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An Estimate for the Kakeya Maximal Operator on Functions of Square Radial Type

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Abstract. The small Kakeya maximal operator, $M_{a,N}$, in $\mathbf{R}^d$ is defined by averages on cylinders with the width a and the height Na. We show that the inequality $\|M_{a,N}f\|_d \le C\log N\|f\|_d$ holds for the functions of square radialy type, where C is a constant depending only on d.

1. Introduction and theorem.

In this note we shall prove an estimate for the Kakeya maximal operator on functions of square radial type by applying the idea which is used in [Ta].

Fix $N \gg 1$. For a real number $a>0$ let $\mathscr{B}_{a,N}$ be the family of all cylinders in the d-dimensional Euclidean space $\mathbf{R}^d$, $d \ge 2$, which are congruent to

$$\left\{x=(x_1, \cdots, x_d)\in\mathbf{R}^d \,\middle|\, |x_1|<\frac{Na}{2}, \left(\sum_{l=2}^{d} x_l^2\right)^{1/2}<\frac{a}{2}\right\}$$

but with arbitrary directions and centers. Note that this cylinder has the height Na and the width a.

The small Kakeya maximal operator $M_{a,N}$ is defined for function f on $\mathbf{R}^d$ by

$$(M_{a,N}f)(x)=\sup_{x\in R\in\mathscr{B}_{a,N}} \frac{1}{|R|}\int_R |f(y)|dy\,,$$

where $|A|$ represents the Lebesgue measure of a set A.

For $x=(x_1, \cdots, x_d)$ in $\mathbf{R}^d$ let $|x|_m$ be the maximal norm of x defined by

$$|x|_m=\max_l |x_l|\,.$$

If a function on $\mathbf{R}^d$ is of the form $f(x)=f_0(|x|_m)$ we will call f a function of square radial type, where f_0 is a function on $[0, \infty)$.

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It is conjectured that $M_{a,N}$ is bounded on $L^d(\mathbf{R}^d)$ with the norm which grows no faster than $O((\log N)^{\alpha_d})$ for some $\alpha_d>0$ as $N\to\infty$. This conjecture was solved affirmatively in the case $d=2$ by Córdoba [Co] with $\alpha_d=1/2$ but it seems to remain unsolved for $d\geq 3$. But for $d\geq 3$ this conjecture is known to be true for radial functions (cf. Carbery, Hernández and Soria [CHS] and Igari [Ig2]) and for functions of the form $f(x)=\prod_{l=1}^{d} f_l(x_l)$ (cf. Igari [Ig1] and also Tanaka [Ta]) with some constants α_d.

The purpose of this note is to show that this conjecture is also true for functions of square radial type.

THEOREM 1. *Let $d\geq 2$. There exists a constant C depending only on the dimension d such that*

$$\|M_{a,N}f\|_d\leq C\log N\|f\|_d$$

holds for all square radial functions f in $L^d(\mathbf{R}^d)$. Here $\|f\|_d$ denotes the L^d-norm of f.

The methods to prove this theorem will be applicable to functions of polygonally radial type, but may not be applicable to functions of radial type considered by [CHS].

In the followong C's will denote constants which may be different in each occasion but depend only on the dimension d.

I would like to express my gratitude to professor S. T. Kuroda who simplified my proof of the last part of Section 3.

2. Proof of Theorem 1.

In this section we shall prove Theorem 1. The method used here is an application of the idea which is used in [Ta].

We may assume that $f_0\geq 0$ and N is a positive integer. By dilation invariance it suffices to consider only the case $a=1$. We write $M_{1,N}$ as M_N. We will linearize the maximal function first. We divide $\mathbf{R}^d$ into unit cubes Q_i which have center at lattice points $i\in\mathbf{Z}^d$ and whose sides are parallel to the axes. By the local integrability of f we can find for every cube Q_i a cylinder R_i in $\mathscr{B}_{1,N}$ such that

$$Q_i\cap R_i\neq\varnothing$$

and

$$(M_Nf)(x)\leq\frac{2}{|R_i|}\int_{R_i}f(y)dy\,,\qquad \forall x\in Q_i\,. \tag{1}$$

Obviously, this shows that for proving the theorem it is sufficient to estimate

$$\sum_{i\in\mathbf{Z}^d}\frac{1}{N}\int_{R_i}f(y)dy\cdot\chi_{Q_i}(x)\,. \tag{2}$$

In the proof we use the following notations.

$$\gamma_i = \{j \in \mathbf{Z}^d \mid Q_j \cap R_i \neq \emptyset\},$$

$$S_k = \begin{cases} \{x \in \mathbf{R}^d \mid |x|_m \leq \frac{1}{2}\}, & k = 0, \\ \{x \in \mathbf{R}^d \mid k - \frac{1}{2} < |x|_m \leq k + \frac{1}{2}\}, & k \geq 1. \end{cases}$$

First we note that

$$\begin{aligned} N^d \int_{\mathbf{R}^d} \Big(\frac{1}{N} \sum_{i \in \mathbf{Z}^d} \int_{R_i} f(y) dy \cdot \chi_{Q_i}(x) \Big)^d dx \\ \leq \sum_{i \in \mathbf{Z}^d} \Big(\int_{R_i} f(y) dy \Big)^d \leq \sum_{i \in \mathbf{Z}^d} \Big(\sum_{j \in \gamma_i} \int_{Q_j} f(y) dy \Big)^d. \end{aligned} \tag{3}$$

In the next step we will use the inequality

$$\int_{Q_j} f_0(|y|_m) dy \leq d \int_{|j|_m - 1/2}^{|j|_m + 1/2} f_0(|r|) dr.$$

This inequality is verified easily. Now we obtain

$$\begin{aligned} \sum_{i \in \mathbf{Z}^d} \Big(\sum_{j \in \gamma_i} \int_{Q_j} f(y) dy \Big)^d \\ \leq d^d \sum_{i \in \mathbf{Z}^d} \Big(\sum_{k=0}^{\infty} \operatorname{card}(\{j \in \gamma_i \mid Q_j \cap S_k \neq \emptyset\}) \int_{k-1/2}^{k+1/2} f_0(|r|) dr \Big)^d. \end{aligned} \tag{4}$$

Let I_1 and I_2 be

$$I_1 = \{i = (i_1, \cdots, i_d) \in \mathbf{Z}^d \mid i_l \geq 0,\ l = 1, \cdots, d,\ |i|_m \leq 3N\},$$

$$I_2 = \{i = (i_1, \cdots, i_d) \in \mathbf{Z}^d \mid i_l \geq 0,\ l = 1, \cdots, d,\ |i|_m > 3N\}.$$

Then by the symmetry we may restrict the first sum of the right-hand side of (4) to $I_1 \cup I_2$. Let $c(i, k)$ be

$$c(i, k) = \operatorname{card}(\{j \in \gamma_i \mid Q_j \cap S_k \neq \emptyset\}).$$

We shall prove the following two inequalities:

$$\sum_{i \in I_1} \Big(\sum_{k=0}^{5N} c(i, k) \int_{k-1/2}^{k+1/2} f_0(|r|) dr \Big)^d \leq C N^d (\log N)^d \sum_{k=0}^{5N} \int_{S_k} f^d dx, \tag{5}$$

$$\sum_{i \in I_2} \Big(\sum_{k=N}^{\infty} c(i, k) \int_{k-1/2}^{k+1/2} f_0(r) dr \Big)^d \leq C N^d \log N \sum_{k=N}^{\infty} \int_{S_k} f^d dx. \tag{6}$$

If this can be done, we will finish the proof of the theorem by (1)–(4).

1. Proof of (6). If all components of $j \in \mathbf{Z}^d$ are mutually different, then we easily see that

$$\int_{|j|_m-1/2}^{|j|_m+1/2} f_0(r)dr=\int_{Q_j} f_0(|y|_m)dy\,.$$

Using Hölder's inequality, this equation and the fact that the number of Q_j, $Q_j\subset S_k$, such that all components of j are mutually different is $2d(2k-1)^{d-1}$, we have that

$$\begin{aligned}
&\sum_{i\in I_2}\left(\sum_{k=N}^{\infty} c(i,k)\int_{k-1/2}^{k+1/2} f_0(r)dr\right)^d\\
&=\sum_{i\in I_2}\left\{\sum_{k=N}^{\infty} c(i,k)^{1-1/d}\cdot c(i,k)^{1/d}\int_{k-1/2}^{k+1/2} f_0(r)dr\right\}^d\\
&\le\sum_{i\in I_2}\left\{\sum_{k=N}^{\infty} c(i,k)\right\}^{d-1}\cdot\left\{\sum_{k=N}^{\infty} c(i,k)\int_{k-1/2}^{k+1/2} f_0(r)^d dr\right\}\\
&\le CN^{d-1}\sum_{i\in I_2}\sum_{k=N}^{\infty} c(i,k)(2d(2k-1)^{d-1})^{-1}\cdot\left\{2d(2k-1)^{d-1}\int_{k-1/2}^{k+1/2} f_0(r)^d dr\right\}\\
&\le CN^{d-1}\sum_{i\in I_2}\sum_{k=N}^{\infty}\frac{c(i,k)}{k^{d-1}}\int_{S_k} f^d dx=CN^{d-1}\sum_{k=N}^{\infty}\sum_{i\in I_2}\frac{c(i,k)}{k^{d-1}}\int_{S_k} f^d dx\,.
\end{aligned}$$

Now if we can prove that

$$\sum_{i\in I_2}\frac{c(i,k)}{k^{d-1}}\le CN\log N\,,\qquad \forall k\ge N\,, \tag{7}$$

then we will obtain the proof of (6).

Let $S_{l,k}$, $l=1,\cdots,d$, be the part of S_k defined by

$$S_{l,k}=\left(\mathbf{R}^{l-1}\times\left(k-\frac{1}{2},k+\frac{1}{2}\right]\times\mathbf{R}^{d-l}\right)\cap S_k\,,\qquad k\ge 1\,.$$

Let $c_l(i,k)$ be

$$c_l(i,k)=\operatorname{card}(\{j\in\gamma_i \mid Q_j\cap S_{l,k}\ne\varnothing\})\,,\qquad k\ge 1\,.$$

Then, if we can prove

$$\sum_{i\in I_2}\frac{c_l(i,k)}{k^{d-1}}\le CN\log N\,,\qquad \forall k\ge N\,, \tag{8}$$

for $l=1,\cdots,d$, (7) will follow by the symmetry of the problem and by the fact that the union of $S_{l,k}$ is the half of S_k. Without the loss of generality we may assume that $l=1$.

Let $D_{k,q}$, $q=1,2$, be the set of lattice points defined by

$$D_{k,q}=([k-2N,k+2N]\times[0,k+2N]^{d-1})\cap I_q\,,\qquad k\ge 1\,.$$

Then we may restrict the sum in (8) to $D_{k,2}$. The rest of the proof will rely on the next geometric inequality:

$$c_1(i, k) \le C \frac{N}{1+|k-i_1|} . \tag{9}$$

Proof of (9). Let the band-like domain Ω be

$$\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbf{R}^{d-1} .$$

Choose an $i=(i_1, \cdots, i_d)$ from $\mathbf{Z}^d$ as $i_1 > 10$. Let the line segment L_i be

$$L_i = \{i + t\omega \mid \omega = (\omega_1, \cdots, \omega_d) \in S^{d-1}, \ -N \le t \le N\} .$$

Then, if this line segment L_i penetrates Ω, we have

$$|\Omega \cap L_i| = \frac{1}{|\omega_1|} ,$$

$$i_1 + \frac{1}{2} \le |\omega_1| N$$

and hence

$$|\Omega \cap L_i| \le \frac{N}{i_1 + 1/2} \le 2 \frac{N}{i_1 + 1} .$$

(9) follows easily from this inequality. □

Inserting (9) into the left hand side of (8), we have

$$\sum_{i \in I_2} \frac{c_1(i, k)}{k^{d-1}} = \sum_{i \in D_{k,2}} \frac{c_1(i, k)}{k^{d-1}} \le C \left(\frac{k+2N}{k}\right)^{d-1} \sum_{i_1 = k-2N}^{k+2N} \frac{N}{1+|k-i_1|} \le CN \log N .$$

Thus, we obtain (6).

2. Proof of (5). It follows by Hölder's inequality and the definition of $c(i, k)$ that

$$\sum_{i \in I_1} \left(\sum_{k=0}^{5N} c(i, k) \int_{k-1/2}^{k+1/2} f_0(|r|) dr \right)^d$$

$$\le CN^d \int_{S_0} f^d dx + C \sum_{i \in I_1} \left(\sum_{k=1}^{5N} c(i, k) \int_{k-1/2}^{k+1/2} f_0(r) dr \right)^d . \tag{10}$$

In the same way as in the first part of the proof of (6) we have

$$\sum_{i \in I_1} \left(\sum_{k=1}^{5N} c(i, k) \int_{k-1/2}^{k+1/2} f_0(r) dr \right)^d$$

$$= \sum_{i \in I_1} \left\{ \sum_{k=1}^{5N} \left(\frac{c(i, k)}{2k-1} \right)^{1-1/d} \cdot (c(i, k)(2k-1)^{d-1})^{1/d} \int_{k-1/2}^{k+1/2} f_0(r) dr \right\}^d$$

$$\leq C\sum_{i\in I_1}\left\{\sum_{k=1}^{5N}\frac{c(i,k)}{k}\right\}^{d-1}\cdot\left\{\sum_{k=1}^{5N}c(i,k)\int_{S_k}f^d dx\right\}. \tag{11}$$

We shall prove the following two inequalities:

$$\sum_{k=1}^{5N}\frac{c(i,k)}{k}\leq C\log N, \qquad \forall i\in I_1; \tag{12}$$

$$\sum_{i\in I_1}c(i,k)\leq CN^d\log N, \qquad \forall k, \quad 1\leq k\leq 5N. \tag{13}$$

If this can be done, we will finish the proof of (5) by (10) and (11).

(13) follows in the same way as in the last part of the proof of (6). Indeed, for every $1\leq k\leq 5N$ we obtain

$$\sum_{i\in I_1}c_1(i,k)=\sum_{i\in D_{k,1}}c_1(i,k)\leq C(k+2N)^{d-1}\sum_{i_1=k-2N}^{k+2N}\frac{N}{1+|k-i_1|}\leq CN^d\log N.$$

3. Proof of (12). As in the proof of (6) it is sufficient to show that

$$\sum_{l=1}^{d}\sum_{k=1}^{5N}\frac{c_l(i,k)}{k}\leq C\log N, \qquad \forall i\in I_1. \tag{14}$$

Fix $i\in I_1$ and fix l. Let $\omega_i=(a_1,\cdots,a_d)$ be a unit vector which is parallel to the axis of R_i. Then we easily see that

$$c_l(i,k)\leq C\min\left\{k,\frac{1}{|a_l|}\right\}. \tag{15}$$

If $S_{l,k}\cap R_i=\varnothing$, $\forall k\in[1,5N]$, we have nothing to prove. Therefore, we assume that $S_{l,k}\cap R_i\neq\varnothing$, for some $k\in[1,5N]$. All such $S_{l,k}$ can be listed as S_{l,k_1}, S_{l,k_1+1}, $\cdots$, S_{l,k_2-1}, S_{l,k_2}. If $k_2-k_1\leq 2d$, then we have

$$\sum_{k=k_1}^{k_2}\frac{c_l(i,k)}{k}\leq C(k_2-k_1+1)\leq C$$

by (15). And if $|a_l|\geq 1/(2\sqrt{d})$, then we have

$$\sum_{k=k_1}^{k_2}\frac{c_l(i,k)}{k}\leq C\sum_{k=1}^{5N}\frac{1}{k}\leq C\log N$$

by (15).

Now, we assume that $k_2-k_1>2d$ and $|a_l|<1/(2\sqrt{d})$. We assume also that $|a_j|=|\omega_i|_m$. Since $|a_j|\geq 1/\sqrt{d}$, we have $l\neq j$. Let $B(x,r)$ be the open ball of radius r centered at x. Let $p=(p_1,\cdots,p_d)$ be the center of R_i. We note that

$$R_i\subset\left\{b+t\omega_i\,\middle|\,b\in B\left(p,\frac{1}{2}\right),\ -\frac{N}{2}<t<\frac{N}{2}\right\}. \tag{16}$$

We shall examine the projections of $S_{l,k}$ and R_i to the (x_l, x_j)-plain in $\mathbf{R}^d$. In this proof we write $(0, \cdots, 0, x_l, 0, \cdots, 0, x_j, 0, \cdots, 0)$ as (x_l, x_j). $S'_{l,k}$ and R'_i will denote the projections to (x_l, x_j)-plain of $S_{l,k}$ and R_i, respectively. Without the loss of generality we may assume that $a_l, a_j > 0$. We see that R'_i is contained in the strip of width 1 by (16). Let R be the strip of width 1 which is parallel to (a_l, a_j) and whose left side meets the lower right corner of S_{l,k_1}. Then by a simple computation we see that the condition $R \cap S_{k_2} \neq \varnothing$ is equivalent to

$$\frac{2k_2+1}{2} \geq \frac{a_j}{a_l}(k_2-k_1-1) - \frac{2k_1+1}{2} - \frac{(a_l^2+a_j^2)^{1/2}}{a_l}.$$

Hence it follows from $a_j \geq 1/\sqrt{d}$ that

$$\frac{k_2-k_1-2d}{\sqrt{d}\,a_l} \leq k_1+k_2+1. \tag{17}$$

Using $|a_l| < 1/(2\sqrt{d})$ and (17), we first see that

$$2(k_2-k_1-2d) \leq k_1+k_2+1,$$

and hence

$$k_2 \leq 3k_1 + C.$$

Inserting this inequality to the right hand side of (17) we have

$$\frac{k_2-k_1-2d}{\sqrt{d}\,a_l} \leq k_1+k_2+1 \leq 4k_1+C$$

and hence

$$k_2-k_1-2d \leq \sqrt{d}\,(4k_1+C)a_l.$$

From these inequalities and (15) we obtain

$$\sum_{k=k_1}^{k_2} \frac{c_l(i,k)}{k} = \sum_{k=k_1}^{k_2-2d-1} \frac{c_l(i,k)}{k} + \sum_{k=k_2-2d}^{k_2} \frac{c_l(i,k)}{k}$$

$$\leq C\frac{1}{a_l}\left(\sum_{k=k_1}^{k_2-2d-1} \frac{1}{k}\right) + C \leq C\frac{1}{a_l} \cdot (k_2-k_1-2d) \cdot \frac{1}{k_1} + C \leq C.$$

Thus, we finish the proof of (14).

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