TOKYO J. MATH.
VOL. 22, NO. 2, 1999

Presheaves Associated to Modules over Subrings of Dedekind Domains

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(Communicated by K. Shinoda)

Introduction.

Let A be a commutative ring with unity. For a subset E of $\mathrm{Spec}A$, we put

(1) $$S_E = \bigcap_{\mathfrak{p} \in E} (A \backslash \mathfrak{p}) \quad (S_\phi = A).$$

Then S_E is a saturated multiplicatively closed set.

To an A-module M, we associate a presheaf $\bar{M}$ in the following way. By putting

(2) $$\bar{M}(U) = S_U^{-1} M$$

for an open subset U of $\mathrm{Spec}A$, we define a presheaf $\bar{M}$ of modules on $\mathrm{Spec}A$. Here $\bar{M}$ is not a sheaf in general. But the sheafification of $\bar{M}$ turns out to be the quasi-coherent $\tilde{A}$-module $\tilde{M}$. Then we ask the question: When is the presheaf $\bar{M}$ actually a sheaf?

Noting that $\bar{M}$ is a sheaf if and only if $\bar{M} = \tilde{M}$, we introduce the following three conditions for a ring A:

(S.1) $\bar{M} = \tilde{M}$ for any A-module M.

(S.2) $\bar{\mathfrak{a}} = \tilde{\mathfrak{a}}$ for any ideal $\mathfrak{a}$ of A.

(S.3) $\bar{A} = \tilde{A}$.

In the previous paper, the following facts are shown (see [5]):

FACT 1. *Suppose that A is a valuation ring. Then*
(i) *A satisfies the condition* (S.3).
(ii) (S.1) $\Leftrightarrow$ (S.2) $\Leftrightarrow$ $\mathrm{Spec}A$ *is a noetherian topological space.*

FACT 2. *Let A be a Dedekind domain. Then*

(S.1) $\Leftrightarrow$ (S.2) $\Leftrightarrow$ (S.3) $\Leftrightarrow$ *the ideal class group of A is torsion.*

FACT 3. *Suppose that A is a unique factorization domain. Then*
(i) *A satisfies the condition* (S.3).

Received November 13, 1997

(ii) (S.1)$\Leftrightarrow$(S.2)$\Leftrightarrow$*A is a principal ideal domain.*

Next we introduce the topological conditions (T.1), (T.2) and (T.3).
For a ring A, we put

$$\Sigma=\{D(f) \mid f\in A\},$$
$$\Sigma_1=\{D(\mathfrak{a}_\alpha) \mid \alpha\in QA\}\cup\{\phi\}.$$

Here $D(\mathfrak{a})=\{\mathfrak{p}\in\operatorname{Spec}A \mid \mathfrak{a}\not\subset\mathfrak{p}\}$, QA is the total quotient ring of A and $\mathfrak{a}_\alpha=\{b\in A \mid b\alpha\in A\}$. Moreover for any subset E of $\operatorname{Spec}A$, we put

$$\tilde{E}=\bigcap_{\substack{U\in\Sigma\\ U\supset E}} U=\left\{\mathfrak{p}\in\operatorname{Spec}A \mid \mathfrak{p}\subset\bigcup_{\mathfrak{p}'\in E}\mathfrak{p}'\right\}, \tag{3}$$

$$\tilde{E}^1=\bigcap_{\substack{V\in\Sigma_1\\ V\supset E}} V. \tag{4}$$

Then we introduce the following conditions for the topology of $\operatorname{Spec}A$.

(T.1) For any open subset U of $\operatorname{Spec}A$, there exists $f\in A$ such that $U=D(f)$.

(T.2) For any open subset U of $\operatorname{Spec}A$, $U=\tilde{U}$.

(T.3) For any open subset U of $\operatorname{Spec}A$, $\tilde{U}^1=\tilde{U}$.

The main results of this paper are as follows.

THEOREM 1. *For an integral ring A, we obtain*

$$\begin{array}{ccccc} (S.1) & \Leftrightarrow & (S.2) & \Rightarrow & (S.3)\\ \Updownarrow & & \Updownarrow & & \Updownarrow\\ (T.1) & \Leftrightarrow & (T.2) & \Rightarrow & (T.3). \end{array}$$

THEOREM 2. *Let A be a ring consisting of algebraic integers with quadratic quotient field. Then A satisfies the condition* (S.1).

THEOREM 3. *Let k be a field of characteristic $p\geqq 0$, $s(t)$ a monic polynomial of $k[t]$ where $\deg s\geqq 2$, and $A=k\oplus s(t)k[t]$. If $s(t)=\prod_{i=1}^{m}(t-\alpha_i)^{e_i}$ is the irreducible polynomial decomposition in $\bar{k}[t]$ where $\bar{k}$ is an algebraic closure of k, then*

$$\begin{cases} (S.1)\Longleftrightarrow(S.2)\Longleftrightarrow(S.3)\Longleftrightarrow p\neq 0 & \text{if } m=1,\\ (S.1)\Longleftrightarrow(S.2)\Longleftrightarrow(S.3)\Longleftrightarrow p\neq 0 \text{ and } k \text{ is algebraic over } \mathbf{F}_p & \text{if } m\geqq 2. \end{cases}$$

The authors wish to express their thanks to Professor Hideo Wada and Doctor Koji Sekiguchi for their advices and warm encouragement.

1. In this section we shall prove Theorem 1.

LEMMA 1. *Let A be a ring and $\mathfrak{a}$ an ideal of A. Then*

(i) $f \notin \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p} \Longleftrightarrow \mathfrak{a} \subset \sqrt{(f)}$, *for any element* f *of* A.

(ii) *The following four conditions are equivalent*:
 (a) $\mathfrak{a} \not\subset \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p}$.
 (b) *There exists* $f \in A$ *such that* $\sqrt{\mathfrak{a}} = \sqrt{(f)}$.
 (c) *If* $\mathfrak{a} \subset \bigcup_{\mathfrak{p} \in E} \mathfrak{p}$, *then there exists* $\mathfrak{p} \in E$ *such that* $\mathfrak{a} \subset \mathfrak{p}$ *for any subset* E *of* Spec$A$.
 (d) *If* $\mathfrak{a} \subset \bigcup_{\mathfrak{p} \in U} \mathfrak{p}$, *then there exists* $\mathfrak{p} \in U$ *such that* $\mathfrak{a} \subset \mathfrak{p}$ *for any open subset* U *of* Spec$A$.

PROOF. (i)
$$\begin{aligned} f \notin \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p} &\Longleftrightarrow f \notin \mathfrak{p} \quad \text{for any} \quad \mathfrak{p} \in D(\mathfrak{a}) \\ &\Longleftrightarrow \mathfrak{p} \in D(f) \quad \text{for any} \quad \mathfrak{p} \in D(\mathfrak{a}) \\ &\Longleftrightarrow D(\mathfrak{a}) \subset D(f) \\ &\Longleftrightarrow \sqrt{\mathfrak{a}} \subset \sqrt{(f)} \\ &\Longleftrightarrow \mathfrak{a} \subset \sqrt{(f)}\,. \end{aligned}$$

(ii)(a) $\Leftrightarrow$ (b):
$$\begin{aligned} \mathfrak{a} \not\subset \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p} &\Longleftrightarrow \text{there exists } f \in A \text{ such that } f \in \mathfrak{a},\ f \notin \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p} \\ &\Longleftrightarrow \text{there exists } f \in A \text{ such that } f \in \mathfrak{a},\ \mathfrak{a} \subset \sqrt{(f)} \\ &\Longleftrightarrow \text{there exists } f \in A \text{ such that } \sqrt{\mathfrak{a}} = \sqrt{(f)}\,. \end{aligned}$$

(a) $\Leftrightarrow$ (c):
$$\begin{aligned} \mathfrak{a} \not\subset \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p} &\Longleftrightarrow \text{if } E \subset D(\mathfrak{a}), \text{ then } \mathfrak{a} \not\subset \bigcup_{\mathfrak{p} \in E} \mathfrak{p} \text{ for any subset } E \text{ of Spec} A \\ &\Longleftrightarrow \text{if } \mathfrak{a} \subset \bigcup_{\mathfrak{p} \in E} \mathfrak{p}, \text{ then } E \not\subset D(\mathfrak{a}) \text{ for any subset } E \text{ of Spec} A \\ &\Longleftrightarrow \text{if } \mathfrak{a} \subset \bigcup_{\mathfrak{p} \in E} \mathfrak{p}, \text{ then there exists } \mathfrak{p} \in E \text{ such that } \mathfrak{a} \subset \mathfrak{p} \\ &\qquad \text{for any subset } E \text{ of Spec} A\,. \end{aligned}$$

(a) $\Leftrightarrow$ (d): Since D($\mathfrak{a}$) is an open set, the proof is clear. Q.E.D.

For a ring A, we introduce the following conditions (I.1), (I.2), (I.1)′ and (I.2)′.

(I.1) For any ideal $\mathfrak{a}$ of A, there exists $f \in A$ such that $\sqrt{\mathfrak{a}} = \sqrt{(f)}$.

(I.2) For any $\mathfrak{p} \in \mathrm{Spec}A$, there exists $f \in A$ such that $\mathfrak{p} = \sqrt{(f)}$.

(I.1)′ For any ideal $\mathfrak{a}$ of A and any subset E of Spec$A$, if $\mathfrak{a} \subset \bigcup_{\mathfrak{p}' \in E} \mathfrak{p}'$, then there exists $\mathfrak{p}' \in E$ such that $\mathfrak{a} \subset \mathfrak{p}'$.

(I.2)′ For any $\mathfrak{p} \in \operatorname{Spec} A$ and any subset E of $\operatorname{Spec} A$, if $\mathfrak{p} \subset \bigcup_{\mathfrak{p}' \in E} \mathfrak{p}'$, then there exists $\mathfrak{p}' \in E$ such that $\mathfrak{p} \subset \mathfrak{p}'$.

PROPOSITION 1. *For a ring A, the conditions* (S.1), (S.2), (T.1), (T.2), (I.1), (I.2), (I.1)′ *and* (I.2)′ *are all equivalent.*

PROOF. By Lemma 1, we have that (T.1)$\Leftrightarrow$(I.1)$\Leftrightarrow$(I.1)′ and (T.2)$\Leftrightarrow$(I.2)$\Leftrightarrow$(I.2)′.

Next we shall prove that (I.1)′$\Leftrightarrow$(I.2)′. It is sufficient to prove that (I.2)′$\Rightarrow$(I.1)′. Since $\mathfrak{a} \cap S_E = \phi$, there exists $\mathfrak{p} \in \operatorname{Spec} \bar{A}(E)$ such that $\mathfrak{a} \subset \mathfrak{p} \subset \bigcup_{\mathfrak{p}' \in E} \mathfrak{p}'$, By assumption, there exists $\mathfrak{p}' \in E$ such that $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{p}'$.

Finally we shall prove that (T.1)$\Leftrightarrow$(S.1)$\Leftrightarrow$(S.2). It is sufficient to prove that (S.2)$\Rightarrow$(I.1). Therefore, we shall prove that if A does not satisfy the condition (I.1), then A does not satisfy the condition (S.2). We can assume that A satisfies the condition (S.3). By Lemma 1, A does not satisfy the condition (I.1) if and only if there exists an ideal $\mathfrak{a}$ of A such that $\mathfrak{a} \subset \bigcup_{\mathfrak{p} \in D(\mathfrak{a})} \mathfrak{p}$. Here we fix such an ideal $\mathfrak{a}$ of A and we put $U = D(\mathfrak{a})$. Then we shall prove $\bar{\mathfrak{a}}(U) \neq \tilde{\mathfrak{a}}(U)$. From $\mathfrak{a} \subset \bigcup_{\mathfrak{p} \in U} \mathfrak{p}$, we obtain $\bar{\mathfrak{a}}(U) \subsetneqq \bar{A}(U)$. On the other hand we have $\mathfrak{a}_{\mathfrak{p}} = A_{\mathfrak{p}}$ for any $\mathfrak{p} \in U$. Since $\tilde{\mathfrak{a}}$ is a sheaf, we obtain $\tilde{\mathfrak{a}}(U) = \tilde{A}(U)$. From $\bar{A}(U) = \tilde{A}(U)$, we have $\bar{\mathfrak{a}}(U) \neq \tilde{\mathfrak{a}}(U)$. Therefore A does not satisfy the condition (S.2). Q.E.D.

Then the proof of Theorem 1 is easy from Proposition 1 and [5], Lemma 8.

EXAMPLE 1. Let k be a field and $t_1, t_2, \cdots$ indeterminates over k. We put $A_0 = k$, $A_i = A_{i-1} + t_i(QA_{i-1})[[t_i]]$ $(i \geqq 1)$, and $A = \bigcup_{i=0}^{\infty} A_i$. Then A is a valuation ring of infinite dimension and every non-zero prime is of finite depth, so $\operatorname{Spec} A$ is a noetherian topological space. By Fact 1, A satisfies the condition (S.1).

The following lemma is needed in section 3.

LEMMA 2. *Let A_1 and A_2 be integral rings such that* $\dim A_1 \leqq 1$, $A_1 \subset A_2$ *and* $\operatorname{Spec} A_2 \to \operatorname{Spec} A_1$ *is injective. If A_1 satisfies the condition* (S.1), *then A_2 satisfies the condition* (S.1).

PROOF. For any $\mathfrak{P} \in \operatorname{Spec} A_2$, we put $\mathfrak{p} = \mathfrak{P} \cap A_1 \in \operatorname{Spec} A_1$. Since A_1 satisfies the condition (I.2), there exists $f \in A_1$ such that $\mathfrak{p} = \sqrt{(f)}$ in A_1. Since $\operatorname{Spec} A_2 \to \operatorname{Spec} A_1$ is injective and $\dim A_1 \leqq 1$, $\mathfrak{P} = \sqrt{(f)}$ in A_2. Therefore A_2 satisfies the condition (I.2), and hence (S.1). Q.E.D.

2. In this section we shall prove Theorem 2.

LEMMA 3. *Let K be an algebraic number field, B the ring of integers of K and A a subring of B with quotient field K.*

(i) *Then A is of finite index $n = (B : A)$.*

(ii) *If p is a prime number which dose not divide the index $n = (B : A)$ and $\mathfrak{p}$ is a prime ideal of A which contains p, then there exists $f \in A$ such that $\mathfrak{p} = \sqrt{(f)}$.*

PROOF. (i) is well-known.

(ii) Take any $\mathfrak{P} \in \operatorname{Spec} B$ which contains $\mathfrak{p}$. Then there exists a positive integer h such that $\mathfrak{P}^h = (g)$ in B. Since p dose not divide n, g is a unit in $B/(n)$. Then there exists a positive integer l such that $g^l - 1 \in (n)$ in B. Therefore $f = g^l \in A$ and $\mathfrak{p} = \sqrt{(f)}$.

Q.E.D.

PROOF OF THEOREM 2. We shall prove that A satisfies the condition (I.2). Let $(p) = \mathbf{Z} \cap \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} A$. By Lemma 3, we can assume p divides n. If the prime ideal (p) dose not split in B, where B is the ring of integers of QA, then $\mathfrak{p}$ is a unique prime ideal of A which contains (p). Therefore $\mathfrak{p} = \sqrt{(p)}$.

Next we assume (p) splits in B. Then $(p) = (p, \alpha)(p, \alpha')$ in B, where $\mathrm{N}(\alpha) = \alpha\alpha' = pm$ and $(p, m) = 1$. There exists an intermediate module M of B/A such that $(B : M) = p$. Then we shall prove $(p, \alpha) \cap M = (p, \alpha') \cap M$. Since (p) splits in B, we have $\alpha \notin M$. Let $(p, \alpha) \cap M \ni y = ap + b\alpha$. Then $p | b$. If put $b = pb_1$, then

$$my = amp + b\alpha m = amp + b_1\alpha\alpha'\alpha$$
$$= amp + b_1(c\alpha + d)\alpha' = (am + b_1cm)p + b_1d\alpha' ,$$

where $\alpha^2 = c\alpha + d$. Since $(p, m) = 1$, $y \in (p, \alpha') \cap M$. Therefore $(p, \alpha) \cap A = (p, \alpha') \cap A$ and $\mathfrak{p} = \sqrt{(p)}$.

Q.E.D.

EXAMPLE 2. Let $m_1, \cdots, m_s$ be square free integers such that $(m_i, m_j) = 1$ for any $i \neq j$. Then $A = \mathbf{Z}[\sqrt{m_1}, \cdots, \sqrt{m_s}]$ satisfies the condition (S.1).

PROOF. We shall prove that A satisfies the condition (I.2). First we shall compute the index $(B : A)$, where B is the ring of integers of QA. Let p be an odd prime which divides $m_1 m_2 \cdots m_s$. Since the 2^s elements $1, \sqrt{m_1}, \cdots, \sqrt{m_1 m_2}, \cdots, \sqrt{m_1 m_2 \cdots m_s}$, we put $\alpha_1, \cdots, \alpha_{2^s}$, form a $\mathbf{Z}$-basis of A, the p-part of the discriminant of A is $p^{2^{s-1}}$. On the other hand, the group of *Dirichlet* characters associated to $K = \mathbf{Q}(\sqrt{m_1}, \cdots, \sqrt{m_s})$ is generated by $\{\chi_{m_1}, \cdots, \chi_{m_s}\}$, where χ_{m_i} is a quadratic character with conductor m_i or $4m_i$. Therefore the conductor-discriminant formula says the p-part of discriminant of K and that of A coincide, and hence $(B : A) = \sqrt{d(\alpha_1, \cdots, \alpha_{2^s})/d_K}$ is a power of 2.

Therefore $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathfrak{p} \not\ni 2$ imply $\mathfrak{p} = \sqrt{(f)}$ for some $f \in A$ by Lemma 3. Since $\sqrt{m_i}$ and their conjugates are congruent modulo 2, A has only one prime ideal $\mathfrak{p}_2$ which contains 2. Then $\mathfrak{p}_2 = \sqrt{(2)}$.

Q.E.D.

3. In this section we shall prove Theorem 3 and consider affine coordinate rings of singular rational curves.

Let k be a field of characteristic $p \geqq 0$ and $k[t]$ a polynomial ring with variable t. For any non constant polynomial $s(t)$ of $k[t]$, we put

$$A = k \oplus s(t)k[t] \subset k[t] . \tag{5}$$

Then A is a subring of $k[t]$ and $k[t]$ is integral over A. Therefore $\operatorname{Spec} k[t] \to \operatorname{Spec} A$ is surjective. Moreover for any polynomial $f(t)$ of $k[t]$, we put

(6) $$\mathfrak{I}_f = f(t)k[t] \cap A .$$

Then $\mathfrak{I}_f$ is an ideal of A.

LEMMA 4. *Let $f_1(t)$ and $f_2(t)$ be irreducible monic polynomials of $k[t]$. Then*
(i) $f_1(t) \mid s(t) \Rightarrow \mathfrak{I}_{f_1} = \mathfrak{I}_s$.
(ii) $f_1(t) \nmid s(t), f_1(t) \neq f_2(t) \Rightarrow \mathfrak{I}_{f_1} \neq \mathfrak{I}_{f_2}$.

PROOF. (i) It is sufficient to prove $\mathfrak{I}_{f_1} \subset \mathfrak{I}_s$. For any $g(t) \in \mathfrak{I}_{f_1}$, we put $g(t) = c + s(t)g_1(t)$. Then $c = 0$ from $f_1(t) \mid g(t)$. Therefore $g(t) = s(t)g_1(t) \in \mathfrak{I}_s$.

(ii) If we assume that $\mathfrak{I}_{f_1} = \mathfrak{I}_{f_2}$, then $s(t)f_2(t) \in (f_1(t))$ in $k[t]$. Since $k[t]$ is a principal ideal domain and $f_1(t) \nmid s(t)$, we have $f_1(t) = f_2(t)$. This is a contradiction.

Q.E.D.

COROLLARY. *Let $s(t) = s_1(t) \cdots s_n(t)$ be the irreducible polynomial decomposition in $k[t]$. Then $\mathfrak{I}_{s_1} = \cdots = \mathfrak{I}_{s_n} = \mathfrak{I}_s$. Moreover, the mapping*

$$\operatorname{Spec} k[t] \setminus \{(s_1), \cdots, (s_n)\} \longrightarrow \operatorname{Spec} A \setminus \{\mathfrak{I}_s\}$$

is bijective.

The following two lemmas are easy to prove from Corollary of Lemma 4.

LEMMA 5. *Let U be an open set of* Spec A. *If $U \not\ni \mathfrak{I}_s$, then there exists $g(t) \in A$ such that $U = \mathrm{D}(g)$.*

LEMMA 6. *Let $f_1(t), \cdots, f_m(t)$ be irreducible polynomials of $k[t]$ such that $f_i(t) \nmid s(t)$ $(1 \leqq i \leqq m)$ and $U = \operatorname{Spec} A \setminus \{\mathfrak{I}_{f_1}, \cdots, \mathfrak{I}_{f_m}\}$. Then there exists $g(t) \in A$ such that $U = \mathrm{D}(g)$ if and only if there exist positive integers $l_1, \cdots, l_m$ such that $\prod_{i=1}^{m} f_i(t)^{l_i} \in A$.*

LEMMA 7. *Let U be an open set of* Spec A. *Then we obtain $\tilde{U}^1 = U$.*

PROOF. For any irreducible polynomial $f(t)$ of $k[t]$, we put $U_f = \operatorname{Spec} A \setminus \{\mathfrak{I}_f\}$. It is sufficient to prove $\tilde{U}_f^1 = U_f$. If $f(t) \mid s(t)$, then we obtain $U_f = \mathrm{D}(s) = \mathrm{D}(\mathfrak{a}_{\frac{1}{s}})$; otherwise since $\mathfrak{I}_f \subset \mathfrak{a}_{\frac{s}{f}} \subsetneqq A$ and $\dim A = 1$, $\mathfrak{I}_f = \mathfrak{a}_{\frac{s}{f}}$. Therefore $\tilde{U}_f^1 = U_f$. Q.E.D.

COROLLARY. *For a ring $A = k \oplus s(t)k[t]$, all the conditions* (S.1), (S.2) *and* (S.3) *are equivalent.*

PROOF OF THEOREM 3. From Proposition 1, Lemmas 5, 6 and Corollary of Lemma 7, A satisfies the conditions (S.1), (S.2), (S.3) $\Leftrightarrow$ for any irreducible polynomial $f(t) \in k[t]$ such that $f(t) \nmid s(t)$, there exists a positive integer n such that $f(t)^n \in A$.

First we suppose that $m = 1$ and put $s(t) = (t - \alpha_1)^{e_1}$, $e_1 \geqq 2$. By the statement, we consider only two cases for the characteristic of k:

(i) $p = 0$. We put $f(t) = t - c$ for some $c \in k \setminus \{\alpha_1\}$. Since $(f(t)^n)' = n(t-c)^{n-1}$, we have $(f^n)'(\alpha_1) \neq 0$ for any positive integer n. Hence $f(t)^n \notin A$ for any positive integer n.

Therefore A does not satisfy the condition (S.3).

(ii) $p\neq 0$. Let $e_1=p^a e$, where $(p, e)=1$. Then $\alpha_1^{p^a}\in k$ because

$$s(t)=(t-\alpha_1)^{p^a e}=(t^{p^a}-\alpha_1^{p^a})^e=t^{e_1}-e\alpha_1^{p^a}t^{p^a(e-1)}+\cdots+(-\alpha_1)^{e_1}\in k[t]\,.$$

Here for any polynomial $f(t)\in k[t]$, we put $f(t)=f(\alpha_1)+(t-\alpha_1)g(t)$ where $g(t)\in\bar{k}[t]$. Then $f(t)^{p^n}=f(\alpha_1)^{p^n}+(t-\alpha_1)^{p^n}g(t)^{p^n}$ for any positive integer n. Since $f(\alpha_1)^{p^n}\in k$ for any $n\geqq a$, if $p^n\geqq e_1$, then $s(t)\mid(f(t)^{p^n}-f(\alpha_1)^{p^n})$ in $k[t]$ and then $f(t)^{p^n}\in A$. Therefore A satisfies the condition (S.1).

Next we suppose that $m\geqq 2$. For any irreducible polynomial $f(t)\in k[t]$ such that $f(t)\nmid s(t)$,

$$f(t)^n\in A \Longrightarrow f(\alpha_1)^n=f(\alpha_2)^n \Longrightarrow \frac{f(\alpha_1)^n}{f(\alpha_2)^n}=1 \Longrightarrow \frac{f(\alpha_1)}{f(\alpha_2)} \text{ is a root of unity}\,.$$

Hence if $f(\alpha_1)/f(\alpha_2)$ is not a root of unity, then $f(t)^n\notin A$ for any positive integer n. Therefore, if there exists $c\in k$ such that $s(c)\neq 0$ and $(\alpha_1-c)/(\alpha_2-c)$ is not a root of unity, then A does not satisfy the condition (S.3). Here by the statement, we consider only three cases for the characteristic and the transcendental degree of k:

(i) $p=0$. Since the mapping $\mathbf{Q}\setminus\{\alpha_1,\cdots,\alpha_m\}\to\mathbf{Q}(\alpha_1,\alpha_2)$ defined by $c\mapsto(\alpha_1-c)/(\alpha_2-c)$ is injective and the set of root of unity in $\mathbf{Q}(\alpha_1,\alpha_2)$ is finite, there exists $c\in\mathbf{Q}$ such that $s(c)\neq 0$ and $(\alpha_1-c)/(\alpha_2-c)$ is not a root of unity. Therefore A does not satisfy the condition (S.3).

(ii) $p\neq 0$ and $\mathrm{tr.deg}_{\mathbf{F}_p}k=0$. For any irreducible polynomial $f(t)\in k[t]$ such that $f(t)\nmid s(t)$, there exist a positive integer a such that $f(\alpha_1)^a=\cdots=f(\alpha_m)^a=1$. Then $f(t)^a=1+(t-\alpha_1)\cdots(t-\alpha_m)g(t)$ where $g(t)\in\bar{k}[t]$. Here we put $e=\max(e_1,\cdots,e_m)$. Then $f(t)^{ap^n}=1+(t-\alpha_1)^{p^n}\cdots(t-\alpha_m)^{p^n}g(t)^{p^n}$ for any positive integer n. Therefore if $p^n\geqq e$, then $s(t)\mid(f(t)^{ap^n}-1)$ in $k[t]$. Hence if $p^n\geqq e$, then $f(t)^{ap^n}\in A$. Therefore A satisfies the condition (S.1).

(iii) $p\neq 0$ and $\mathrm{tr.deg}_{\mathbf{F}_p}k\geqq 1$. If both α_1 and α_2 are algebraic elements over $\mathbf{F}_p$, then $(\alpha_1-c)/(\alpha_2-c)$ is a transcendental element for any transcendental element c. If α_1 is an algebraic element and α_2 a transcendental element, then $(\alpha_1-c)/(\alpha_2-c)$ is a transcendental element for any algebraic element $c\neq\alpha_1$. If both α_1 and α_2 are transcendental elements, then α_1/α_2 or $(\alpha_1-1)/(\alpha_2-1)$ is a transcendental element. Therefore A does not satisfy the condition (S.3). Q.E.D.

LEMMA 8. *Let $A_1=k\oplus s_1(t)k[t]$ and $A_2=k\oplus s_2(t)k[t]$, where $s_1(t)$ and $s_2(t)$ are polynomials of $k[t]$ such that $\deg s_1$, $\deg s_2\geqq 2$. Then*

(i) $A_1\subset A_2\Leftrightarrow s_2(t)\mid s_1(t)$.

(ii) *Suppose that $A_1\subset A_2$. Then* $\mathrm{Spec}\,A_2\to\mathrm{Spec}\,A_1$ *is injective if and only if* $\sqrt{(s_1(t))}=\sqrt{(s_2(t))}$ *in* $k[t]$.

(iii) *Let A be an intermediate ring of $k[t]/A_1$ such that* $\mathrm{Spec}\,A\to\mathrm{Spec}\,A_1$ *is injective. Then $\tilde{U}^1=U$ for any open set U of* $\mathrm{Spec}\,A$.

PROOF. (i) We put $s_1(t)=c+s_2(t)g(t)$. Then $s_1(t)t=ct+s_2(t)g(t)t\in A_2$. Hence $ct\in A_2$. Then $c=0$ from $\deg s_2\geqq 2$. Therefore $s_2(t)\mid s_1(t)$.

(ii) The proof is easy from Corollary of Lemma 4.

(iii) The proof is similar to that of Lemma 7. Q.E.D.

COROLLARY. *For an intermediate ring A of $k[t]/A_1$ such that* $\operatorname{Spec} A\to\operatorname{Spec} A_1$ *is injective, all the conditions* (S.1), (S.2) *and* (S.3) *are equivalent.*

In particular, we suppose that $s_2(t)\mid s_1(t)$ and $\sqrt{(s_1(t))}=\sqrt{(s_2(t))}$ in $k[t]$. Then for an intermediate ring A of A_2/A_1, all the conditions (S.1), (S.2) *and* (S.3) *are equivalent.*

PROPOSITION 2. *Let A_1 and A_2 be as in Lemma 8 such that $s_2(t)\mid s_1(t)$ and $\sqrt{(s_1(t))}=\sqrt{(s_2(t))}$ in $k[t]$, and A an intermediate ring of A_2/A_1. Then*

$$A \text{ satisfies the condition (S.1)} \Longleftrightarrow A_1 \text{ satisfies the condition (S.1)}$$
$$\Longleftrightarrow A_2 \text{ satisfies the condition (S.1)}.$$

The proof is easy from Theorem 3, Lemma 2, and Corollary of Lemma 8.

LEMMA 9. *For any monotone increasing sequence of natural number $\{a_j\}_{j\geqq 0}$ such that $a_0=0$, we put $A=\bigoplus_{j=0}^{\infty}kt^{a_j}\subset k[t]$ and $G=\{a_j\mid j\geqq 0\}\subset\mathbf{N}$. Then*

(a) *A is a ring if and only if G is an additively closed set.*

(b) *Suppose that A is a ring with characteristic* 0. *For any irreducible polynomials $f_1(t),\cdots,f_m(t)$ of $k[t]$ such that $f_i(t)\neq t$ and any positive integers $l_1,\cdots,l_m$, we put $f(t)=\prod_{i=1}^{m}f_i(t)^{l_i}$ and $h(t)=\sum_{i=1}^{m}l_if_i'(t)/f_i(t)$. Then*

$$f(t)\in A \Longleftrightarrow f^{(n)}(0)=0 \quad \text{for any} \quad n\in\mathbf{N}\setminus G$$
$$\Longleftrightarrow h^{(n-1)}(0)=0 \quad \text{for any} \quad n\in\mathbf{N}\setminus G,$$

where $f^{(n)}(t)$ is the n-th derivative of $f(t)$.

(c) *If A is a ring and $\mathbf{N}\setminus G$ is a non-empty finite set, then*

$$(\text{S.1}) \Longleftrightarrow (\text{S.2}) \Longleftrightarrow (\text{S.3}) \Longleftrightarrow p\neq 0.$$

Moreover we shall determine open sets U of $\operatorname{Spec} A$ *such that $\bar{A}(U)=\tilde{A}(U)$.*

(I) *$U=\operatorname{Spec} A$ or $U=\phi\Rightarrow\bar{A}(U)=\tilde{A}(U)$.*

(II) *$U\notin\mathfrak{I}_t\Rightarrow\bar{A}(U)=\tilde{A}(U)$.*

(III) *For any irreducible polynomials $f_1(t),\cdots,f_m(t)$ of $k[t]$ such that $f_i(t)\neq t$ $(1\leqq i\leqq m)$, we put $U=\operatorname{Spec} A\setminus\{\mathfrak{I}_{f_1},\cdots,\mathfrak{I}_{f_m}\}$ where $\mathfrak{I}_{f_i}=f_i(t)k[t]\cap A$. Then $\bar{A}(U)=\tilde{A}(U)\Leftrightarrow$*

$$\exists l_1,\cdots,l_m\geqq 1 \quad \text{such that} \quad h^{(n-1)}(0)=0 \quad \text{for any} \quad n\in\mathbf{N}\setminus G. \tag{7}$$

In particular, if $f_i(t)=t-\gamma_i$, then the condition (7) *can be replaced by the following condition* (8);

(8) $$\exists l_1, \cdots, l_m \geqq 1 \quad \textit{such that} \quad \sum_{i=1}^{m} \frac{l_i}{\gamma_i^n} = 0 \qquad \textit{for any} \quad n \in \mathbf{N} \setminus G .$$

PROOF. (a) The proof is easy.

(b) It is sufficient to prove that $f^{(n)}(0)=0$ for any $n \in \mathbf{N} \setminus G \Leftrightarrow h^{(n-1)}(0)=0$ for any $n \in \mathbf{N} \setminus G$.

First we shall prove "only if part". Since $f^{(1)}(t)=h(t)f(t)$ and $f(0) \neq 0$, $f^{(1)}(0)=0 \Leftrightarrow h(0)=0$. Here for any $n' \leqq n$ such that $n' \in \mathbf{N} \setminus G$, we assume that $h^{(n'-1)}(0)=0$. If $n+1 \in \mathbf{N} \setminus G$, then $r+1 \in \mathbf{N} \setminus G$ or $n-r \in \mathbf{N} \setminus G$ for $0 \leqq {}^\forall r \leqq n$. Therefore $h^{(r)}(0)=0$ or $f^{(n-r)}(0)=0$ for $0 \leqq {}^\forall r < n$. Since $f^{(n+1)}(t)=\sum_{r=0}^{n} {}_nC_r h^{(r)}(t) f^{(n-r)}(t)$, $h^{(n)}(0)=0$.

The proof of "if part" is similar to that of "only if part".

(c) (S.1) $\Leftrightarrow$ (S.2) $\Leftrightarrow$ (S.3) $\Leftrightarrow p \neq 0$: Obvious from Proposition 2.

Next we shall determine open sets U of $\operatorname{Spec} A$ such that $\bar{A}(U)=\tilde{A}(U)$. It is sufficient to consider that $p=0$ and $U \neq \phi$. By Lemma 8 and [5], Lemma 5, $\bar{A}(U)=\tilde{A}(U)$ if and only if $U=\tilde{U}$. Since $\tilde{U}^C$ is a finite set, there exists $f \in A$ such that $\tilde{U}=\mathrm{D}(f)$. Therefore, $\bar{A}(U)=\tilde{A}(U)$ if and only if there exists $f \in A$ such that $U=\mathrm{D}(f)$. From (b), we obtain (I), (II) and (III). Q.E.D.

Next we consider the affine coordinate ring of singular rational curves as applications of Theorem 3 and Lemma 9.

EXAMPLE 3. Let $A=k[x, y]/(y^2+axy-bx^2-x^3)$, where $a, b \in k$, then A is the affine coordinate ring of the singular Weierstrass curve $C: y^2+axy=x^3+bx^2$.

From the k-algebra homomorphism $\varphi: k[x, y] \to k[t]$ defined by $x \mapsto t^2+at-b$, $y \mapsto t^3+at^2-bt$, we obtain $A \cong k \oplus s(t)k[t]$, where $s(t)=t^2+at-b$. Hence we can apply Theorem 3. Therefore,

$$\begin{cases} (\mathrm{S}.1) \Longleftrightarrow (\mathrm{S}.2) \Longleftrightarrow (\mathrm{S}.3) \Longleftrightarrow p \neq 0 & \text{if the singular point is a cusp ,} \\ (\mathrm{S}.1) \Longleftrightarrow (\mathrm{S}.2) \Longleftrightarrow (\mathrm{S}.3) & \text{if the singular point is a node .} \\ \qquad \Longleftrightarrow p \neq 0 \text{ and } k \text{ is algebraic over } \mathbf{F}_p & \end{cases}$$

Moreover we shall determine open sets U of $\operatorname{Spec} A$ such that $\bar{A}(U)=\tilde{A}(U)$. Let $s(t)=(t-\alpha_1)(t-\alpha_2)$ be the irreducible polynomial decomposition in $\bar{k}[t]$.

(I) $U=\operatorname{Spec} A$ or $U=\phi \Rightarrow \bar{A}(U)=\tilde{A}(U)$.

(II) $U \notin \mathfrak{I}_s \Rightarrow \bar{A}(U)=\tilde{A}(U)$.

(III) For any irreducible polynomials $f_1(t), \cdots, f_m(t)$ of $k[t]$ such that $f_i(t) \nmid s(t)$ $(1 \leqq i \leqq m)$, we put $U=\operatorname{Spec} A \setminus \{\mathfrak{I}_{f_1}, \cdots, \mathfrak{I}_{f_m}\}$. Then $\bar{A}(U)=\tilde{A}(U) \Leftrightarrow$

(9) $$\begin{cases} \exists l_1, \cdots, l_m \geqq 1 \text{ such that } \prod_{i=1}^{m} f_i(\alpha_1)^{l_i} \in k,\ \prod_{i=1}^{m} l_i f_i'(\alpha_1)/f_i(\alpha_1)=0 & \text{if } \alpha_1=\alpha_2 , \\ \exists l_1, \cdots, l_m \geqq 1 \text{ such that } \prod_{i=1}^{m} f_i(\alpha_1)^{l_i}=\prod_{i=1}^{m} f_i(\alpha_2)^{l_i} \in k & \text{if } \alpha_1 \neq \alpha_2 . \end{cases}$$

In particular, if k is an algebraically closed field, then by putting $f_i(t)=t-\gamma_i$, the condition (9) can be replaced by the following condition (10);

$$\text{(10)}\quad \begin{cases} {}^{\exists}l_1, \cdots, l_m \geqq 1 \text{ such that } \sum_{i=1}^{m} l_i/(\alpha_1-\gamma_i)=0 & \text{if } \alpha_1=\alpha_2\,, \\ {}^{\exists}l_1, \cdots, l_m \geqq 1 \text{ such that } \prod_{i=1}^{m} ((\alpha_1-\gamma_i)/(\alpha_2-\gamma_i))^{l_i}=1 & \text{if } \alpha_1 \neq \alpha_2\,. \end{cases}$$

EXAMPLE 4. Let $A=k[x, y, z]/(x^3-yz, x^2y-z^2, y^2-xz)$, then A is the affine coordinate ring of the singular rational curve C: $x^3-yz=0$, $x^2y-z^2=0$, $y^2-xz=0$.

From the k-algebra homomorphism $\varphi: k[x, y, z] \to k[t]$ defined by $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$, we obtain $A \cong k \oplus t^3k[t]$. Hence we can apply Lemma 9. Therefore,

$$(\text{S.1}) \Longleftrightarrow (\text{S.2}) \Longleftrightarrow (\text{S.3}) \Longleftrightarrow p \neq 0\,.$$

Moreover we shall determine open sets U of Spec$A$ such that $\bar{A}(U)=\tilde{A}(U)$.

(I) $U=\operatorname{Spec}A$ or $U=\phi \Rightarrow \bar{A}(U)=\tilde{A}(U)$.

(II) $U \notin \mathfrak{I}_t \Rightarrow \bar{A}(U)=\tilde{A}(U)$.

(III) For any irreducible polynomials $f_1(t), \cdots, f_m(t)$ of $k[t]$ such that $f_i(t) \neq t$ $(1 \leqq i \leqq m)$, we put $U=\operatorname{Spec}A \setminus \{\mathfrak{I}_{f_1}, \cdots, \mathfrak{I}_{f_m}\}$. Then $\bar{A}(U)=\tilde{A}(U) \Leftrightarrow$

$$\text{(11)}\qquad {}^{\exists}l_1, \cdots, l_m \geqq 1 \text{ such that } h(0)=h^{(1)}(0)=0\,.$$

In particular, if k is an algebraically closed field, then by putting $f_i(t)=t-\gamma_i$, the condition (11) can be replaced by the following condition (12);

$$\text{(12)}\qquad {}^{\exists}l_1, \cdots, l_m \geqq 1 \text{ such that } \sum_{i=1}^{m} \frac{l_i}{\gamma_i} = \sum_{i=1}^{m} \frac{l_i}{\gamma_i^2} = 0\,.$$

EXAMPLE 5. Let $A=k[x, y]/(x^4-y^3)$, then A is the affine coordinate ring of the singular rational curve C: $y^3=x^4$.

From the k-algebra homomorphism $\varphi: k[x, y] \to k[t]$ defined by $x \mapsto t^3$, $y \mapsto t^4$, we obtain $A \cong k \oplus t^3k \oplus t^4k \oplus t^6k[t]$. Hence we can apply Lemma 9. Therefore,

$$(\text{S.1}) \Longleftrightarrow (\text{S.2}) \Longleftrightarrow (\text{S.3}) \Longleftrightarrow p \neq 0\,.$$

Moreover we shall determine open sets U of Spec$A$ such that $\bar{A}(U)=\tilde{A}(U)$.

(I) $U=\operatorname{Spec}A$ or $U=\phi \Rightarrow \bar{A}(U)=\tilde{A}(U)$.

(II) $U \notin \mathfrak{I}_t \Rightarrow \bar{A}(U)=\tilde{A}(U)$.

(III) For any irreducible polynomials $f_1(t), \cdots, f_m(t)$ of $k[t]$ such that $f_i(t) \neq t$ $(1 \leqq i \leqq m)$, we put $U=\operatorname{Spec}A \setminus \{\mathfrak{I}_{f_1}, \cdots, \mathfrak{I}_{f_m}\}$. Then $\bar{A}(U)=\tilde{A}(U) \Leftrightarrow$

$$\text{(13)}\qquad {}^{\exists}l_1, \cdots, l_m \geqq 1 \text{ such that } h(0)=h^{(1)}(0)=h^{(4)}(0)=0\,.$$

In particular, if k is an algebraically closed field, then by putting $f_i(t)=t-\gamma_i$, the condition (13) can be replaced by the following condition (14);

$$\text{(14)}\qquad {}^{\exists}l_1, \cdots, l_m \geqq 1 \text{ such that } \sum_{i=1}^{m} \frac{l_i}{\gamma_i} = \sum_{i=1}^{m} \frac{l_i}{\gamma_i^2} = \sum_{i=1}^{m} \frac{l_i}{\gamma_i^5} = 0\,.$$

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