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J-Holomorphic Curves of a 6-Dimensional Sphere

Hideya HASHIMOTO*

Nippon Institute of Technology

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1. Introduction.

It is well known that a 6-dimensional sphere S^6 can be considered as a homogeneous space $G_2/SU(3)$ where G_2 is the Lie group of automorphisms of the octonions $\mathbf{O}$. From this representation, we can define an almost Hermitian structrure $(J, \langle\ ,\ \rangle)$ on a 6-dimensional sphere by making use of the vector cross product of the octonions. Also it is known that the almost Hermitian structure of S^6 satisfy the nearly Kähler condition $((D_X J)X = 0)$ where D is the Riemannian connection of S^6 with respect to the canonical metric and X is a tangent vector of S^6. A submanifold M in an almost Hermitian manifold N is called an almost complex submanifold if each tangent space of M is invariant under the almost complex structure of N. Almost complex submanifolds of S^6 were studied by many authors, for example, K. Sekigawa ([Se]), J. Bolton et al. ([Bo1]), R. L. Bryant ([Br1]), F. Dillen et al. ([D-V-V]), and A. Gray ([G]). A. Gray proved that there exists no 4-dimenional almost complex submanifold of S^6. Hence the dimension of almost complex submanifold of S^6 is either 2 or 6. In particular, we call a 2-dimensional almost complex submanifold a J-holomorphic curve. R. L. Bryant ([Br1]) constructed superminimal J-holomorphic curves of any compact Riemann surface to S^6 by using twistor methods with respect to the G_2-moving frame. Also, J. Bolton et al. ([Bo1, 2]) constructed non-superminimal J-holomorphic curves of 2-dimensional tori to S^6 by using the soliton theory. Curvature properties of J-holomorphic curves of S^6 were studied by K. Sekigawa ([Se]) and F. Dillen et al. ([D-V-V]). In this paper, we unify their results about J-holomorphic curves, making use of G_2-moving frame methods by R. L. Bryant and a Lemma of Eschenburg et al. [E-G-T] (also see ([Ch])), and give some results of curvcature properties of a J-holomorphic curve of S^6. Also we give two partial differential equations with respect to the Gauss curvature and the third fundamental form, and we obtain some G_2 rigidity theorem of J-holomorphic curves of S^6, genus formula (which

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is obtained by R. L. Bryant) by making use of another elementary methods, and give some existence theorem of superminimal points by applying this genus formula.

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2. Preliminaries.

2.1. Notations. We denote by $M_{p\times q}(\mathbf{C})$ the set of $p \times q$ complex matrices and $[a] \in M_{3\times 3}(\mathbf{C})$ is given by

$$[a] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$$

where $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in M_{3\times 1}(\mathbf{C})$. Then we have

$$[a]b + [b]a = 0$$

where $a, b \in M_{3\times 1}(\mathbf{C})$. Let $\langle\, ,\, \rangle$ be the canonical inner product of $\mathbf{O}$. For any $x \in \mathbf{O}$, we denote by $\bar{x}$ the conjugate of x. We remark that the octonians may be regarded as the direct sum $\mathbf{H} \oplus \mathbf{H}$ where $\mathbf{H}$ is the quaternions.

2.2. Structure equation of G_2. We recall the structure equations of $(\operatorname{Im}\mathbf{O}, G_2)$ which is established by R. Bryant ([Br1]). The Lie group G_2 is defined by

$$G_2 = \{g \in GL_8(\mathbf{R}) : g(uv) = g(u)g(v) \quad \text{for any } u, v \in \mathbf{O}\}.$$

Now, we set a basis of $\mathbf{C} \otimes_R \operatorname{Im}\mathbf{O}$ by $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$, $E_1 = iN$, $E_2 = jN$, $E_3 = -kN$, $\overline{E_1} = i\bar{N}$, $\overline{E_2} = j\bar{N}$ and $\overline{E_3} = -k\bar{N}$, where $N = (1 - \sqrt{-1}\varepsilon)/2$, $\bar{N} = (1 + \sqrt{-1}\varepsilon)/2 \in \mathbf{C} \otimes_R \mathbf{O}$ and $\{1, i, j, k\}$ is the canonical basis of $\mathbf{H}$. A basis $(u, f, \bar{f})$ of $\mathbf{C} \otimes_R \operatorname{Im}\mathbf{O}$ is said to be admissible, if there exists $g \in G_2 \subset M_{7\times 7}(\mathbf{C})$ such that $(u, f, \bar{f}) = (\varepsilon, E, \bar{E})g$. We identify the element of G_2 with corresponding admissible basis. Then we have

PROPOSITION 2.1. *There exist left invariant 1-forms κ and θ on G_2; $\theta = (\theta^i)$ with values in $M_{3\times 1}(\mathbf{C})$ and $\kappa = ({\kappa_j}^i)$, $1 \le i, j \le 3$, with values in the 3×3 skew Hermitian matrices which satisfy* $\operatorname{tr}\kappa = 0$, *and*

$$\begin{aligned} d(u, f, \bar{f}) &= (u, f, \bar{f}) \begin{pmatrix} 0 & -\sqrt{-1}\,{}^t\bar{\theta} & \sqrt{-1}\,{}^t\theta \\ -2\sqrt{-1}\theta & \kappa & [\bar{\theta}] \\ 2\sqrt{-1}\bar{\theta} & [\theta] & \bar{\kappa} \end{pmatrix} \\ &= (u, f, \bar{f})\Phi\,. \end{aligned} \tag{2.1}$$

Then Φ satisfies $d\Phi = -\Phi \wedge \Phi$, or equivalently,

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta}\,. \tag{2.2}$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - ({}^t\theta \wedge \bar{\theta})I_3\,. \tag{2.3}$$

3. Oriented surfaces in S^6.

In the sequel, we denote by S^6 a 6-dimensional unit sphere with the canonical Riemannian metric $\langle\,,\rangle$. Let $M = (M, x)$ be an oriented surface in S^6 with (isometric) immersion $x : M \to S^6$. We denote by D, ∇ and $\nabla^\perp$ the Riemannian connections of S^6, M and the normal bundle $T^\perp M$, respectivcely. The Gauss and Weingarten formulas are given respectively by

$$D_X Y = \nabla_X Y + \sigma(X, Y)\,, \tag{3.1}$$

$$D_X \xi = -A_\xi X + \nabla^\perp_X \xi\,, \tag{3.2}$$

where σ and A_ξ are the second fundamental form and the shape operator (with respect to a normal vector field ξ), and X, Y are smooth vector fields tangent to M. The second fundamental form σ and the shape operator A_ξ are related by

$$\langle \sigma(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle\,.$$

The Gauss, Codazzi and Ricci equations are given respectively by

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &\quad + \langle\sigma(X, W), \sigma(Y, Z)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle\,, \end{aligned}\tag{3.3}$$

$$(\nabla_X\sigma)(Y, Z) = (\nabla_Y\sigma)(X, Z)\,, \tag{3.4}$$

$$\langle R^\perp(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle\,, \tag{3.5}$$

where

$$(\nabla_X\sigma)(X, Z) = \nabla^\perp_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)\,, \tag{3.6}$$

$$R^\perp(X, Y)\xi = [\nabla^\perp_X, \nabla^\perp_Y]\xi - \nabla^\perp_{[X,Y]}\xi\,, \tag{3.7}$$

$X, Y, Z, W \in X(M)$ ($X(M)$ denotes the Lie algebra of all smooth vector fields tangent to M) and ξ, η are vector fields normal to M (cf. [Sp], Chapter 7).

Let $\{e_1, e_2\}$ be a local orthonormal frame field of M. If the immersion x is minimal (see (1) of Proposition 4.1 in the next section), the Gaussian curvature K is given by

$$K = 1 - (\|\sigma(e_1, e_1)\|^2 + \|\sigma(e_1, e_2)\|^2)\,. \tag{3.8}$$

4. Fundamental properties of J-holomorphic curves of S^6.

In this section, we shall derive some elementary properties of J-holomorphic curves of S^6. First we recall the almost Hermitian structrure of S^6. Let X be a tangent vector of $S^6 \subset \mathrm{Im}\,\mathbf{O}$ at x, the almost complex structure J is defined as follows;

$$JX = X \times x$$

where $\times$ is the vector cross product of $\mathrm{Im}\,\mathbf{O}$. We may observe that this almost complex structure J is orthogonal with respect to the canonical metric on S^6. Hence $S^6 = (S^6, J, \langle\,,\rangle)$ is an almost Hermitian manifold and this structure satisfy nearly Kähler condition $(D_X J)X = 0$ ([Br1]). However the second betti number of S^6 is zero, this almost Hermitian structure is not

Kähler. We denote by $\nu_1 = \mathrm{span}_{\mathbf{R}}\{\sigma(X,Y)\,|\,X, Y \in TM\}$ the first normal space. First we prove the following.

PROPOSITION 4.1. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6.*
(1) *For any vector fields X, Y on M, we have*

$$\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y)\,.$$

In particular the immersion is minimal.
(2) *For a normal vector field $\xi \in \nu_1$, we have*

$$A_{J\xi}(X) = J(A_\xi X)\,.$$

(3) *For a normal vector field $\xi \in \nu_1$, we have*

$$\nabla^\perp_X(J\xi) = J(\nabla^\perp_X \xi) + \xi \times X\,.$$

PROOF. For any vector fields X, Y on M, we have

$$\begin{aligned}\tilde{D}_X(JY) &= \tilde{D}_X(Y \times x) = (\tilde{D}_X Y) \times x + Y \times (\tilde{D}_X x)\\ &= (\nabla_X Y + \sigma(X, Y) - \langle X, Y\rangle x) \times x + Y \times X\\ &= J(\nabla_X Y) + J(\sigma(X, Y)) + Y \times X\,,\end{aligned}$$

where $\tilde{D}$ is the canonical connection of a 7-dimensional Euclidean space $R^7 \simeq \mathrm{Im}\, O$. On the other hand, by the Gauss formula, we get

$$\tilde{D}_X(JY) = \nabla_X(JY) + \sigma(X, JY) - \langle X, JY\rangle x\,.$$

Since $X, Y \in TM$, we have

$$Y \times X = -\langle X, JY\rangle x\,.$$

Therefore we have (1). Next we shall prove (2) and (3). By the weingarten formula, we have

$$D_X(J\xi) = -A_{J\xi}(X) + \nabla^\perp_X(J\xi)\,.$$

From the definition of the almost complex structure of S^6, we get

$$D_X(J\xi) = D_X(\xi \times x) = \tilde{D}_X(\xi \times x) - \langle X, \xi \times x\rangle x\,.$$

Since $\xi \times x = J(\xi)$, we have $\langle X, \xi \times x\rangle = 0$, so we get

$$\begin{aligned}D_X(J\xi) &= \tilde{D}_X(\xi \times x)\\ &= (\tilde{D}_X \xi) \times x + \xi \times (\tilde{D}_X x)\\ &= (-A_\xi(X) + \nabla^\perp_X(\xi)) \times x + \xi \times X\\ &= -JA_\xi(X) + J\nabla^\perp_X(\xi) + \xi \times X\,.\end{aligned}$$

Since $\xi \in \nu_1$, we easily see that $\xi \times X$ is a normal vector field, we get the desired results. □

By Proposition 4.1, we have immediately

COROLLARY 4.2. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6. Then we have*
(1) *The ellipse of curvature $\sigma(X, X)$ ($|X| = 1$) is a circle in the first normal space ν_1.*

(2) *If the point $p \in M$ is not a geodesic one, then we have*

$$T_p{}^{\perp}M = \nu_1(p) \oplus \nu_2(p)$$

where ν_2 denote the second normal space which is spanned by $e_1 \times \xi_1$ and $J(e_1 \times \xi_1)$ for $\xi_1 \in \nu_1$.

PROOF. (1) Let e_1, Je_1 be an orthonormal basis of T_pM at $p \in M$. Any unit vector X can be represented by $X = \cos\theta e_1 + \sin\theta Je_1$. Then we have

$$\begin{aligned}\sigma(X, X) &= \cos^2(\theta)\sigma(e_1, e_1) + \sin(2\theta)\sigma(e_1, Je_1) + \sin^2\theta\sigma(Je_1, Je_1)\\ &= \cos(2\theta)\sigma(e_1, e_1) + \sin(2\theta)\sigma(e_1, Je_1).\end{aligned}$$

Since $\sigma(e_1, Je_1) = J\sigma(e_1, e_1)$, we get desired result.

(2) If we put $\xi_1 = \sigma(e_1, e_1)/|\sigma(e_1, e_1)|$, then we have $\xi_1 \in \nu_1$. By (1) of Proposition 4.1, we have

$$\nu_1 = \mathrm{span}_R\{\xi_1, J\xi_1\}.$$

Also we have

$$\begin{aligned}&\langle e_1 \times \xi_1, x\rangle = 0, \quad \langle e_1 \times \xi_1, e_1\rangle = 0, \quad \langle e_1 \times \xi_1, Je_1\rangle = 0,\\ &\langle e_1 \times \xi_1, \xi_1\rangle = 0, \quad \langle e_1 \times \xi_1, J\xi_1\rangle = 0.\end{aligned}$$

Hence we have (2). □

COROLLARY 4.3. *If a point p is not a geodesic one, then the shape operators A_{ξ_α} are given by the following form*

$$A_{\xi_1} = \begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}, \quad A_{J\xi_1} = \begin{pmatrix}0 & \lambda\\ \lambda & 0\end{pmatrix}, \quad A_{e_1\times\xi_1} = 0, \quad \textit{and} \quad A_{J(e_1\times\xi_1)} = 0,$$

where $\xi_1 = \sigma(e_1, e_1)/|\sigma(e_1, e_1)|$ and $\lambda = |\sigma(e_1, e_1)|$ (or equivalently, $\sigma(e_1, e_1) = -\sigma(Je_1, Je_1) = \lambda\xi_1$, $\sigma(e_1, Je_1) = \lambda J\xi_1$).

PROOF. From the definition of ξ_1, we have

$$\begin{aligned}\langle A_{\xi_1}(e_1), e_1\rangle &= \langle\sigma(e_1, e_1), \xi_1\rangle = \left\langle\sigma(e_1, e_1), \frac{\sigma(e_1, e_1)}{|\sigma(e_1, e_1)|}\right\rangle\\ &= \lambda \ (= -\langle A_{\xi_1}(Je_1), Je_1\rangle),\end{aligned}$$

and

$$\begin{aligned}\langle A_{\xi_1}(e_1), Je_1\rangle &= \langle\sigma(e_1, Je_1), \xi_1\rangle\\ &= \langle J\sigma(e_1, e_1), \xi_1\rangle = \left\langle J\sigma(e_1, e_1), \frac{\sigma(e_1, e_1)}{|\sigma(e_1, e_1)|}\right\rangle = 0.\end{aligned}$$

Similary, we have

$$A_{J\xi_1} = \begin{pmatrix}0 & \lambda\\ \lambda & 0\end{pmatrix}.$$

Since the first normal space is perpendicular to the second normal space, we can easily get

$$A_{e_1\times\xi_1} = 0 \quad \text{and} \quad A_{J(e_1\times\xi_1)} = 0. \qquad \square$$

5. G_2 moving frame.

In this section, we shall give the relation between the ordinary surface theory (section 3) and G_2 admissible frame field along the immersion x. We recall that the Lie group G_2 is a principal $SU(3)$ right bundle over S^6. First we define complexified local $SU(3)$-frame field as follows (at points which are not geodesic).

$$f_3 = \frac{1}{2}(e_1 - \sqrt{-1}Je_1)\,, \tag{5.1}$$

$$f_2 = \frac{1}{2}(\xi_1 - \sqrt{-1}J\xi_1)\,, \tag{5.2}$$

$$f_1 = -\frac{1}{2}(e_1 \times \xi_1 - \sqrt{-1}J(e_1 \times \xi_1))\,, \tag{5.3}$$

where e_1, Je_1, ξ_1, $J\xi_1$ and $e_1 \times \xi_1$, $J(e_1 \times \xi_1)$ are local orthonormal frame fields of M, ν_1 and ν_2, respectively. Then, $\{f_1, f_2, f_3\}$ satisfy

$$Jf_i = \sqrt{-1}f_i$$

for any $i = 1, 2, 3$. We can easily see that $\{f_1, f_2, f_3\}$ is a $SU(3)$-frame field, and $\{x, f_1, f_2, f_3, \overline{f_1}, \overline{f_2}, \overline{f_3}\}$ is a local admissible G_2-frame along the immersion x.

Next we shall write down the structure equations of a J-holomorphic curve of S^6 which may admit geodesic points. The left invariant 1-forms on G_2 pull back under the immersion x give forms on the pullback boundle $x^*(G_2)$ which we continue to denote the same letters. We obtain the following

PROPOSITION 5.1. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6. Then we have the following.*

$$dx = f_3(-2\sqrt{-1}\theta^3) + \overline{f_3}(2\sqrt{-1}\,\overline{\theta^3})\,, \tag{5.4}$$

$$\theta^2 = \theta^1 = 0\,, \quad \kappa_3{}^1 = 0\,, \tag{5.5}$$

$$df_3 = x(-\sqrt{-1}\,\overline{\theta^3}) + \sum_{i=1}^{3} f_i \cdot \kappa_3{}^i \quad \textit{Gauss formula})\,, \tag{5.6}$$

$$df_2 = \sum_{i=1}^{3} f_i \cdot \kappa_2{}^i + \overline{f_1}\theta^3\,, \tag{5.7}$$

$$df_1 = \sum_{i=1}^{3} f_i \cdot \kappa_1{}^i - \overline{f_2}\theta^3\,, \quad \textit{Weingarten formula})\,. \tag{5.8}$$

Also we have

$$\kappa_3{}^3 + \kappa_2{}^2 + \kappa_1{}^1 = 0\,, \tag{5.9}$$

$$d\theta^3 + \kappa_3{}^3 \wedge \theta^3 = 0\,, \tag{5.10}$$

$$d\kappa_3{}^3 + \kappa_2{}^3 \wedge \kappa_3{}^2 = 2\theta^3 \wedge \overline{\theta^3}\,, \tag{5.11}$$

$$d\kappa_2{}^2 + \kappa_3{}^2 \wedge \kappa_2{}^3 + \kappa_1{}^2 \wedge \kappa_2{}^1 = -\theta^3 \wedge \overline{\theta^3}\,, \tag{5.12}$$

$$d\kappa_1{}^1 + \kappa_2{}^1 \wedge \kappa_1{}^2 = -\theta^3 \wedge \overline{\theta^3}\,, \tag{5.13}$$

$$d\kappa_3{}^2 + (\kappa_2{}^2 - \kappa_3{}^3) \wedge \kappa_3{}^2 = 0\,, \tag{5.14}$$

$$d\kappa_2{}^1 + (\kappa_1{}^1 - \kappa_2{}^2) \wedge \kappa_2{}^1 = 0\,. \tag{5.15}$$

We note that $\kappa_3{}^2$, $\kappa_2{}^1 \in \Lambda^{(1,0)}$ *where* $\Lambda^{(1,0)}$ *is a space of* 1*-forms of type* $(1,0)$ *with respect to the complex structure* J *of* M.

PROOF. By (5.5), we get

$$d\theta^1 = -\kappa_3{}^1 \wedge \theta^3 = 0\,, \quad d\theta^2 = -\kappa_3{}^2 \wedge \theta^3 = 0\,,$$

so the 1-forms $\kappa_3{}^1$, $\kappa_3{}^2$ are $(1,0)$-forms on M. Since $x(M)$ is a minimal surface of S^6, we can take the vector field f_1 is an orthogonal complement of the complex vector space $\mathrm{span}_C\{f_2, f_3\}$ (with respect to the Hermitian inner product), we have

$$\kappa_3{}^1(X) = 2\langle df_3(X), \overline{f_1}\rangle = 2\langle \sigma(X, f_3), \overline{f_1}\rangle = 0\,.$$

We get the desired result. □

We put the connection 1-form $\kappa_3{}^3 = \sqrt{-1}\rho_1$ of M, the connection $\kappa_2{}^2 = \sqrt{-1}\rho_2$ of the 1st normal bundle ν_1, the connection $\kappa_1{}^1 = \sqrt{-1}\rho_3$, of the second normal bundle ν_2, respectively. If the immersion x does not have a geodesic point, then we have

$$\rho_1(X) = \langle \nabla_X e_1, Je_1\rangle\,, \tag{5.16}$$

$$\rho_2(X) = \langle \nabla^\perp_X \xi_1, J\xi_1\rangle\,, \tag{5.17}$$

$$\rho_3(X) = \langle \nabla^\perp_X e_1 \times \xi_1, J(e_1 \times \xi_1)\rangle \tag{5.18}$$

for any $X \in TM$.

LEMMA 5.2. *If the immersion* x *does not have a geodesic point, then we have*

$$\kappa_3{}^3 = \sqrt{-1}\rho_1\,, \quad \kappa_2{}^2 = \sqrt{-1}\rho_2\,, \quad \kappa_1{}^1 = \sqrt{-1}\rho_3\,, \tag{5.19}$$

$$\kappa_3{}^2 = \lambda(-2\sqrt{-1}\theta^3)\,, \tag{5.20}$$

$$\kappa_2{}^1 = \frac{2}{\lambda}\langle (\nabla_{f_3}\sigma)(f_3, f_3), \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\,, \tag{5.21}$$

where $\lambda = \sqrt{(1-K)/2}$.

PROOF. From the structure equations (5.6)–(5.8), and (5.16)–(5.18), we have (5.19). Next we show (5.20). Since $\kappa_3{}^2 \in \Lambda^{(1,0)}$, we get

$$\begin{aligned}
\kappa_3{}^2 &= \kappa_3{}^2(f_3)(-2\sqrt{-1}\theta^3) \\
&= 2\langle D_{f_3} f_3, \overline{f_2}\rangle(-2\sqrt{-1}\theta^3) \\
&= 2\langle \sigma(f_3, f_3), \overline{f_2}\rangle(-2\sqrt{-1}\theta^3) \\
&= \frac{1}{4}\langle \sigma(e_1 - \sqrt{-1}Je_1, e_1 - \sqrt{-1}Je_1), \xi_1 + \sqrt{-1}J\xi_1\rangle(-2\sqrt{-1}\theta^3) \\
&= \langle \sigma(e_1, e_1) - \sqrt{-1}\sigma(e_1, Je_1), \xi_1 + \sqrt{-1}J\xi_1\rangle(-\sqrt{-1}\theta^3) \\
&= \lambda(-2\sqrt{-1}\theta^3)\,.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}\kappa_2{}^1 &= \kappa_2{}^1(f_3)(-2\sqrt{-1}\theta^3)\\ &= 2\langle\nabla^{\perp}_{f_3} f_2, \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\\ &= \frac{1}{2}\langle\nabla^{\perp}_{e_1-\sqrt{-1}Je_1}(\xi_1 - \sqrt{-1}J\xi_1), \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\\ &= \frac{1}{2}\langle\nabla^{\perp}_{e_1-\sqrt{-1}Je_1}\frac{1}{\lambda}(\sigma(e_1, e_1) - \sqrt{-1}J\sigma(e_1, e_1)), \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\\ &= \frac{1}{\lambda}\langle(\nabla_{e_1}\sigma)(e_1, e_1) - \sqrt{-1}(\nabla_{Je_1}\sigma)(e_1, e_1), \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\\ &= \frac{2}{\lambda}\langle(\nabla_{f_3}\sigma)(e_1, e_1), \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\,.\end{aligned}$$

On the other hand, we have $(\nabla_{f_3}\sigma)(\overline{f_3}, \overline{f_3}) = (\nabla_{\overline{f_3}}\sigma)(f_3, \overline{f_3}) = 0$. This yields

$$\kappa_2{}^1 = \frac{2}{\lambda}\langle(\nabla_{f_3}\sigma)(f_3, f_3), \overline{f_1}\rangle(-2\sqrt{-1}\theta^3)\,.$$

We get (5.21). □

PROPOSITION 5.3. *$x : M \to S^6$ be a J-holomorphic curve of S^6.*

(1) $$d\rho_1 = (2|\kappa_3{}^2(f_3)|^2 - 1)\Omega = -K\Omega\,,$$

(2) $$d\rho_2 = \frac{1}{2}(1 - 4|\kappa_3{}^2(f_3)|^2 + 4|\kappa_2{}^1(f_3)|^2)\Omega\,,$$

(3) $$d\rho_3 = \frac{1}{2}(1 - 4|\kappa_2{}^1(f_3)|^2)\Omega\,,$$

where $\Omega = 2\sqrt{-1}\theta^3 \wedge \overline{\theta^3}$ is a volume element of M.

PROOF. By (5.11) and (5.20), we get

$$\begin{aligned}d\rho_1 &= -\sqrt{-1}\,\overline{\kappa_3{}^2} \wedge \kappa_3{}^2 - 2\sqrt{-1}\theta^3 \wedge \overline{\theta^3}\\ &= (4\sqrt{-1}\kappa_3{}^2(f_3)\overline{\kappa_3{}^2(f_3)} - 2\sqrt{-1})\theta^3 \wedge \overline{\theta^3}\\ &= (2|\kappa_3{}^2(f_3)|^2 - 1)\Omega\\ &= -K\Omega\,.\end{aligned}$$

Hence we get (1). Similarly, we have (2) and (3). □

REMARK. Each second cohomology class $[-d\rho_i/2\pi] \in H^2(M^2, Z)$ is a first Chern class of the corresponding complex line bundle.

Next we show that the function $|\kappa_2{}^1(f_3)|^2$ is well defined at isolated geodesic point. We recall the definition the holomorphic line bundles $\nu_1{}^{(1,0)}$ and $\nu_2{}^{(1,0)}$. First, we shall define the $su(3)$ connection on $T^{(1,0)}S^6$ as follows;

Let $\{f_1, f_2, f_3\}$ be the $SU(3)$-frame field of $T^{(1,0)}S^6$. A section s of the bundle $T^{(1,0)}S^6$ can be represented by

$$s = \sum_{i=1}^{3} f_i \otimes s^i .$$

We set the operator $\tilde{\nabla}$ on $T^{(1,0)}S^6$ such that

$$\tilde{\nabla}_s = \sum_{i=1}^{3} f_i \otimes (ds^i + \kappa_j{}^i s^j) .$$

Then $\tilde{\nabla}$ is the connection which satisfies $\tilde{\nabla} J = 0$ and preserve the Hermitian inner product of $T^{(1,0)}S^6$ (see [Br1] or [H]).

Let $x : M \to S^6$ be a J-holomorphic curve of S^6 which is not totally geodesic. We denote by $T^{\perp}M$ the normal bundle of rank 4 over the J-holomorphic curve. Since the tangent bundle of M is invariant under the almost complex structure, so is $T^{\perp}M$. We denote by $T^{\perp(1,0)}M$ the $(1,0)$ part of the complexified normal bundle $T^{\perp}M \otimes C$. Since we take f_3 as a section of $T^{(1,0)}M$, $\{f_1, f_2\}$ is a local unitary frame of $T^{\perp(1,0)}M$. We can define the induced conneciton of $T^{\perp(1,0)}M$ from the above $su(3)$-connection as follows. A (local) section s of the bundle $T^{\perp(1,0)}M$ can be represented by

$$s = \sum_{i=1}^{2} f_i \otimes s^i .$$

We set the operator $\tilde{\nabla}$ on $T^{\perp(1,0)}M$ such that

$$\tilde{\nabla} s = \sum_{i=1}^{2} f_i \otimes (ds^i + \kappa_j{}^i s^j) .$$

Since M is a Riemmann surface, it can be shown that $\tilde{\nabla}$ defines the compatible holomorphic structure on $T^{\perp(1,0)}M$ by the Proposition (3.7) (in [K: page 9]). We call a (local) section s holomorphic one if

$$\tilde{\nabla}_{\overline{\partial_z}} s = 0 ,$$

where $\overline{\partial_z} = \partial/\partial\bar{z}$. We show that $\sigma(\partial_z, \partial_z)$ is a holomorphic section of $T^{\perp(1,0)}$ where $\partial_z = \partial/\partial z$. In fact, by (5.7) and (5.8), we have

$$\begin{aligned} \tilde{\nabla}_{\overline{\partial_z}}(\sigma(\partial_z, \partial_z)) &= \nabla^{\perp}{}_{\overline{\partial_z}}(\sigma(\partial_z, \partial_z)) = (\nabla_{\overline{\partial_z}}\sigma)(\partial_z, \partial_z) \quad (\text{since } \nabla_{\overline{\partial_z}}\partial_z = 0) \\ &= (\nabla_{\partial_z}\sigma)(\overline{\partial_z}, \partial_z) = 0 , \end{aligned}$$

the 3rd equality holds by the Codazzi equation and the last equality holds by (1) of Corollary 4.2. Since we assume that σ is not identically zero, the geodesic points are isolated. Let z_0 be an isolated geodesic point on M and (U, z) is an isothermal coordinate of M centered at z_0, then the metric is given by

$$ds^2 = \rho^2 dz \circ \overline{dz} .$$

Since $\sigma(\partial_z, \partial_z)$ is a holomorphic section of $T^{\perp(1,0)}M$, we may put

$$\sigma(\partial_z, \partial_z) = (z - z_0)^m \xi(z),$$

on U, where $\xi(z_0)$ $(\neq 0) \in T^{\perp(1,0)}$. We define $(1,0)$ part of the first normal bundle ${\nu_1}^{(1,0)}$ as follows

$$\text{fiver of } {\nu_1}^{(1,0)} \text{ over } \{z\} = \begin{cases} \mathrm{span}_{\mathbf{C}}\{\sigma(\partial_z, \partial_z)\} & \text{for any} \quad z \in U \setminus \{z_0\} \\ \mathrm{span}_{\mathbf{C}}\{\xi(z_0)\} & \text{at} \quad z_0 . \end{cases}$$

Then ${\nu_1}^{(1,0)}$ is well defined on M and it is a holomorphic line bundle. Since the geodesic point z_0 is isolated, we may put

$$f_2 = \frac{\xi(z)}{\sqrt{2}|\xi(z)|}$$

where $\xi(z)$ is a $T^{\perp(1,0)}$-valued holomorphic section and satisfy $\xi(z) \neq 0$ on U. In this case, we have

$$\begin{aligned} {\kappa_3}^2(f_3) &= \frac{2}{\rho}\langle D_{\partial_z} f_3, \overline{f_2}\rangle = \frac{\sqrt{2}}{\rho}\left\langle D_{\partial_z}\left(x_*\left(\frac{1}{\rho}\partial_z\right)\right), \overline{f_2}\right\rangle . \\ &= \frac{\sqrt{2}}{\rho^2|\xi(z)|}\langle \sigma(\partial_z, \partial_z), \overline{\xi(z)}\rangle = \frac{\sqrt{2}(z - z_0)^m |\xi(z)|}{\rho^2} . \end{aligned}$$

In the same way,

$$\text{fiver of } {\nu_2}^{(1,0)} \text{ over } \{z\} = \begin{cases} \mathrm{span}_{\mathbf{C}}\{-x_*(\partial_{\bar{z}}) \times \overline{\sigma(\partial_z, \partial_z)}\} & \text{for any} \quad z \in U \setminus \{z_0\} \\ \mathrm{span}_{\mathbf{C}}\{-x_*(\partial_{\bar{z}}) \times \overline{\xi(z_0)}\} & \text{at} \quad z_0 . \end{cases}$$

Then ${\nu_2}^{(1,0)}$ is also we defined. Also the bundle ${\nu_2}^{(1,0)}$ can be considered as the quotient bundle

$$T^{\perp(1,0)}/{\nu_1}^{(1,0)} = {\nu_2}^{(1,0)} .$$

Next we shall define $|{\kappa_2}^1(f_3)|^2$ at the geodesic point as follows. We note that

$${\kappa_2}^1(f_3) = \frac{2}{\rho}\langle D_{\partial_z} f_2, \overline{f_1}\rangle .$$

Since the vector field f_1 is well defined at z_0, we have

$${\kappa_2}^1(f_3) = \frac{-1}{\rho^2|\xi(z)|^2}\langle D_{\partial_z}\xi, x_*(\partial_z) \times \xi(z)\rangle .$$

Therefore,

$$|{\kappa_2}^1(f_3)|^2 = \frac{1}{\rho^4|\xi(z)|^4}\langle D_{\partial_z}\xi, x_*(\partial_z) \times \xi(z)\rangle\overline{\langle D_{\partial_z}\xi, x_*(\partial_z) \times \xi(z)\rangle} .$$

We can easily see that the function $|{\kappa_2}^1(f_3)|^2$ does not depend on the choice of the frame fields, so we can define $|{\kappa_2}^1(f_3)|^2$ whole on M (if M has only isolated geodesic points). We put $|\mathbf{III}|^2 = |{\kappa_2}^1(f_3)|^2$ (in the extended sense). We recall the definition of functions of holomorphic type and absolute value type.

DEFINITION 5.4. A smooth complex valued function $p : M \to \mathbf{C}$ is called a one of holomorphic type if locally $p = p_0 \cdot p_1$, where p_0 is a holomorphic function and p_1 is smooth without zeros. A non-negative function $f : M \to \mathbf{R}_{\geq 0}$ is called a one of absolute value type, if there exists a function g of holomorphic type with $f = |g|$. The zero set of such function is either isolated or the whole of M, and outside its zero, the function is smooth.

Then we have

PROPOSITION 5.5. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6 which is not totally geodesic. We assume that the induced metric is given by the following form (locally)*

$$ds^2 = \rho^2 dz \circ \overline{dz}\,.$$

Then the functions and $a_2^1 = \kappa_2{}^1(\partial_z)$ are of holomorphic type and hence $|\kappa_3{}^2(f_3)| = \sqrt{(1-K)/2}$ and $|\kappa_2{}^1(f_3)| = |\mathbf{III}|$ are of absolute value type. Moreover they satisfy the following equations.

$$4|\mathbf{III}|^2 - 1 = \Delta \log(1-K) - 6K\,, \tag{1}$$

$$\Delta \log |\mathbf{III}| = 1 - 4|\mathbf{III}|^2\,, \tag{2}$$

$$\Delta \log\{(1-K)|\mathbf{III}|\} = 6K\,, \tag{3}$$

outside the corresponding zero sets. The 1-forms ρ_1, ρ_2, ρ_3 satisfy the following

$$\rho_1 - \rho_2 = -2\,\mathrm{Im}\{\partial_z(\log(\overline{a_3{}^2}))dz\}\,, \tag{4}$$

$$\rho_2 - \rho_3 = -2\,\mathrm{Im}\{\partial_z(\log(\overline{a_2{}^1}))dz\}\,. \tag{5}$$

In order to prove Proposition 5.5, we recall the following fundamental lemma which is obtained by Eschenberg et al. ([E-G-T]) or S. S. Chern ([Ch]).

LEMMA 5.6. *Let (M, ds^2) be a Riemann surface and (U, z) be an isothermal coordinate system. Let $\mathbf{p} : U \to \mathbf{C}$ be a smooth complex valued function which is not identically zero, and ω be a real valued 1-form on M. The function $\mathbf{p}$ and a 1-form ω satisfy the following equality*

$$(d\mathbf{p} - \sqrt{-1}\mathbf{p}\omega) \wedge dz = 0\,,$$

if and only if

(1) $\mathbf{p}$ *is a function of holomorphic type.*

$$\omega = -2\,\mathrm{Im}\{\partial_z(\log(\bar{\mathbf{p}}))dz\}\,. \tag{2}$$

In particular, by (2), we have

$$d\omega = -\Delta \log |\mathbf{p}| \Omega\,.$$

Now we are in a position to prove Proposition 5.5.

PROOF OF PROPOSITITION 5.5. Since the 1-forms $\kappa_3{}^2, \kappa_2{}^1 \in \Lambda^{(1,0)}(M)$, there exist local functions $a_3{}^2, a_2{}^1$ such that

$$\kappa_3{}^2 = a_3{}^2 dz\,, \quad \kappa_2{}^1 = a_2{}^1 dz\,.$$

From the assumption, $a_3{}^2$, $a_2{}^1$ are smooth. By (5.14) and (5.15), the following equalities hold

$$\begin{aligned} &\{da_3{}^2 - \sqrt{-1}(\rho_1 - \rho_2)a_3{}^2\} \wedge dz = 0\,, \\ &\{da_2{}^1 - \sqrt{-1}(\rho_2 - \rho_3)a_2{}^1\} \wedge dz = 0\,. \end{aligned} \tag{5.22}$$

We can apply (2) of Lemma 5.6 to (5.22), we get (4) and (5). By applying (3) of Lemma 5.6 to (5.22), we get (1)–(3). □

LEMMA 5.7. *For any real valued positive function* $f : M \to \mathbf{R}_{>0}$, *we have*

$$J^*(d\log f) = 2\,\mathrm{Im}\{\partial_z(\log f)dz\}\,,$$

where J^* *is defined by* $J^*\alpha(X) = -\alpha(JX)$ *for any* 1*-form* α *and* $X \in TM$.

PROOF.

$$\begin{aligned} J^*(d\log f) &= J^*\{(\partial_z \log f)dz + (\overline{\partial_z} \log f)d\bar{z}\} \\ &= \partial_z \log f(-\sqrt{-1}dz) + \partial_{\bar{z}} \log f(\sqrt{-1}d\bar{z}) = 2\,\mathrm{Im}\left(\frac{\partial}{\partial z}(\log f)dz\right). \end{aligned}$$ □

LEMMA 5.8. *Let* $x : M \to S^6$ *be a J-holomorphic curve of* S^6. *We assume that the induced metric is given by the following form (locally)*

$$ds^2 = \rho^2 dz \circ \overline{dz}\,.$$

Then the connection 1*-forms are given by the following*

$$\rho_1 = J^*d\log\rho\,, \tag{5.23}$$

$$\rho_2 = 2\,\mathrm{Im}\{\partial_z \log(\rho\overline{a_3^2})dz\}\,, \tag{5.24}$$

$$\rho_3 = -2\,\mathrm{Im}\{\partial_z \log(\rho^2\overline{a_3^2})dz\}\,, \tag{5.25}$$

$$0 = \mathrm{Im}\{\partial_z(\log(\rho^3(\overline{a_3^2})^2\overline{a_2{}^1}))dz\}\,. \tag{5.26}$$

In particular, we have

$$\begin{aligned} d\rho_1 &= (\Delta\log\rho)\Omega = -K\Omega = (2\rho^2|a_3^2|^2 - 1)\Omega\,, \\ d\rho_2 &= (\Delta\log\rho|a_3^2|)\Omega = (-K + \Delta\log|a_3^2|)\Omega\,, \\ d\rho_3 &= -(\Delta\log\rho^2|a_3^2|)\Omega = (2K - \Delta\log|a_3^2|)\Omega\,. \end{aligned}$$

PROOF. Since the induced metric is given by the above form, we get

$$\rho_1 = J^*d\log\rho\,.$$

By (4) of Proposition 5.5 and Lemma 5.7, we have

$$\rho_2 = \rho_1 + 2\,\mathrm{Im}\{\partial_z \log(\overline{a_3^2})dz\}\,.$$

Hence, we get (5.24). By (5.9), (5.23) and (5.24), we obtain (5.25). By (5.24), (5.25) and (5) of Proposition 5.5, we get (5.26). □

REMARK. The equation (5.26) gives an important role to define geometrical invariants of G_2.

6. Local existence and rigidity.

We first remark that a J-holomorphic curve M of S^6 is a totally geodesic, if and only if, the Gauss curvature of M is identically 1. Therefore we may consider the case that the Gauss curvature of M is *not* identically 1. We shall prove the following existence theorem.

THEOREM 6.1 *Let M be a connected, simply connected Riemann surface with the metric of the form*

$$ds^2 = \rho^2 dz \circ \overline{dz}.$$

We assume that the Gauss curvature K is not identically 1 and $K \leq 1$. Then functions $\sqrt{1-K}$ and $|\mathbf{III}|$ are of absolute value type and that the following equations are satisfied

$$\Delta \log(1-K) = 6K - 1 + 4|\mathbf{III}|^2, \tag{6.1}$$

$$\Delta \log |\mathbf{III}| = 1 - 4|\mathbf{III}|^2, \tag{6.2}$$

outside its zero set. Then there exists a J-holomorphic curve of $\varphi : M \to S^6$ with the Gauss curvature $K = -\Delta \log \rho$ and $|\mathbf{III}| = |\kappa_2{}^1(\partial_z)/\rho|$.

In order to prove Theorem 6.1, we recall the following theorem.

PROPOSITION 6.2 ([Gri]). *Let G be a Linear Lie group and $\mathcal{G}$ denote its Lie algebra. Let*

$$\omega = g^{-1} dg$$

be the Maurer-Cartan form where $g = (g_{ij})$ is a variable non-singular matrix, and N be a connected, simply connected n-dimensional manifold. If there exists a $\mathcal{G}$ valued 1-form ψ such that

$$d\psi + \frac{1}{2}[\psi, \psi] = 0, \quad (\textit{integarability condition})$$

then there exists a map $f : N \to G$ such that

$$\psi = f^* \omega.$$

PROOF OF THEOREM 6.1. By Proposition 6.2, to show Theorem 6.1, we may prove that there is a $\mathcal{G}_2$ valued 1-form on M which satisfy the integarability condition. First we define $\mathcal{G}_2$ valued 1-form on surface M. Let (U, z) be a local isothermal coordinate system of M. From the assumption, there exists holomorphic functions $h_0(z)$, $g_0(z)$ and nowhere-zero (complex valued) functions $h_1(z)$, $g_1(z)$ such that $\rho\sqrt{(1-K)/2} = |h_0(z)h_1(z)|$ and $\rho|\mathbf{III}| = |g_0(z)g_1(z)|$. If z_0 is a geodesic point, then we have $h_0(z) = (z-z_0)^m \alpha_0(z)$ where $\alpha_0(z) \neq 0$ on U. If z_1 is a zero point of $|\mathbf{III}|$, i.e., $|\mathbf{III}|(z_1) = 0$ (which will be called a super-minimal point in the later of this paper), then we have $g_0(z) = (z-z_1)^k \beta_0(z)$ where $\beta_0(z) \neq 0$ on U. By (6.1) and (6.2), we have

$$\Delta \log(|\alpha_0(z)\beta_0(z)h_1(z)g_1(z)|\rho^6) = 0.$$

Since the function $\alpha_0(z)\ \beta_0(z)\ h_1(z)\ g_1(z)$ does not have a zero point on U, there exists a holomorphic function $f(z)$ such that $\rho^6\alpha_0(z)\ \beta_1(z)\ h_1(z)\ g_1(z) = e^{\mathrm{Re} f(z)}$. We put

$$\begin{aligned}
&\theta^3 = \frac{1}{2}\sqrt{-1}\rho dz\,, \quad \theta^1 = \theta^2 = 0\,,\\
&\kappa_3{}^3 = \sqrt{-1}\rho_1 = \sqrt{-1}(J^* d\log\rho)\,,\\
&\kappa_2{}^2 = \sqrt{-1}\rho_2 = 2\sqrt{-1}\,\mathrm{Im}\{\partial_z \log(\rho\overline{h_0(z)h_1(z)})dz\}\,,\\
&\kappa_1{}^1 = \sqrt{-1}\rho_3 = -2\sqrt{-1}\,\mathrm{Im}\{\partial_z \log(\rho^2\overline{h_0(z)h_1(z)})dz\}\,,\\
&\kappa_3{}^2 = h_0(z)h_1(z)dz = -\overline{\kappa_2{}^3}\,,\\
&\kappa_3{}^1 = 0 = \kappa_1{}^3\,,\\
&\kappa_2{}^1 = \frac{(z-z_1)^k e^{\mathrm{Re} f(z)}}{\rho^6\alpha_0(z)h_1(z)}dz = -\overline{\kappa_1{}^2}\,,
\end{aligned}$$

on U, where $h_0(z) = (z - z_0)^m\alpha_0(z)$ if z_0 is a geodesic point in U. By (6.1), (6.2) and direct calculation, we can easily see that the integrability conditions are satisfied. By Proposition 6.2, there exists a map $\tilde{\varphi} : M \to G_2$. From the definition, the image of $\tilde{\varphi}$ transverse to $SU(3)$. We get the desired result. □

REMARK. From the above observation, we can obtain a J-holomorphic curve associated to $\tilde{\varphi}$ as follows

$$\pi \circ \tilde{\varphi}(p) = \text{the first column of the matrix } (\varepsilon, E, \bar{E})\varrho(\tilde{\varphi}(p))$$

where $\varrho : G_2 \to SO(7) \subset \mathrm{End}_{\mathbf{R}}\mathbf{C}^7$ is a faithful representation, $\varepsilon = (0, 1) \in \mathrm{Im}\,\mathbf{O}$ and $p \in M$.

We shall prove the rigidity theorem with respect to G_2. First, we shall determine the geometrical invariants up to the action of G_2.

Let $x : M \to S^6$ be a J-holomorphic curve of S^6 and $\{f_3, f_2, f_1\}$ (resp. $\{f_3', f_2', f_1'\}$) a special unitary frame on U (resp. on V) where U, V are sufficiently small open subset of M such that $U \cap V \neq \emptyset$. Then there exists a $\theta \in S^1$ such that $f_3' = e^{\iota\theta} f_3$, $f_2' = e^{2\iota\theta} f_2$, $f_1' = e^{-3\iota\theta} f_1$ where $\iota = \sqrt{-1}$. From this relation, we have the following relations about 1-forms on $U \cap V$

$$\begin{aligned}
&\theta'^3 = e^{-\iota\theta}\theta^3\,,\\
&\kappa_3'^3 = \kappa_3{}^3 + \sqrt{-1}d\theta\,, \quad \kappa_2'^2 = \kappa_2{}^2 + 2\sqrt{-1}d\theta\,, \quad \kappa_1'^1 = \kappa_1{}^1 - 3\sqrt{-1}d\theta\,,\\
&\kappa_3'^2 = e^{-\iota\theta}\kappa_3{}^2\,, \quad \kappa_3'^1 = \kappa_3{}^1 = 0\,, \quad \kappa_2'^1 = e^{5\iota\theta}\kappa_2{}^1\,.
\end{aligned}$$

Therefore we can easily see that

$$\Lambda = 4\sqrt{-1}(\kappa_3{}^2)^2 \otimes \kappa_2{}^1 \otimes (\theta^3)^3$$

is independent of the choice of the special unitary frame. We call Λ a *geometrical invariant* of a J-holomorphic curve of S^6 with respect to the action of G_2. We remark that Λ is a holomorphic section of the bundle $\otimes^6 T^{*(1,0)}M$. In particular, by (5.26), Λ is a globally defined holomorphic 6-differential on M. In fact, we may put $f_3 = (1/\rho)\partial_z$, then we have

$$\Lambda = \begin{cases} -\langle \sigma_2(\partial_z, \partial_z, \partial_z), x_*(\partial_z) \times \sigma(\partial_z, \partial_z)\rangle (dz)^6 & \text{(without geodesic point)} \\ -(z-z_0)^{2m}\langle D_{\partial_z}\xi(z), x_*(\partial_z) \times \xi(z)\rangle (dz)^6 & \text{(where } z_0 \text{ is a geodesic point)} \end{cases} \tag{6.3}$$

where $\sigma_2(X, Y, Z)$ is the components of the 2nd normal space of $(\nabla_X \sigma)(Y, Z)$ for any tangent vectors X, Y, Z of M. From the above representation of Λ, we may write $\Lambda = F(z)dz^6$ where $F(z)$ is a holomorphic function on (U, z). By direct calculation, we have

$$|\Lambda| = \frac{(1-K)|\mathbf{III}|}{32}.$$

LEMMA 6.3. *Let M be a connected surface and $x_1, x_2 : M \to S^6$ be two J-holomorphic curves with same induced metric. Let Λ_1 and Λ_2 be the corresponding holomorphic differentials. Then there exists an element $g \in G_2$ such that $g \circ x_1 = x_2$ if and only if $\Lambda_1 = \Lambda_2$.*

PROOF. We may assume that $ds_1^2 = ds_2^2 = \rho^2 dz \circ \overline{dz}$ and $\Lambda_1 = \Lambda_2$ on sufficiently small neighborhood (U, z) of M. We can take the common complexified tangent vector field $f_3 = f_3'$. Then we have $\theta^3 = \theta'^3$ and $\kappa_3{}^3 = \kappa_3'^3$. Since $|a_3{}^2| = |a_3'^2| = \rho\sqrt{(1-K)/2}$, there exists a real valued differentiable function φ such that $a_3'^2 = e^{\iota\varphi}a_3{}^2$. If we change the adapted frame field of x_2, from $\{f_3' = f_3, f_2', f_1'\}$ to $\{f_3' = f_3, e^{\iota\varphi}f_2', e^{-\iota\varphi}f_1'\}$, we may assume that $a_3'^2 = a_3{}^2$. By (5.24) and (5.25), we have $\kappa_1{}^1 = \kappa_1'^1, \kappa_2{}^2 = \kappa_2'^2$. Also, since $\Lambda = \rho^3(a_3{}^2)^3 a_2{}^1(dz)^6$, $\Lambda_1 = \Lambda_2$ implies that $a_2'^1 = a_2{}^1$. Therefore we have $\kappa_2{}^1 = \kappa_2'^1$. From the following Proposition 6.4, we see that the subset of M where x_2 and $g \circ x_1$ agree, is closed and open subset in M, it is coincide with the whole of M. The converse statement is clear. □

PROPOSITION 6.4 ([Gri]). *Let $f, \tilde{f} : N \to G$ be two smooth maps of a connected manifold N into G. Then we have*

$$f = g \circ \tilde{f}$$

for fixed $g \in G$, if and only if

$$f^*\omega = \tilde{f}^*\omega$$

where ω is the Maurer-Cartan forms on G.

THEOREM 6.5. *Let $x : M \to S^6$ be a J-holomorphic curve and M is a connected surface.*

(1) *We assume that x is totally geodesic. Let $x' : M \to S^6$ be a J-holomorphic curve with the same induced metric as x. Then there exists a $g \in G_2$ such that $x' = g \circ x$.*

(2) *We assume that x is not totally geodesic and $|\mathbf{III}| \equiv 0$. Let $x' : M \to S^6$ be a J-holomorphic curve with the same induced metric as x. Then there exists a $g \in G_2$ such that $x' = g \circ x$.*

(3) *We assume that x is not totally geodesic and $|\mathbf{III}|$ is not identically zero. Let $x' : M \to S^6$ be a J-holomorphic curve with the same induced metric as x. Then there exists a one parameter family of J-holomorphic curves of $x_\theta : M \to S^6$ $(\theta \in S^1)$ with same induced*

metric as x. Moreover, any J-holomorphic curve with the same induced metric belongs to this family, up to the action of G_2.

PROOF. (1) By Lemma 4.4 in [Br1], this is proved.

(2) By Lemma 6.3, we get (2).

(3) By (1) of Proposition 5.5, the function $|\mathbf{III}|$ is determined by the induced metric. By the assumption, we can easily see that there exists a real valued function $\theta \in S^1$ such that $\Lambda' = e^{\iota\theta}\Lambda$. Since Λ', Λ are holomorphic 6-differential on M, θ is a constant. By Theorem 6.1, there exists a 1-parameter family of J-holomorphic curves with the same induced metric of x and $e^{\iota\theta}\Lambda$ where θ is a constant. By Lemma 6.3, any J-holomorphic curve with the same induced metric of x, is congruent to this family up to the action of G_2. We get the desired result. □

7. Some theorems associated to curvature.

In this section, we give some theorems as an application of Proposition 5.5 and unify some results obtained by K. Sekigawa ([Se]) and F. Dillen et al. ([D-V-V]) concerned with curvatures of M. It was proved that the Veronese immersion of $S^2_{\frac{1}{6}}$ to S^6 and the Kenmotsu surface T^2 to $S^5 \subset S^6$, are J-holomorphic curves of S^6 ([Se], [Bo2]). First we prove the following.

THEOREM 7.1. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6. If its induced metric is complete and the Gauss curvature K satisfies the following conditions*

(1) *K is bounded from below,*

(2) *$K \leq 1/6$,*

(3) *$\int_M (K^-)\Omega$ is finite, where $K^-(x) = \max\{-K(x), 0\}$,*

(4) *$|\mathbf{III}|^2 < 1/4$. Then the immersion x is congruent to the Veronese immersion of $S^2_{\frac{1}{6}}$ to S^6 up to the action of G_2.*

In order to prove Theorem 7.1, we prepare the following

PROPOSITION 7.2. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6. The Gauss curvature K and the length of third fundamental form $|\mathbf{III}|^2$ satisfy the following differential equations*

$$\frac{1}{2}\Delta(1-K)^2 = 2\|\operatorname{grad} K\|^2 + (1-K)^2(6K - 1 + 4|\mathbf{III}|^2), \tag{1}$$

$$\frac{1}{2}\Delta|\mathbf{III}|^4 = 2\|\operatorname{grad}|\mathbf{III}|^2\|^2 + |\mathbf{III}|^4(1 - 4|\mathbf{III}|^2), \tag{2}$$

$$\frac{1}{2}\Delta\{(1-K)^2|\mathbf{III}|^2\}^2 = 2\|\operatorname{grad}(1-K)^2|\mathbf{III}|^2\|^2 + 12K(1-K)^4|\mathbf{III}|^4. \tag{3}$$

PROOF. First we shall prove (1). Since the immersion x is minimal, the Gauss curvature K satrisfy $K \leq 1$. If $K \equiv 1$, then (1) is automorphically satisfied. Hence we may consider the case that K is not identically 1. Then the geodesic points are isolated. On the other hand, we have

$$\Delta \log f = \frac{f \Delta f - \|\mathrm{grad}\, f\|^2}{f^2}$$

outside the corresponding zero sets, for any non-negative function f which is not identically zero. This formula and (1) of Proposition 5.5, we get (1) for the regular points. Since the L.H.S and the R.H.S of (1) are continuous functions, the equatlity holds on M. In the same way, we get (2) and (3). □

Now we are in a position to prove Theorem 7.1.

PROOF OF THEOREM 7.1. From the condition (2), we see that the immersion does not have a geodesic point. If M is a non-compact complete Riemann surface, then the conditions (1) and (3) imply M is a parabolic Riemann surface by Huber's theorem ([Hu, Theorem 15]). By the condition (4) of Theorem 7.1, and the equation (2) of Proposition 7.2, $|\mathbf{III}|^2$ is a bounded subharmonic function on M, and hence it is constant. By (2) of Proposition 7.2 and the assumption (4), we have $|\mathbf{III}|^2 = 0$. By (1) of Proposition 5.5, conditions (1) and (2), we get $-\log(1-K)$ is also a bounded subharmonic function on M, the Gauss curvature K must be constant. Again by (1) of Proposition 7.2, we have $K \equiv 1/6$. By (2) of Theorem 6.5, we get the desired result. □

We give another proof of the following theorems concerning to the curvcature properties of J-holomorphic curve of S^6.

THEOREM 7.3 (of Sekigawa [S] and F. Dillen et al. [D]). *Let M be a J-holomorphic curve of S^6.*

(1) *M is complete and $1/6 \leq K \leq 1$ then $K \equiv 1/6$ or $K \equiv 1$,*

(2) *If M is compact and $0 \leq K \leq 1/6$ then $K \equiv 0$ or $K \equiv 1/6$.*

PROOF. (1) By Myers' Theorem and the assumption, M is diffeomorphic to 2-dimensional sphere. By Theorem 4.6 in ([Br1]), we have $|\mathbf{III}|^2 = 0$. By (1) of Proposition 7.2, we get the desired results.

(2) If the genus of M is zero, then by Theorem 4.6 in ([Br1]), $|\mathbf{III}|^2 = 0$. Applying Theorem 7.1, we have $K \equiv 1/6$. If the genus of M is one, by the Gauss Bonnet Theorem, we get $K \equiv 0$. □

We give somewhat generalization of Theorem 7.3.

THEOREM 7.4. *Let M be a complete J-holomorphic curve of S^6. If the Gauss curvature K is nonnegative and $|\mathbf{III}|^2$ is bounded from above, then we have the one of the following*

(1) *$K \equiv 0$ and congruent to the one parameter family of Kenmotsu surface $T^2 \to S^5$, up to the actionof G_2.*

(2) $(1-K)^2|\mathbf{III}|^2 = 0$.

PROOF. From the assumption, we see that M is a parabolic Riemann surface by Huber's theorem ([Hu, Theorem 15]). If K is not identically zero, by (3) of Proposition 7.2, $(1-K)^2|\mathbf{III}|^2$ is a bounded subharmonic function, therefore it is constant on M. We get $(1-K)^2|\mathbf{III}|^2 = 0$. □

Also we have the following

THEOREM 7.5. *Let $x : M \to S^6$ be a compact J-holomorphic curves in S^6. Then we have the following inequality*

$$2\int_M |\mathrm{grad}\, K|^2 \Omega \leq \int_M (1-6K)(1-K)^2 \Omega\,.$$

The equality holds if and only if

$$(1-K)^2|\mathbf{III}|^2 \equiv 0\,.$$

PROOF. By (1) Proposition 7.2, we get the desired result. □

THEOREM 7.6. *Let $x : M \to S^6$ be a J-holomorphic curve of S^6. If its induced metric is complete and the Gauss curvature K satisfies the following conditions*

(1) *K is bounded from below,*
(2) *$K \leq 0$,*
(3) *$\int_M (K^-)\Omega$ is finite,*
(4) *$|\mathbf{III}|^2 \leq 1/4$.*

Then $K \equiv 0$, $|\mathbf{III}|^2 \equiv 1/4$ and the immersion x is congruent to the one parameter family of Kenmotsu surface $T^2 \to S^5$ up to the action of G_2.

PROOF. By (2) of Proposition 7.2, $|\mathbf{III}|^4$ is a bounded subharmonic function on M. Since M is a parabolic Riemann surface by Huber's theorem ([Hu]), $|\mathbf{III}|^2$ is a constant function on M. Also, by (2) of Proposition 7.2, we have $|\mathbf{III}|^2 \equiv 0$ or 1/4. If $|\mathbf{III}|^2 \equiv 0$, by (1) of Proposition 5.5, we have $\Delta \log(1-K) = 6K - 1$, therefore $1/(1-K)$ is a bounded subharmonic function, K is constant on M. Also, by (1) of Proposition 5.5, we have $K \equiv 1/6$. This contradicts the assumption. Hence $|\mathbf{III}|^2 \equiv 1/4$. By (3) of Proposition 5.5, we have $\Delta \log(1-K) = 6K$. Hence $1-K$ is a super harmonic function on M. Since M is a parabolic, K is constant, so $K \equiv 0$. We get the desired result by (2) of Theorem 6.5 and Theorem 3.1 in ([Br2]). □

8. Genus formulas of R. L. Bryant ([Br1]).

In this section we give another proof of R. L. Bryant's Divisor formula. We recall the Bryant's formula in our situation.

THEOREM 8.1. *Let $x : M \to S^6$ be a compact J-holomorphic curve in S^6. If the Gauss curvature K is not identically* 1 *and $|\mathbf{III}|$ is not identically* 0. *Then we have*

(1) $$\chi(\nu_1) + \chi(\nu_2) + \chi(M) = 0\,,$$

$$\chi(\nu_1) = 2\chi(M) + N(\lambda), \tag{2}$$

$$\chi(\nu_2) = -3\chi(M) + N(\lambda) = \frac{1}{2}N(|\mathbf{III}|) \tag{3}$$

(or equivalently, $6\chi(M) + 2N(\lambda) + N(|\mathbf{III}|) = 0$), where $\chi(M)$ is the Euler number of TM and $\chi(\nu_i)$ is the degree of the i-th normal bundle, and $N(f)$ is the sum of all orders for all zeros of f.

REMARK. R. L. Bryant showed the divisor formulas for any branched J-holomorphic curve of S^6. We use the formulas for the case that J-holomorphic curve has no branched point in this paper.

To prove Theorem 8.1, we recall the following elementary lemma which is obtained by Eschenberg et al. ([E-G-T]).

LEMMA 8.2. *Let f be a non-negative function of an absolute type on M. Then we have*

$$\int_M \Delta \log f\Omega = -2\pi N(f).$$

PROOF OF THEOREM 8.1. Since λ and $|\mathbf{III}|$ are functions of absolute value type, we can apply to Lemma 8.2 to the functions λ and $|\mathbf{III}|$. From the definition of the degree, we have

$$\begin{aligned} 2\pi\chi(\nu_1) &= -\int_M d\rho_2 = \int_M (2K - \Delta\log\lambda)\Omega \\ &= 4\pi\chi(M) - \int_M (\Delta\log\lambda)\Omega \\ &= 4\pi\chi(M) + 2\pi N(\lambda). \end{aligned}$$

By (1) and (2) of Proposition 5.5, we get

$$\begin{aligned} 2\pi\chi(\nu_2) &= -\int_M d\rho_3 = -\int_M (3K - \Delta\log\lambda)\Omega \\ &= -6\pi\chi(M) - 2\pi N(\lambda) \\ &= -\frac{1}{2}\int_M (1 - 4|\mathbf{III}|^2)\Omega \\ &= -\frac{1}{2}\int_M \Delta\log|\mathbf{III}|\Omega = \pi N(|\mathbf{III}|). \end{aligned}$$

By (5.9) of Propostion 5.1, we get (1). Hence we get the desired results. □

COROLLARY 8.3. *Let $x : M \to S^6$ be a compact J-holomorphic curve of S^6. If the Gauss curvature $K < 1/6$. Then we have*

$$12(g-1) = N(|\mathbf{III}|)$$

where g is a genus of M. In particular, $g \geq 2$, then there exists a superminimal point. Moreover if the immersion does not have a geodesic point and $|\mathbf{III}| \neq 0$ then the genus of $M = 1$.

PROOF. In order to prove Corollary 8.3, we show that $|\mathbf{III}|$ does not identically zero. If the function $|\mathbf{III}|$ is identically zero, then (1) of Proposition 5.5, we have

$$\int_M (6K-1)\Omega = 0\,.$$

This contradicts the assumption. Hence we see that $|\mathbf{III}|$ is not identically zero. So we can apply (3) of Theorem 8.1, we get the desired result.

Since the functions λ and $|\mathbf{III}|$ are positive functions of M, by (1) and (2) of Proposition 5.4, we have $g=1$. □

9. Existence of superminimal points and the genus of M.

In this section, we consider the relation between the existence of superminimal points and the genus of M without geodesic points. First we give some equivalent conditions of a superminimal point.

DEFINITION 9.1. The point $p \in M$ is called a superminimal one of J-holomorphic curve if $|\mathbf{III}(p)| = 0$.

We have the following.

PROPOSITION 9.2. *Let $x : M \to S^6$ be a J-holomorphic curve in S^6 which is not totally geodesic. For a point p of M which is not geodesic, the following conditions are equivalent.*

(1) *p is a superminimal point.*
(2) *$|\sigma_2(X,X,X)|$ is constant for any unit tangent vector $X \in T_pM$.*
(3) *The holomorphic 6-differential*

$$\Lambda = -\langle \sigma_2(\partial_z,\partial_z,\partial_z), \partial_z \times \sigma(\partial_z,\partial_z)\rangle (dz)^6$$

is zero at $p \in M$, where z is a local isothermal coordinate system centered at p which is compatible with the given orientation.

PROOF. Since $|\mathbf{III}| = 2|F(z)|/(\lambda^2\rho^6)$ and (6.4), the condition (1) is equivalent to (3). The 3rd fundamental form σ_2 is given by

$$\begin{aligned}\sigma_2 =& 2\langle(\nabla_{f_3}\sigma)(f_3,f_3),\overline{f_1}\rangle f_1 \otimes (-2\sqrt{-1}\theta^3)^3\\ &+ 2\langle(\nabla_{f_3}\sigma)(f_3,f_3),f_1\rangle \overline{f_1} \otimes (-2\sqrt{-1}\theta^3)^3\\ &+ 2\langle(\nabla_{\overline{f_3}}\sigma)(\overline{f_3},\overline{f_3}),\overline{f_1}\rangle f_1 \otimes (2\sqrt{-1}\,\overline{\theta^3})^3\\ &+ 2\langle(\nabla_{\overline{f_3}}\sigma)(\overline{f_3},\overline{f_3}),f_1\rangle \overline{f_1} \otimes (2\sqrt{-1}\,\overline{\theta^3})^3\,.\end{aligned}$$

We may put $f_3 = (1/\rho)\partial z$, then

$$\langle(\nabla_{f_3}\sigma)(f_3,f_3),\overline{f_1}\rangle = \frac{1}{\lambda\rho^6}F(z)\,,$$

where $F(z)$ is a holomorphic function on some neighborhood near p. Since f_1 is an element of the second normal space, we have

$$\begin{aligned}\langle(\nabla_{f_3}\sigma)(f_3, f_3), f_1\rangle &= \langle\nabla^\perp{}_{f_3}(\sigma(f_3, f_3)), f_1\rangle - 2\langle\sigma(\nabla_{f_3}f_3, f_3), f_1\rangle \\ &= -\langle\sigma(f_3, f_3), \nabla^\perp{}_{f_3}f_1\rangle .\end{aligned}$$

By (5.8), we get

$$\begin{aligned}\nabla^\perp{}_{f_3}f_1 &= f_1\kappa_1{}^1(f_3) + f_2\kappa_1{}^2(f_3) - \overline{f_2}\theta^3(f_3) \\ &= f_1\kappa_1{}^1(f_3) + f_2\kappa_1{}^2(f_3) - \frac{\iota}{2}\overline{f_2} .\end{aligned}$$

This yields

$$\langle(\nabla_{f_3}\sigma)(f_3, f_3), f_1\rangle = \frac{\iota\lambda}{4} .$$

Hence

$$\begin{aligned}\sigma_2 =& \frac{2}{\lambda\rho^6}F(z)f_1 \otimes (-2\sqrt{-1}\theta^3)^3 + \frac{\iota\lambda}{2}\overline{f_1} \otimes (-2\sqrt{-1}\theta^3)^3 \\ &- \frac{\iota\lambda}{2}f_1 \otimes (2\sqrt{-1}\,\overline{\theta^3})^3 + \frac{2}{\lambda\rho^6}\overline{F(z)}\,\overline{f_1} \otimes (2\sqrt{-1}\,\overline{\theta^3})^3 .\end{aligned}$$

So we get

$$\sigma_2(f_3, f_3, f_3) = \frac{2F(z)}{\lambda\rho^6}f_1 + \frac{\iota\lambda}{2}\overline{f_1} .$$

Since we can put $X = f_3e^{\iota\theta} + \overline{f_3}e^{-\iota\theta}$, then

$$\begin{aligned}\sigma_2(X, X, X) &= \sigma_2(f_3, f_3, f_3)e^{3\iota\theta} + \sigma_2(\overline{f_3}, \overline{f_3}, \overline{f_3})e^{-3\iota\theta} \\ &= \left(\frac{2}{\lambda\rho^6}F(z)f_1 + \frac{\iota\lambda}{2}\overline{f_1}\right)e^{3\iota\theta} + \left(\frac{2}{\lambda\rho^6}\overline{F(z)}\,\overline{f_1} - \frac{\iota\lambda}{2}f_1\right)e^{-3\iota\theta} \\ &= \left(\frac{2}{\lambda\rho^6}F(z)e^{3\iota\theta} - \frac{\iota\lambda}{2}e^{-3\iota\theta}\right)f_1 + \left(\frac{2}{\lambda\rho^6}\overline{F(z)}e^{-3\iota\theta} + \frac{\iota\lambda}{2}e^{3\iota\theta}\right)\overline{f_1} .\end{aligned}$$

Finally, we have

$$|\sigma_2(X, X, X)|^2 = \frac{4|F(z)|^2}{\lambda^2\rho^{12}} + \frac{\lambda^2}{4} - 2\frac{|F(z)|}{\rho^6}\sin(\alpha + 6\theta) ,$$

where $F(z) = |F(z)|e^{\iota\alpha}$. From this we see that condition (2) is equivalent to (3). Hence we get desired results.

THEOREM 9.3. *Let $x : M \to S^6$ be a compact J-holomorphic curve in S^6 with genus g. Then we have*

(1) *The immersion x is superminimal if $g = 0$.* ([Br1]).

(2) *The immersion x is superminimal or otherwise the immersion x is nowhere superminimal on M if $g = 1$.*

(3) *If* $K < 1$, $g \geq 2$ *and there exists a point at which the immersion is not superminimal, then the multiplicity of each zero of the third fundamental form is divisible by* 6 *and furthermore the equality*

$$\sum_{i=1}^{l} k_i = 12(g-1)$$

holds, where p_i $(1 \leq i \leq l)$ *are superminimal points of* x *with multiplicity* k_i.

PROOF. By Riemann Roch theorem, we can get (1) and (2). By Theorem 8.1, we may show that k_i is divisible by 6. In fact, if $F(p_i) = 0$ then we can put

$$F(z) = (z - p_i)^{k_i} g(z)$$

where $g(z) \neq 0$ on an isothermal coordinate system (U, z) centered at p_i where U is a sufficiently small simply connected neighborhood of p_i. Since we have

$$|\sigma_2(X, X, X)|^2 = \frac{4|F(z)|^2}{\lambda^2 \rho^{12}} + \frac{\lambda^2}{4} - 2\frac{|F(z)|}{\rho^6} \sin(\alpha + 6\theta)$$

where $X = (1/\rho)\{e^{\iota\theta}\partial_z + e^{-\iota\theta}\overline{\partial_z}\}$ and $F(z) = |F(z)|e^{\iota\alpha}$, where $\alpha \in \mathbf{R}\ (\mathrm{mod}\, 2\pi)$. If we put

$$e_1 = \frac{1}{\rho}\{e^{\iota\theta_0}\partial_z + e^{-\iota\theta_0}\overline{\partial_z}\}$$

where $\theta_0 = -(\alpha/6) - (\pi/12)$. Then the vector field e_1 is differentiable on $U \setminus \{p_i\}$ and satisfy

$$|\sigma_2(e_1, e_1, e_1)| = \max_{|X|=1} |\sigma_2(X, X, X)|.$$

Since

$$\alpha = -\frac{\iota}{2}\{\log F(z) - \log \overline{F(z)}\},$$

we have

$$d\theta_0 = -\frac{1}{6}d\alpha = \frac{\iota}{12}(d \log F(z) - d \log \overline{F(z)}) = \frac{\iota}{12}\left(\frac{F'(z)}{F(z)} - \overline{\frac{F'(z)}{F(z)}}\right)$$

on $U \setminus \{p_i\}$. Therefore we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial B_r} d\theta_0 &= -\frac{1}{12\pi}\int_{\partial B_r} d\alpha \\ &= -\frac{\iota}{24\pi}\left\{\int_{\partial B_r} \frac{F'(z)}{F(z)}dz - \overline{\int_{\partial B_r} \frac{F'(z)}{F(z)}dz}\right\}.\end{aligned}$$

Since $F(z)$ is a holomorphic function, we have

$$\frac{1}{2\pi\iota}\int_{\partial B_r} \frac{F'(z)}{F(z)}dz = k \in \mathbf{Z}_+.$$

So we get

$$\frac{1}{2\pi}\int_{\partial B_r} d\theta_0 = \frac{1}{6}k \in \mathbf{Z}_+. \qquad \square$$

COROLLARY 9.4. *Let $x : M \to S^6$ be a compact J-holomorphic curve in S^6 with genus g. If $K < 1/6$ and $g \geq 2$, there exist at most $2(g-1)$ superminimal points.*

PROOF. Since $K < 1/6$, superminimal points are isolated. In fact, if $|\mathbf{III}|^2 \equiv 0$, then, by (1) of Proposition 5.5 the Gauss curvature K is constant (because M is compact), and hence $K \equiv 1/6$. This contradicts the assumption. By (3) of Theorem 9.3, we get the desired result. □

References

[Bo1] J. BOLTON, F. PEDIT and L. WOODWARD, Minimal surfaces and the affine Toda field model J. Reine Angew. Math. **459** (1995), 119–150.

[Bo2] J. BOLTON, L. VRANCKEN and L. WOODWARD, On almost complex curves in the nearly Kähler 6-sphere, Quart. J. Math. **45** (1994), 407–427.

[Br1] R. L. BRYANT, Submanifolds and special structrures on the octonians, J. Differential Geom. **17** (1982), 185–232.

[Br2] R. L. BRYANT, Minimal surfaces of constant curvature in S^n, Trans. Amer. Math. Soc. **290** (1985), 259–271.

[Ch] S. S. CHERN, On the minimal immersions of the two sphere in a space of constant curvature (edited by C. Gunning), Princeton Univ. Press (1970).

[D-V-V] F. DILLEN, L. VERSTRALEN and L. VRANKEN, Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geq \frac{1}{16}$, J. Math. Soc. Japan **42** (1990), 565–584.

[E-G-T] J-H. ESCHENBURG, I. GUADAULUPE and R. TRIBUZY, Fundamental Equations of minimal surfaces in CP^2, Math. Ann. **270** (1985), 571–598.

[Gra] A. GRAY, Almost complex submanifolds of Six sphere, Proc. Amer. Math. Soc. (1969), 277–279.

[Gri] P. GRIFFITH, On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, Duke Math. J. **41** (1974), 775–814.

[H-l] R. HARVEY and H. B. LAWSON, Calibrated geometries. Acta Math. **148** (1982), 47–157.

[H] H. HASHIMOTO, Characteristic classes of oriented 6-dimensional submanifolds in the octonians, Kodai Math. J. **16** (1993), 65–73.

[H-S] H. HASHIMOTO and K. SEKIGAWA, Minimal surfaces in a 4-dimensional sphere, Houston J. Math. **21** (1995), 449–464.

[Hu] A. HUBER, On subharmonic functions and differential geometry in the large, Comment. Math. Helv. **32** (1957), 13–72.

[K] S. KOBAYASHI, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan **15** (1987), Iwanami/Princeton Univ. Press.

[Se] K. SEKIGAWA, Almost complex submanifolds of a 6-dimensional sphere, Kodai Math. J. **6** (1983), 174–185.

[Sp] M. SPIVAK, *A comprehensive introduction to differential geometry IV*, Publish or Perish (1975).

Present Address:
NIPPON INSTITUTE OF TECHNOLOGY,
MIYASHIRO, MINAMI-SAITAMA GUN, SAITAMA 345–8501, JAPAN.
e-mail: hideya@nit.ac.jp