,, \\ \dfrac{\partial\psi^{(k)}}{\partial n}\bigg|_S=u^{(k)}\cdot n\,. \end{cases} \tag{3.10}$$

LEMMA 3.3. *The sequence $\{v^{(k)}\}_k$ is bounded in $C^0([0, T_2]; W_p^2(\Omega))$ for a sufficiently small $T_2\in(0, T]$.*

PROOF. From the consequences in the previous subsections, we can derive

$$m \leq \rho^{(k)}(x, t) \leq M ,$$

$$\|\nabla \rho^{(k)}(t)\|_{W_p^1(\Omega)} \leq \|\nabla \rho_0\|_{W_p^1(\Omega)} \exp\left(c_9 \int_0^t \|v^{(k-1)}(s)\|_{W_p^2(\Omega)} ds \right),$$

$$\|\nabla p^{(k)}(t)\|_{W_p^2(\Omega)} \leq K_4(\|\nabla \rho^{(k)}(t)\|_{W_p^1(\Omega)})(\|f(t)\|_{W_p^2(\Omega)} + \|v^{(k-1)}(t)\|^2_{W_p^2(\Omega)}) ,$$

$$\|u^{(k)}(t)\|_{W_p^2(\Omega)} \leq \exp\left(c_{30} \int_0^t \|v^{(k-1)}(s)\|_{W_p^2(\Omega)} ds \right) [\|v_0\|_{W_p^2(\Omega)}$$

$$+ \int_0^t K_8(\|\nabla \rho^{(k)}(s)\|_{W_p^1(\Omega)})(\|f(s)\|_{W_p^2(\Omega)} + \|v^{(k-1)}(s)\|^2_{W_p^2(\Omega)}) ds] .$$

Since

$$\|v^{(k)}(t)\|_{W_p^2(\Omega)} \leq \|u^{(k)}(t)\|_{W_p^2(\Omega)} + \|\nabla \psi^{(k)}(t)\|_{W_p^2(\Omega)} \leq c_{31} \|u^{(k)}(t)\|_{W_p^2(\Omega)} ,$$

ultimately, we get

$$\text{(3.11)} \qquad \|v^{(k)}(t)\|_{W_p^2(\Omega)} \leq c_{31} \exp\left(c_{30} \int_0^t \|v^{(k-1)}(s)\|_{W_p^2(\Omega)} ds \right) [\|v_0\|_{W_p^2(\Omega)}$$

$$+ \int_0^t K_8(\|\nabla \rho^{(k)}(s)\|_{W_p^1(\Omega)})(\|f(s)\|_{W_p^2(\Omega)} + \|v^{(k-1)}(s)\|^2_{W_p^2(\Omega)}) ds] .$$

Let us choose

$$A_4 \geq 2c_{31}[\|v_0\|_{W_p^2(\Omega)} + K_8(2\|\nabla \rho_0\|_{W_p^1(\Omega)})(T\|f\|_{C^0([0,T];W_p^2(\Omega))} + 1)] ,$$

and define

$$T_2 = \min\{(c_{30}A_4)^{-1} \log 2 ,\ A_4^{-2} ,\ (c_9 A_4)^{-1} \log 2\} .$$

Then we find that

$$\sup_{0 \leq t \leq T_2} \|v^{(k)}(t)\|_{W_p^2(\Omega)} \leq A_4$$

provided that

$$\sup_{0 \leq t \leq T_2} \|v^{(k-1)}(t)\|_{W_p^2(\Omega)} \leq A_4 .$$

Therefore by induction we have the assertion of the lemma. □

By the direct calculation, we get

LEMMA 3.4. *For $k = 1, 2, 3, \cdots$, the estimates*

$$\sup_{0\le t\le T_2} \|\nabla\rho^{(k)}(t)\|_{W_p^1(\Omega)}\le 2\|\nabla\rho_0\|_{W_p^1(\Omega)}\equiv A_5\,,$$

$$\sup_{0\le t\le T_2} \|\nabla p^{(k)}(t)\|_{W_p^2(\Omega)}\le K_4(A_5)(\|f\|_{C^0([0,T];W_p^2(\Omega))}+A_4^2)\equiv A_6\,,$$

$$\sup_{0\le t\le T_2} \|\rho_t^{(k)}(t)\|_{W_p^1(\Omega)}\le A_4A_5\equiv A_7$$

$$\sup_{0\le t\le T_2} \|u_t^{(k)}(t)\|_{W_p^1(\Omega)}\le A_4^2+m^{-1}A_6(1+m^{-1}A_5)+\|f\|_{C^0([0,T];W_p^2(\Omega)}\equiv A_8$$

hold.

3.3. Proof of Theorem 1.2. Set $\sigma^{(k)}=\rho^{(k)}-\rho^{(k-1)}$, $h^{(k)}=u^{(k)}-u^{(k-1)}$, $q^{(k)}=p^{(k)}-p^{(k-1)}$ and $w^{(k)}=v^{(k)}-v^{(k-1)}$. Then we have

$$\text{(3.12)}\qquad \begin{cases} \sigma_t^{(k)}+v^{(k-1)}\cdot\nabla\sigma^{(k)}=-w^{(k-1)}\cdot\nabla\rho^{(k-1)}\,,\\ \sigma^{(k)}|_{t=0}=0\,,\end{cases}$$

$$\text{(3.13)}\qquad \begin{cases} \operatorname{div}\left(\dfrac{1}{\rho^{(k)}}\nabla q^{(k)}\right)=\operatorname{div}\left(\dfrac{\sigma^{(k)}}{\rho^{(k-1)}\rho^{(k)}}\nabla p^{(k-1)}\right)\\ \qquad\qquad -\displaystyle\sum_{i,j=1}^{3}(w_{x_j}^{(k-1),i}v_{x_i}^{(k-1),j}+v_{x_j}^{(k-2),i}w_{x_i}^{(k-1),j})\,,\\ \dfrac{1}{\rho^{(k)}}\dfrac{\partial q^{(k)}}{\partial n}\Big|_S=\displaystyle\sum_{i,j=1}^{3}(w^{(k-1),i}v^{(k-1),j}+v^{(k-2),i}w^{(k-1),j})\phi^{ij}\\ \qquad\qquad -\dfrac{\sigma^{(k)}}{\rho^{(k-1)}\rho^{(k)}}\dfrac{\partial p^{(k-1)}}{\partial n}\Big|_S\,,\end{cases}$$

$$\text{(3.14)}\qquad \begin{cases} h_t^{(k)}+(v^{(k-1)}\cdot\nabla)h^{(k)}+\dfrac{1}{\rho^{(k)}}\nabla q^{(k)}=-(w^{(k-1)}\cdot\nabla)u^{(k-1)}+\dfrac{\sigma^{(k)}}{\rho^{(k-1)}\rho^{(k)}}\nabla p^{(k-1)}\,,\\ h^{(k)}|_{t=0}=0\,.\end{cases}$$

In the same way used for getting the estimates of ρ, p and u, we get

$$\|\sigma^{(k)}(t)\|_{W_p^1(\Omega)}\le c_{32}A_5\exp(c_{32}A_4T_2)\int_0^t\|w^{(k-1)}(s)\|_{W_p^1(\Omega)}ds$$

$$\equiv A_9\int_0^t\|w^{(k-1)}(s)\|_{W_p^1(\Omega)}ds\,,$$

$$\|\nabla q^{(k)}(t)\|_{W_p^1(\Omega)}\le A_{10}(\|\sigma^{(k)}(t)\|_{W_p^1(\Omega)}+\|w^{(k-1)}(t)\|_{W_p^1(\Omega)})\,,$$

$$\|h^{(k)}(t)\|_{W_p^1(\Omega)}\le A_{11}\int_0^t(\|\sigma^{(k)}(s)\|_{W_p^1(\Omega)}+\|\nabla q^{(k)}(s)\|_{W_p^1(\Omega)}+\|w^{(k-1)}(s)\|_{W_p^1(\Omega)})ds\,.$$

From these inequalities, since

$$\|w^{(k)}(t)\|_{W_p^1(\Omega)} \le c_{33} \|h^{(k)}(t)\|_{W_p^1(\Omega)},$$

it follows that

$$\|w^{(k)}(t)\|_{W_p^1(\Omega)} \le A_{12} \int_0^t \|w^{(k-1)}(s)\|_{W_p^1(\Omega)} ds$$

$$\le A_{12}^{k-1} \frac{t^{k-1}}{(k-1)!} \sup_{0 \le s \le t} \|w^{(1)}(s)\|_{W_p^1(\Omega)},$$

consequently,

$$\sup_{0 \le t \le T_2} \|w^{(k)}(t)\|_{W_p^1(\Omega)} \le A_4 A_{12}^{k-1} \frac{(T_2)^{k-1}}{(k-1)!}.$$

Therefore we find that

$$\sum_{k=1}^{\infty} \|w^{(k)}\|_{C^0([0,T_2];W_p^1(\Omega))} < \infty .$$

This implies that $(\rho^{(k)}, p^{(k)}, u^{(k)}, v^{(k)})(x,t) \to (\rho, p, u, v)(x,t)$ as $k \to \infty$, which satisfies equations

$$\text{(3.15)} \qquad \begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \operatorname{div}((v \cdot \nabla) v + \rho^{-1} \nabla p - f) = 0, \\ u_t + (v \cdot \nabla) u + \rho^{-1} \nabla p = f, \\ \Delta \psi = \operatorname{div} u, \\ v = u - \nabla \psi, \end{cases}$$

$$\text{(3.16)} \qquad \begin{cases} ((v \cdot \nabla) v + \rho^{-1} \nabla p - f) \cdot n|_{S_{T_2}} = 0, \\ (u - \nabla \psi) \cdot n|_{S_{T_2}} = 0, \\ \rho|_{t=0} = \rho_0(x), \\ u|_{t=0} = v_0(x). \end{cases}$$

Now let us show that $u = v$. Applying the divergence operator on both sides of $(3.15)_3$ and taking into account $(3.15)_2$, we get

$$(\operatorname{div} u)_t + v \cdot \nabla(\operatorname{div} u) = -\sum_{i,j=1}^{3} v^i_{x_j} \psi_{x_i x_j}.$$

If we take the scalar product of each side of $(3.15)_3$ with n, we obtain

$$(u \cdot n)_t + v \cdot \nabla(u \cdot n) = \sum_{i,j=1}^{3} v^i \psi_{x_j} \phi^{ij}.$$

Noting that $\operatorname{div} v = 0$, $v \cdot n|_S = 0$ and

$$\|\psi\|_{W_p^2(\Omega)} \le c_{34} (\|\operatorname{div} u\|_{L_p(\Omega)} + \|u \cdot n\|_{W_p^{1-1/p}(S)}),$$

we have the inequality

$$\|\mathrm{div}u\|_{L_p(\Omega)}+\|u\cdot n\|_{W_p^{1-1/p}(S)}\le c_{35}\int_0^t(\|\mathrm{div}u\|_{L_p(\Omega)}+\|u\cdot n\|_{W_p^{1-1/p}(S)})ds\,,$$

which means $\mathrm{div}u=0$ and $u\cdot n|_S=0$.

This completes the proof of Theorem.

Part II. Vanishing Viscosity

4. The convergence problem as $\mu\to 0$.

In this section, we shall prove Theorem 1.3.

4.1. A priori estimates. Let $(\rho, v, p)(x, t)$ be a sufficiently regular solution. Hereafter C stands for the generic constant independent of μ.

LEMMA 4.1. *For $\rho(x, t)$, the estimates*

$$m\le\rho(x,t)\le M\,, \tag{4.1}$$

$$\frac{d}{dt}\|\nabla\rho(t)\|_{W_p^1(\mathbf{R}^3)}\le C\|v(t)\|_{W_p^2(\mathbf{R}^3)}\|\nabla\rho(t)\|_{W_p^1(\mathbf{R}^3)} \tag{4.2}$$

hold. Moreover, if we put $\xi(x,t)=\rho(x,t)^{-1}$, then the estimates

$$M^{-1}\le\xi(x,t)\le m^{-1}\,, \tag{4.3}$$

$$\frac{d}{dt}\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}\le C\|v(t)\|_{W_p^2(\mathbf{R}^3)}\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)} \tag{4.4}$$

are valid.

PROOF. Quite similarly to the proof of Lemma 2.2, we can obtain (4.1) and (4.2). If we note that $\xi(x, t)$ satisfies the equation

$$\begin{cases}\xi_t+v\cdot\nabla\xi=0\,,\\ \xi|_{t=0}=\rho_0(x)^{-1}\equiv\xi_0(x)\,,\end{cases}$$

the estimates (4.3) and (4.4) directly follow from (4.1) and (4.2). □

LEMMA 4.2. *Let $\omega(x,t)=\mathrm{rot}\,v(x,t)$. Then the estimate*

$$\frac{d}{dt}\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}\le C(1+\|\mathrm{rot}f\|_{C^0([0,T];W_p^1(\mathbf{R}^3))}) \times(1+\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}+\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}+\|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)})^3 \tag{4.5}$$

is valid.

PROOF. By applying the rotation operator on both sides of $(1.1)_2$, we obtain

$$\omega_t+(v\cdot\nabla)\omega-(\omega\cdot\nabla)v+\nabla\xi\times\nabla p=\mu\xi\Delta\omega+\mu\nabla\xi\times\Delta v+\mathrm{rot}\, f\,. \tag{4.6}$$

If we apply the operator D_x^α to (4.6), multiply the result by $D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}$, integrate over $\mathbf{R}^3$ and sum over $|\alpha|=0, 1$, then we have the equality

$$\begin{aligned}\frac{1}{p}\frac{d}{dp}\|\omega(t)\|^p_{W^1_p(\mathbf{R}^3)}=\sum_{|\alpha|=0,1}\Big[&-\int_{\mathbf{R}^3}D_x^\alpha((v\cdot\nabla)\omega)\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\\&+\int_{\mathbf{R}^3}D_x^\alpha((\omega\cdot\nabla)v)\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx-\int_{\mathbf{R}^3}D_x^\alpha(\nabla\xi\times\nabla p)\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\\&+\mu\int_{\mathbf{R}^3}D_x^\alpha(\xi\Delta\omega)\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx+\mu\int_{\mathbf{R}^3}D_x^\alpha(\nabla\xi\times\Delta v)\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\\&+\int_{\mathbf{R}^3}D_x^\alpha\mathrm{rot}f\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\Big]\equiv\sum_{j=1}^6 I_j\,.\end{aligned} \tag{4.7}$$

Let us estimate each I_j by making use of $\mathrm{div}v=0$ and $\mu\le 1$.

$$\begin{aligned}|I_1|&=\Big|\sum_{|\alpha|=0,1}\int_{\mathbf{R}^3}(v\cdot\nabla)D_x^\alpha\omega\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\\&\qquad+\sum_{|\alpha|=1}\int_{\mathbf{R}^3}(D_x^\alpha v\cdot\nabla)\omega\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\Big|\\&=\Big|\sum_{|\alpha|=1}\int_{\mathbf{R}^3}(D_x^\alpha v\cdot\nabla)\omega\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\Big|\le\sum_{|\alpha|=1}\int_{\mathbf{R}^3}|D_x^\alpha v||D_x^\alpha\omega|^pdx\\&\le C\Big(\sum_{|\alpha|=1}\|D_x^\alpha v(t)\|_{W^1_p(\mathbf{R}^3)}\Big)\|\omega(t)\|^p_{W^1_p(\mathbf{R}^3)}\,.\end{aligned}$$

$$\begin{aligned}|I_2|&=\Big|\sum_{|\alpha|=0,1}\int_{\mathbf{R}^3}(D_x^\alpha\omega\cdot\nabla)v\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\\&\qquad+\sum_{|\alpha|=1}\int_{\mathbf{R}^3}(\omega\cdot\nabla)D_x^\alpha v\cdot D_x^\alpha\omega|D_x^\alpha\omega|^{p-2}dx\Big|\\&\le\sum_{|\alpha|=0,1}\sum_{|\beta|=1}\int_{\mathbf{R}^3}|D_x^\beta v||D_x^\alpha\omega|^pdx\\&\qquad+\sum_{|\alpha|=1}\sum_{|\beta|=2}\int_{\mathbf{R}^3}|D_x^\beta v||\omega||D_x^\alpha\omega|^{p-1}dx\\&\le C\Big(\sum_{|\alpha|=1}\|D_x^\alpha v(t)\|_{W^1_p(\mathbf{R}^3)}\Big)\|\omega(t)\|^p_{W^1_p(\mathbf{R}^3)}\,.\end{aligned}$$

$$|I_3| = \left| \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} (\nabla\xi \times \nabla D_x^\alpha p) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx + \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (\nabla D_x^\alpha \xi \times \nabla p) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \right|$$

$$\leq 2 \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} |\nabla\xi| |\nabla D_x^\alpha p| |D_x^\alpha \omega|^{p-1} dx + 2 \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |\nabla D_x^\alpha \xi| |\nabla p| |D_x^\alpha \omega|^{p-1} dx$$

$$\leq C \|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)} \|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)} \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-1} .$$

$$I_4 = \mu \sum_{|\alpha|=0,1} \int_{\mathbf{R}^3} \xi \Delta D_x^\alpha \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx + \mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} D_x^\alpha \xi \Delta \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \equiv I_{4,1} + I_{4,2} ,$$

$$I_{4,1} = -\mu \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} \xi |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx - \mu(p-2) \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} \xi |D_x^\alpha \omega \cdot D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-4} dx - \mu \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} D_x^\beta \xi D_x^\alpha D_x^\beta \omega \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx .$$

Let the third term of the right hand side be $I_{4,3}$, then

$$|I_{4,2} + I_{4,3}| \leq C\mu \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\beta \xi| |D_x^\alpha D_x^\beta \omega| |D_x^\alpha \omega|^{p-1} dx$$

$$\leq \frac{\mu}{4M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx + C \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\beta \xi|^2 |D_x^\alpha \omega|^p dx$$

$$\leq \frac{\mu}{4M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta \omega|^2 |D_x^\alpha \omega|^{p-2} dx + C \|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}^2 \|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p .$$

$$I_5 = \mu \int_{\mathbf{R}^3} (\nabla\xi \times \Delta v) \cdot \omega |\omega|^{p-2} dx + \mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} D_x^\alpha (\nabla\xi \times \Delta v) \cdot D_x^\alpha \omega |D_x^\alpha \omega|^{p-2} dx \equiv I_{5,1} + I_{5,2} ,$$

$$|I_{5,1}| \le 2\mu \int_{\mathbf{R}^3} |\nabla\xi||\Delta v||\omega|^{p-1}dx$$
$$\le C\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}\|\Delta v(t)\|_{L_p(\mathbf{R}^3)}\|\omega(t)\|_{L_p(\mathbf{R}^3)}^{p-1} .$$

Since

$$I_{5,2} = -\mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} (\nabla\xi \times \Delta v) \cdot D_x^{2\alpha}\omega |D_x^\alpha \omega|^{p-2}dx$$
$$-\mu(p-2) \sum_{|\alpha|=1} \int_{\mathbf{R}^3} [(\nabla\xi \times \Delta v) \cdot D_x^\alpha\omega] D_x^{2\alpha}\omega \cdot D_x^\alpha\omega |D_x^\alpha\omega|^{p-4}dx ,$$

we get

$$|I_{5,2}| \le C\mu \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |\nabla\xi||\Delta v||D_x^{2\alpha}\omega||D_x^\alpha\omega|^{p-2}dx$$
$$\le \frac{\mu}{4M} \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |D_x^{2\alpha}\omega|^2 |D_x^\alpha\omega|^{p-2}dx$$
$$+ C \sum_{|\alpha|=1} \int_{\mathbf{R}^3} |\nabla\xi|^2|\Delta v|^2|D_x^\alpha\omega|^{p-2}dx$$
$$\le \frac{\mu}{4M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta\omega|^2|D_x^\alpha\omega|^{p-2}dx$$
$$+ C\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}^2\|\Delta v(t)\|_{L_p(\mathbf{R}^3)}^2\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-2} .$$

$$|I_6| \le \|\operatorname{rot} f(t)\|_{W_p^1(\mathbf{R}^3)}\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-1} .$$

Hence, using the inequality

$$\sum_{|\alpha|=1} \|D_x^\alpha v(t)\|_{W_p^1(\mathbf{R}^3)} \le C\|\omega(t)\|_{W_p^1(\mathbf{R}^3)} ,$$

we have

$$(4.8) \quad \frac{1}{p}\frac{d}{dt}\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^p + \frac{\mu}{2M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha D_x^\beta\omega|^2|D_x^\alpha\omega|^{p-2}dx$$
$$+ \frac{\mu(p-2)}{M} \sum_{|\alpha|=0,1} \sum_{|\beta|=1} \int_{\mathbf{R}^3} |D_x^\alpha\omega \cdot D_x^\alpha D_x^\beta\omega|^2|D_x^\alpha\omega|^{p-4}dx$$
$$\le C[\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^2 + \|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}\|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)} + \|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}$$
$$+ \|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}^2\|\omega(t)\|_{W_p^1(\mathbf{R}^3)} + \|\operatorname{rot} f(t)\|_{W_p^1(\mathbf{R}^3)}]\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}^{p-1} . \qquad \square$$

LEMMA 4.3. *There exists $T_0 \in (0, T]$ independent of μ such that*

$$\sup_{0\le t\le T_0}(\|\nabla\rho(t)\|_{W_p^1(\mathbf{R}^3)}+\|v(t)\|_{W_p^2(\mathbf{R}^3)}+\|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)})\le C\,. \tag{4.9}$$

PROOF. Similarly to getting the estimate ω, we first obtain

$$\frac{1}{p}\frac{d}{dt}\|v(t)\|_{L_p(\mathbf{R}^3)}\le C[\|\nabla p(t)\|_{L_p(\mathbf{R}^3)}+\|\nabla\xi(t)\|^2_{W_p^1(\mathbf{R}^3)}\|v(t)\|_{L_p(\mathbf{R}^3)}+\|f(t)\|_{L_p(\mathbf{R}^3)}]\,.$$

Next, from the equation

$$\operatorname{div}(\xi\nabla p)=\operatorname{div}f-\sum_{i,j=1}^{3}v^i_{x_j}v^j_{x_i}+\mu\nabla\xi\cdot\Delta v\equiv F\,,$$

we have

$$\begin{aligned}\|\nabla p(t)\|_{W_p^1(\mathbf{R}^3)}&\le K_5(\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)})\|F\|_{L_p(\mathbf{R}^3)}\\&\le CK_5(\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)})\\&\qquad\times(\|\operatorname{div}f\|_{L_p(\mathbf{R}^3)}+\|\omega(t)\|^2_{W_p^1(\mathbf{R}^3)}+\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}\|\omega(t)\|_{W_p^1(\mathbf{R}^3)})\,.\end{aligned}$$

Therefore, if we set

$$Y(t)=1+\|\nabla\xi(t)\|_{W_p^1(\mathbf{R}^3)}+\|v(t)\|_{L_p(\mathbf{R}^3)}+\|\omega(t)\|_{W_p^1(\mathbf{R}^3)}\,,$$
$$B=(1+\|f\|_{C^0([0,T];W_p^2(\mathbf{R}^3))})^2\,,$$

then the above lemmas imply a differential inequality

$$\frac{d}{dt}Y(t)\le CBH(Y(t))\,,$$

where H is a increasing function of $Y(t)$ independent of μ.

Hence we conclude that $Y(t)\le Z(t)$, where $Z(t)$ is the solution of the problem

$$\begin{cases}\dfrac{d}{dt}Z(t)=CBH(Z(t))\,,\\ Z(0)=Y(0)\,,\end{cases}$$

and exists as a continuous function on an interval $[0, T_0]$ with $T_0>0$.

Since T_0 is obviously independent of μ, we obtain the desired result. □

4.2. Proof of Theorem 1.3. First, it follows from Theorems 1.1, 1.2 and Lemma 4.3 that the existence of a unique solution on $[0, T_0]$ with $T_0>0$ independent of μ.

Next we prove (1.22). Subtracting (1.1) with $\mu>0$ from (1.1) with $\mu=0$, we get the following linear system of equations for $\sigma=\rho^0-\rho^\mu$, $w=v^0-v^\mu$ and $q=p^0-p^\mu$:

$$(4.10)\qquad \begin{cases} \sigma_t + v^\mu \cdot \nabla\sigma = -w \cdot \nabla\rho^0 , \\ \rho^\mu[w_t + (v^\mu \cdot \nabla)w] + \nabla q = -\rho^\mu(w \cdot \nabla)v^0 + (\nabla p^0/\rho^0)\sigma - \mu\Delta v^\mu , \\ \operatorname{div} w = 0 , \\ \sigma|_{t=0} = 0 , \\ w|_{t=0} = 0 . \end{cases}$$

In the same way for getting a priori estimates, we have, from (4.9),

$$\|\sigma(t)\|_{L_p(\mathbf{R}^3)} \le C \int_0^t \|w(s)\|_{L_p(\mathbf{R}^3)} ds ,$$

$$\|w(t)\|_{L_p(\mathbf{R}^3)} \le C \int_0^t (\|\sigma(s)\|_{L_p(\mathbf{R}^3)} + \|w(s)\|_{L_p(\mathbf{R}^3)}) ds + \mu C T_0 .$$

Hence, by Gronwall's inequality, we find that

$$(4.11)\qquad \|\sigma(t)\|_{L_p(\mathbf{R}^3)} + \|w(t)\|_{L_p(\mathbf{R}^3)} \le \mu C T_0 \exp(C T_0) .$$

Furthermore, making use of the interpolation inequalities, we have

$$\begin{aligned} \|\nabla\sigma(t)\|_{L_p(\mathbf{R}^3)} &\le C \|\sigma(t)\|^{1/2}_{L_p(\mathbf{R}^3)} \|\nabla\sigma(t)\|^{1/2}_{W^1_p(\mathbf{R}^3)} \\ &\le C \|\sigma(t)\|^{1/2}_{L_p(\mathbf{R}^3)} (\|\nabla\rho^0(t)\|_{W^1_p(\mathbf{R}^3)} + \|\nabla\rho^\mu(t)\|_{W^1_p(\mathbf{R}^3)})^{1/2} \\ &\le C \|\sigma(t)\|^{1/2}_{L_p(\mathbf{R}^3)} , \\ \|\nabla w(t)\|_{L_p(\mathbf{R}^3)} &\le C \|w(t)\|^{1/2}_{L_p(\mathbf{R}^3)} \|\nabla w(t)\|^{1/2}_{W^1_p(\mathbf{R}^3)} \\ &\le C \|w(t)\|^{1/2}_{L_p(\mathbf{R}^3)} (\|\nabla v^0(t)\|_{W^1_p(\mathbf{R}^3)} + \|\nabla v^\mu(t)\|_{W^1_p(\mathbf{R}^3)})^{1/2} \\ &\le C \|w(t)\|^{1/2}_{L_p(\mathbf{R}^3)} . \end{aligned}$$

On the other hand, since $q(x, t)$ satisfies the equation

$$\begin{aligned} \operatorname{div}(\xi^\mu \nabla q) = -\mu\nabla\xi^\mu \cdot \Delta v^\mu - \sum_{i,j=1}^{3} (v^{\mu,i}_{x_j} w^j_{x_i} + v^{0,i}_{x_j} w^j_{x_i}) \\ + \Delta p^0 \xi^0 \xi^\mu \sigma + \nabla p^0 \cdot \nabla\xi^0 \xi^\mu \sigma + \nabla p^0 \cdot \nabla\xi^\mu \xi^0 \sigma + \nabla p^0 \cdot \nabla\sigma \xi^0 \xi^\mu , \end{aligned}$$

we get

$$\|\nabla q(t)\|_{W^1_p(\mathbf{R}^3)} \le C(\|\sigma(t)\|_{W^1_p(\mathbf{R}^3)} + \|w(t)\|_{W^1_p(\mathbf{R}^3)} + \mu) .$$

Thus, owing to (4.11), the proof of Theorem 1.3 is completed.

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