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Remodeling a DS-diagram into one with E-cycle

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1. Introduction

B. G. Casler constructed a standard spine for a 3-manifold with boundary from the polyhedral structure, in [1]. He stated there that two 3-manifolds are homeomorphic if and only if they have a standard spine in common. Standard spines form a good subclass of the spines of 3-manifolds. Later, in [7], Ishii found a better class of spines for closed 3-manifolds. He constructed a spine by making use of a flow on the manifold and called such a spine a flow-spine. Spines of a closed manifold are understood to be the usual ones of the manifold from which a small ball is removed. It is known that the flow-spine form a good subclass of the standard spines. In this paper, we exhibit an algorithm to deform a standard spine to a flow-spine in the given closed manifold by a combinatorial topological method. It is, however, hard to see directly whether a standard spine is a flow-spine or not. By DS-diagrams (see Definition 1.1), we get rid of the difficulty. It is known in [5] that any closed 3-manifold has a DS-diagram constructed from a standard spine. The flow-spines correspond to the DS-diagrams with E-cycle, see [4] and [8]. Thus the problem above can be translated into the remodeling problem of a DS-diagram into one with E-cycle (see Definition 2.2).

The main theorem of this paper can be stated as follows (see Definition 1.2 for the notion of DS-isomorphism).

THEOREM 1.1. *Any DS-diagram is DS-isomorphic to a DS-diagram with E-cycle.*

We prove this theorem by finding a DS-isomorphism to get a DS-diagram with E-cycle algorithmically.

Including the concept of DS-isomorphism, let us review briefly some of the definitions made in [4] through [8] to understand the theorem.

Consider a 2-sphere S^2 and a connected 3-regular graph G embedded in S^2. Let V_G be the set of vertices of G. Then G induces a natural structure of cell complex $K(G)$ on S^2; 0-cells are elements of V_G, 1-cells are the connected components of $G - V_G$ and 2-cells are the connected components of $S^2 - G$. For a definition of cell complexes, see for example, [9].

DEFINITION 1.1. A triple $\Delta = (S^2, G, f)$ is called a *DS-diagram* if

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(1) G is a connected 3-regular graph embedded in S^2.

(2) For a polyhedron P with cell structure $K(P)$, f is a continuous map from S^2 onto P. f is called an *identification map* of Δ,

(3) $f : K(G) \to K(P)$ is a cellular map, that is, for each $\sigma \in K(G)$, $f\,|_\sigma$ is a homeomorphism from σ onto a cell $\lambda = f(\sigma)$ of $K(P)$,

(4) for each k-cell $\lambda^k \in K(P)$, $\sharp f^{-1}(\lambda^2) = 2$, $\sharp f^{-1}(\lambda^1) = 3$ and $\sharp f^{-1}(\lambda^0) = 4$, where $\sharp f^{-1}(\lambda^k)$ means the number of the connected components of $f^{-1}(\lambda^k)$.

We understand that the cells of $K(G)$ and $K(P)$ are oriented so that f is orientation preserving. For each cell $\sigma \in K(G)$, we call the oriented cell $f(\sigma) \in K(P)$ a *label* of σ. We often say that $f(\sigma)$ is a *k-label* of σ if $\dim \sigma = k$. Usually we say $\sigma \in K(G)$ a *cell* in Δ and $f(\sigma) \in K(P)$ a *label* in Δ.

Let $\Delta = (S^2, G, f)$ be a DS-diagram with an identification map $f : S^2 \to P$. The identification space $S^2/f = P$ is necessarily a closed fake surface (for the definition of a closed fake surface, see [2]). Let B^3 be a 3-ball with boundary $\partial B^3 = S^2$. Then the identification space B^3/f is automatically a closed 3-manifold. We will denote B^3/f by $M(\Delta)$ and call it *the manifold associated with the DS-diagram* Δ.

We explain here the terminology "DS-isomorphism" briefly, see [6] for detail. It should be remarked that if Δ' is DS-isomorphic to Δ, then a manifold $M(\Delta')$ associated with Δ' is homeomorphic to $M(\Delta)$. It is not hard to see that the replacements stated below correspond to well-known deformations of a spine keeping the manifold fixed.

DEFINITION 1.2. Let v be a 0-label of a DS-diagram Δ_1 and

$$\Sigma_1(v) = \{A^+B^+, C^+D^+, A^+C^+, B^+D^+, A^+D^+, B^+C^+\}$$

the surroundings around v in Δ_1; A^+ means the head part of an arrow indicating a 1-label A. We can consider three pairs $\{A^+B^+, C^+D^+\}$, $\{A^+C^+, B^+D^+\}$, $\{A^+D^+, B^+C^+\}$. Choose one of them, say $\{A^+B^+, C^+D^+\}$. Replacing $\Sigma_1(v)$ by

$$\Sigma_2(EF^{-1}, GH^{-1}) = \{EF^{-1}, GH^{-1}; A^+B^+, C^+D^+ \\ A^+GEC^+, B^+GFC^+, A^+HED^+, B^+HFD^+\},$$

we obtain another DS-diagram Δ_2 from Δ_1. Then $\Delta_1 \Leftrightarrow \Delta_2$ is called an *elementary deformation of Type I* (or briefly, *I-deformation*, see Figure 1-a). We use the notation $\Phi = \Phi(A^+B^+, C^+D^+) : \Delta_1 \Rightarrow \Delta_2$ and $\Phi^{-1} = \Phi^{-1}(EF^{-1}, GH^{-1}) : \Delta_2 \Rightarrow \Delta_1$. We say Φ is *of type* I^+ and Φ^{-1} is *of type* I^-.

Let A be a 1-label of a DS-diagram Δ_3 with surroundings

$$\Sigma_3(A) = \{P^+AS^-, Q^+AT^-, R^+AU^-, Q^+R^+, R^+P^+, P^+Q^+, T^-U^-, U^-S^-, S^-T^-\};$$

P^+ (or S^-) means the head part (or the tail part) of an arrow indicating a 1-label P (or S, respectively), and so on.

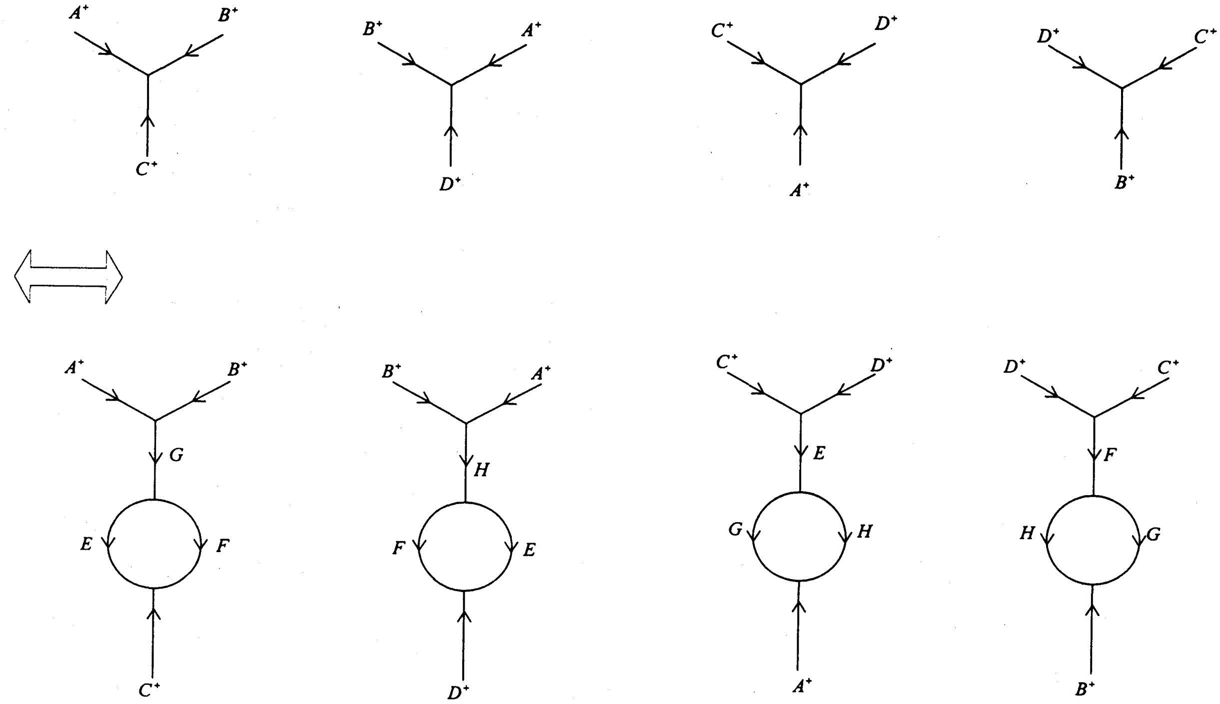


FIGURE 1-a. Elementary deformation of type I.

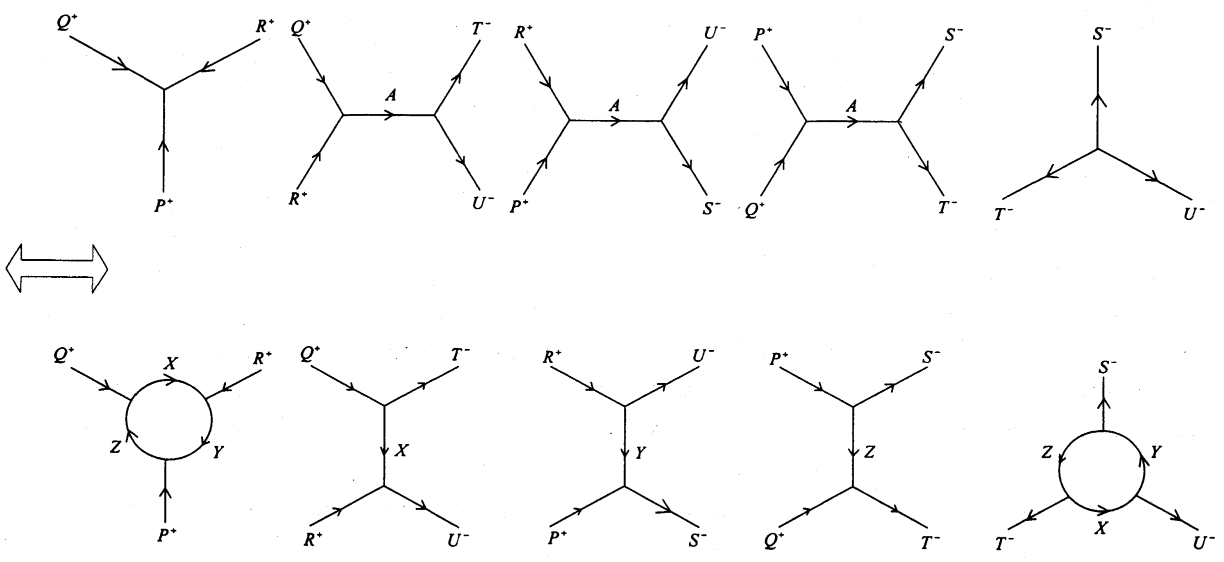


FIGURE 1-b. Elementary deformation of type II.

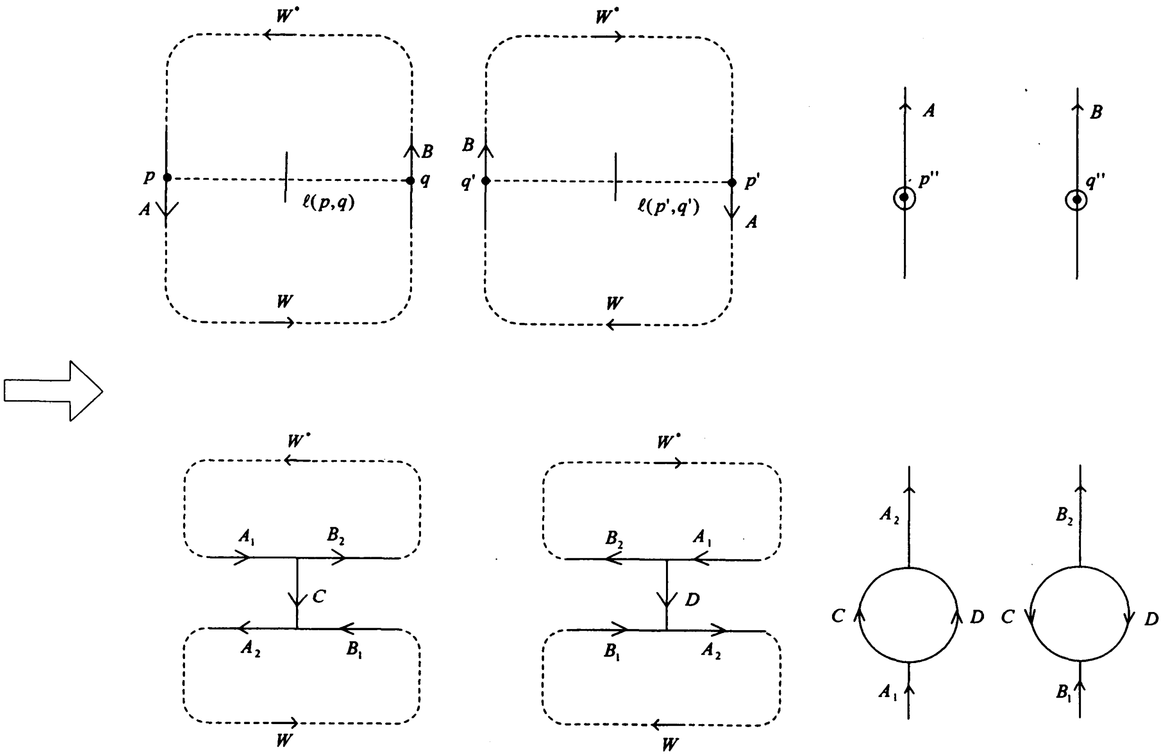


FIGURE 2. Piping.

Suppose Δ_4 be obtained from Δ_3 by only replacing only $\Sigma_3(A)$ by

$$\Sigma_4(XYZ) = \{XYZ;\ P^+S^-, Q^+T^-, R^+U^-, Q^+XR^+, R^+YP^+,\\ P^+ZQ^+, T^-XU^-, U^-YS^-, S^-ZT^-\}.$$

Then $\Delta_3 \Leftrightarrow \Delta_4$ is called an *elementary deformation of type II* (or briefly, *II-deformation*, see Figure 1-b). We use the notation $\Psi = \Psi(A) : \Delta_3 \Rightarrow \Delta_4$ and $\Psi^{-1} = \Psi^{-1}(XYZ) : \Delta_4 \Rightarrow \Delta_3$. We say Ψ is *of type* II^+ and Ψ^{-1} is *of type* II^-.

A 1-label said to be of *loop-type* if the closure is a loop, and of *arc-type* otherwise. We note that II-deformation is available if A, X, Y, Z are all 1-labels of arc-type.

A finite application of elementary deformations is called a *DS-deformation*. Suppose Δ and Δ' are DS-diagrams. We say Δ' is *DS-isomorphic to* Δ if Δ' is obtained from Δ by a DS-deformation.

The DS-isomorphism, called a *piping*, on a DS-diagram plays an important role in this paper. We explain here this deformation.

DEFINITION 1.3. Let $\Delta = (S^2, G, f)$ be a DS-diagram. Let α^+ and α^- be 2-cells in Δ with the same 2-label α. Choosing 1-cells P, Q (possibly $P = Q$) on the boundary $\partial\alpha^+$ of α^+, we can denote $\partial\alpha^+$ as

$$\partial\alpha^+ = P\tau_1\tau_2\cdots\tau_n Q\tau_1^*\tau_2^*\cdots\tau_m^*$$

where τ_i, τ_j^* are 1-cells on $\partial\alpha^+$. Choose two points $p \in P$ and $q \in Q$ so that $f(p) \neq f(q)$. Let $\ell^+(p, q)$ be a proper arc in a 2-cell α^+ joining p with q. Put $x = f(p)$, $y = f(q)$ and $\ell(x, y) = f(\ell^+(p, q))$. Let A, B, J_i, J_j^* be 1-labels of P, Q, τ_i, τ_j^* respectively; that is, $A = f(P)$, $B = f(Q)$, $J_i = f(\tau_i)$, $J_j^* = f(\tau_j^*)$. Then we can write

$$\partial\alpha = AJ_1J_2 \cdots J_nBJ_1^*J_2^* \cdots J_m^* = AwBw^*,$$

where $w = J_1J_2 \cdots J_n$ and $w^* = J_1^*J_2^* \cdots J_m^*$. Then, the surroundings around $A \cup B$ are

$$\Sigma(A \cup B) = \{AwBw^*; \cdots A \cdots, \cdots B \cdots, \cdots A \cdots, \cdots B \cdots\}.$$

Consider a DS-diagram Δ' obtained from Δ by replacing $\Sigma(A \cup B)$ by

$$\Sigma' = \{CD^{-1}, A_2wB_1, B_2w^*A_1; A_1CA_2 \cdots, \cdots B_1C^{-1}B_2 \cdots, \cdots A_1DA_2 \cdots, \cdots B_1D^{-1}B_2 \cdots\}.$$

We call $L = L(A, B) : \Delta \Rightarrow \Delta'$ a *piping along* $\ell(x, y)$, see Figure 2. We showed in [6] the fact that a manifold $M(\Delta')$ associated with Δ' is homemorphic to $M(\Delta)$. Suppose that there exists a 2-gon on a DS-diagram just like CD^{-1} in Σ'. Then we can consider the inverse $L^{-1} : \Delta' \Rightarrow \Delta$. We often use the notation $\delta(CD^{-1})$ instead of L^{-1}, and call it a *2-gon collapsing*. For detail, see [6].

2. Remodeling a DS-diagram into a splittable one.

The main purpose of this section is to show the following.

THEOREM 2.1. *For any DS-diagram Δ, there exists a splittable DS-diagram* (*see Definition* 2.1 *for "splittable"*) *which is DS-isomorphic to Δ.*

Let $\Delta = (S^2, G, f)$ be a DS-diagram. Consider a pair of 2-cells in Δ with the same label α. We denote one of them α^+ and the other α^-. In this way, we can separate whole 2-cells in Δ into two classes $\{\alpha_1^+, \alpha_2^+, \cdots, \alpha_{n+1}^+\}$ and $\{\alpha_1^-, \alpha_2^-, \cdots, \alpha_{n+1}^-\}$.

DEFINITION 2.1. The closure Z^+ of $\alpha_1^+ \cup \alpha_2^+ \cup \cdots \cup \alpha_{n+1}^+$ (or Z^- of $\alpha_1^- \cup \alpha_2^- \cup \cdots \cup \alpha_{n+1}^-$) is called the *positive zone* (or the *negative zone*, respectively). We will call (Z^+, Z^-) a *bicoloring* of the DS-diagram Δ. We will call (Z^+, Z^-) a *split bicoloring* of Δ if both of Z^+ and Z^- are connected. A DS-diagram Δ is *splittable* if Δ has a split bicoloring.

DEFINITION 2.2. Let Δ be a splittable DS-diagram with a split bicoloring (Z^+, Z^-). Let $a_1, a_2, \cdots, a_m$ be a sequence of 1-cells on a simple loop $Z^+ \cap Z^-$ such that $cl(a_1 \cup a_2 \cup \cdots \cup a_m) = Z^+ \cap Z^-$, where $cl(X)$ means the closure of X. Let A_i be the label of a_i, $1 \leq i \leq m$. Then we say that $\Lambda = A_1A_2 \cdots A_m$ is a *splitting cycle* of Δ associated with (Z^+, Z^-). We will call a splitting cycle $\Lambda = A_1A_2 \cdots A_m$ an *E-cycle* if $A_i \neq A_j$ for each $i \neq j$.

DEFINITION 2.3. Let σ be a cell in Δ. We say that σ is *positive* if $\sigma \subset \operatorname{Int} Z^+$, *negative* if $\sigma \subset \operatorname{Int} Z^-$, and *neutral* if $\sigma \subset Z^+ \cap Z^-$. We will use the notation $\nu(\alpha)$ for the number of neutral cells with label α.

It is easy to see the following.

PROPOSITION 2.1. *If α is a 2-label, then $\nu(\alpha) = 0$. If A is a 1-label, then $\nu(A)$ is either 1 or 3. If v is a 0-label, then $2 \leq \nu(v) \leq 4$.*

DEFINITION 2.4. A 1-label A appearing in Δ is said to be *bordered* with respect to (Z^+, Z^-) if each of three 1-cells with the label A is neutral. Otherwise, A is said to be *distributed* with respect to (Z^+, Z^-). Note that for a distributed 1-label A, there are three 1-cells with the 1-label A such that one of them is positive, another is negative and the other is neutral.

We will introduce a new DS-deformation, named digging.

DEFINITION 2.5. Consider a DS-diagram Δ with a bicoloring (Z^+, Z^-). Let σ and τ be 1-cells in a connected component R to Z^+. Suppose $A = f(\sigma)$ and $B = f(\tau)$, where f is the identification map associated with a DS-diagram Δ. Let $p \in \sigma$ and $q \in \tau$ be two points chosen so that $f(p) \neq f(q)$. Then there is a simple arc $\ell^+ = \ell_0^+ \cup \ell_1^+ \cup \ell_2^+ \cup \cdots \cup \ell_m^+$ transverse to 1-cells in R such that

(1) each ℓ_r^+ is a directed arc with the initial point p_{r-1} and the terminal point p_r, where $p_0 = p$ and $p_m = q$,

(2) $\ell_r^+ \cap \ell_{r+1}^+ = p_r$,

(3) the interior $\mathrm{Int}\,\ell_r^+$ of ℓ_r^+ is in a positive 2-cell for each r, and p_r is in a positive 1-cell if $r \neq 0, m$,

(4) $f(p_r) \neq f(p)$ and $f(q)$ if $r \neq 0, m$.

We will call such a simple arc $\ell^+ = \ell_0^+ \cup \ell_1^+ \cup \ell_2^+ \cup \cdots \cup \ell_m^+$ a *mark-line joining σ with τ*. Note that, for each simple arc ℓ_r^+, there is the spouse ℓ_r^- of ℓ_r^+ so that $f(\ell_r^+) = f(\ell_r^-)$ and $\ell_r^- \cap (\ell_0^+ \cup \ell_1^+ \cup \ell_2^+ \cup \cdots \cup \ell_m^+) = \emptyset$.

See Figure 3. Figure 3-c is obtained from Figure 3-a via Figure 3-b. We will say $d(\ell^+)$: Figure 3-a $\Rightarrow$ Figure 3-c is a *digging along a mark-line* ℓ^+, or simply a *digging* if there is no confusion.

THEOREM 2.2. *A digging is a DS-deformation.*

PROOF. Suppose $d(\ell^+)$ is a digging along a mark-line $\ell^+ = \ell_0^+ \cup \ell_1^+ \cup \ell_2^+ \cup \cdots \cup \ell_m^+$. If $m = 0$, $d(\ell^+) = d(\ell_0^+)$ is nothing but a piping, and hence $d(\ell^+)$ is a DS-deformation. If $m = 1$, $d(\ell^+)$ is established by applying pipings twice. In general, $d(\ell^+)$ is a consequence of $m + 1$ times of applications of piping. □

Suppose $\{R_1, R_2, \cdots, R_p\}$ and $\{S_1, S_2, \cdots, Sq\}$ are the set of connected components of Z^+ and Z^-. We will call R_i a *positive region* and S_j a *negative region*. In this situation, we will denote $\rho(Z^+, Z^-) = p+q$. Note that regions R_i and S_j are 2-disks with or without holes since a DS-diagram Δ is a diagram on a 2-sphere S^2 associated with a 3-regular connected graph. Hence at least one element of $\{R_i\} \cup \{S_j\}$ is a 2-disk. Without loss of generality we may assume S_q is a 2-disk. Suppose R_p is the positive region adjacent to S_q. If $p + q \geq 3$, there is another negtive region, say S_{q-1}, adjacent to R_p.

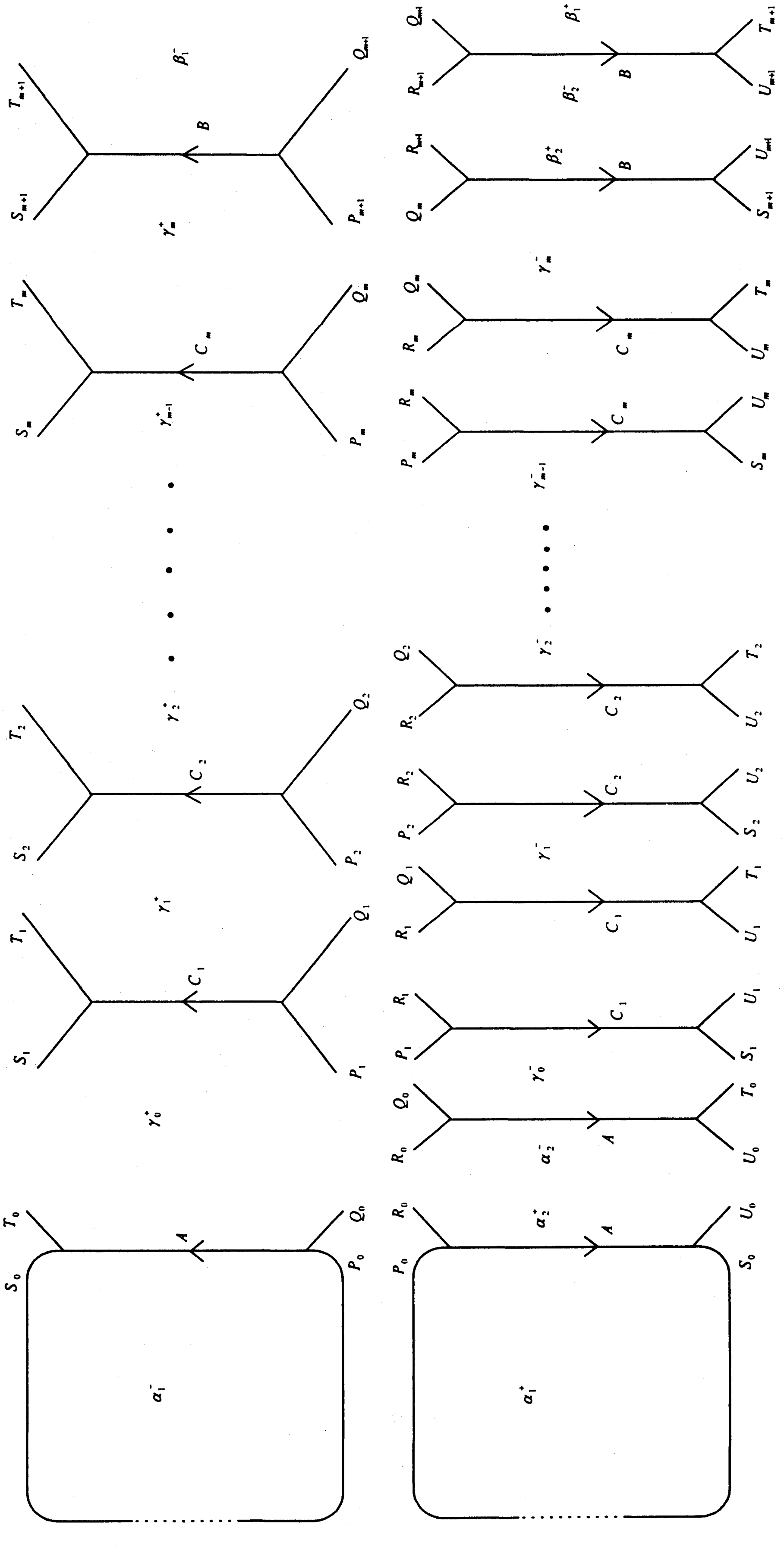


FIGURE 3-a. Before applying a digging $d(\ell^+)$.

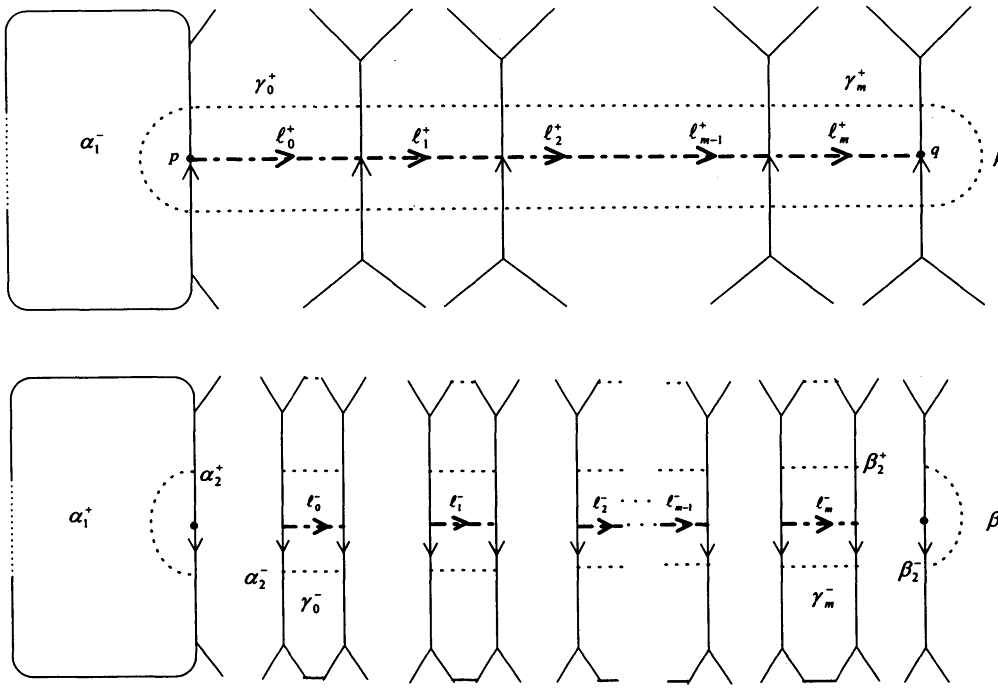


FIGURE 3-b. Mark line ℓ^+.

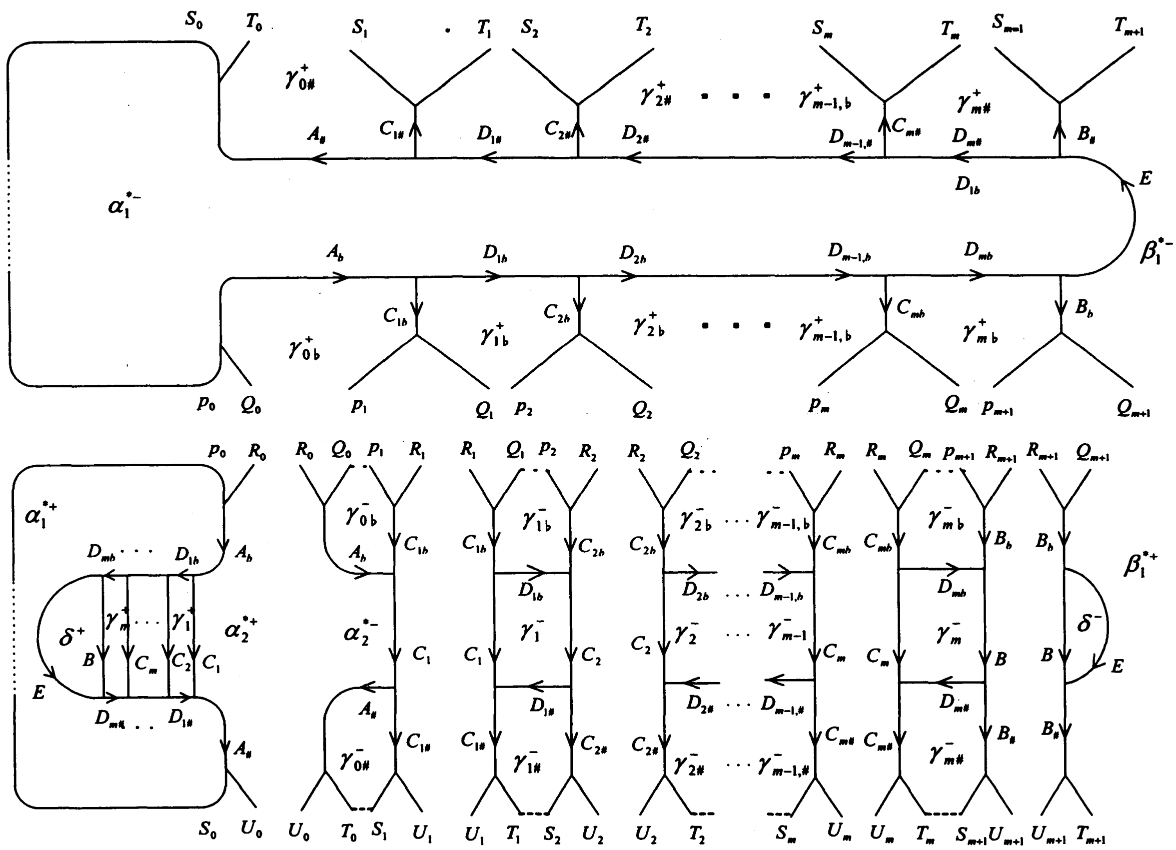


FIGURE 3-c. After applying a digging $d(\ell^+)$.

LEMMA 2.1. *Suppose $p+q \geq 3$. Let $\sigma \subset R_p \cap S_q$ and $\tau \subset R_p \cap S_{q-1}$ be neutral 1-cells. Suppose the label of σ is distributed and the label of τ is bordered. Then there is a DS-diagram Δ' with a bicoloring (Z'^+, Z'^-) such that Δ' is DS-isomorphic to Δ and $\rho(Z'^+, Z'^-) = p+q-1$.*

PROOF. Applying a digging along a proper arc in R_p joining 1-cells σ and τ, we will be able to obtain a required DS-diagram Δ' as follows. Let A and B be 1-labels of σ and τ. Suppose $\alpha_1^{\pm}, \alpha_2^{\pm}, \beta_1^{\pm}, \beta_2^{\pm}, \gamma_r^{\pm}$ $(0 \leq r \leq m)$ are 2-cells in Δ as shown in Figure 3-a where $\alpha_1^- \subset S_q$, $\gamma_0^+ \cup \gamma_1^+ \cup \cdots \cup \gamma_m^+ \subset R_p$, $\beta_1^- \subset S_{q-1}$. Let ℓ^+ be the closure of $\ell_0^+ \cup \ell_1^+ \cup \ell_2^+ \cdots \cup \ell_m^+$ which is a proper arc in the closure of $\gamma_0^+ \cup \gamma_1^+ \cup \cdots \cup \gamma_m^+$ joining a point $p \in \sigma$ to a point $q \in \tau$ (see Figure 3-b). By a digging $d(\ell^+)$ along ℓ^+, α_i and β_i $(i = 1, 2)$ are replaced by α_i^* and β_i^*, and γ_i is replaced by two 2-labels $\gamma_{i\sharp}$ and $\gamma_{i\flat}$ and further, a new 2-label δ with $\partial\delta = BE^{-1}$ is born. Note that $\gamma_{0\sharp}^+ \cup \gamma_{1\sharp}^+ \cup \cdots \cup \gamma_{m\sharp}^+$ and $\gamma_{0\flat}^+ \cup \gamma_{1\flat}^+ \cup \cdots \cup \gamma_{m\flat}^+$ are in the same region, say R_p', since S_q is a 2-cell. We can see the resulting DS-diagram Δ' (Figure 3-c) has a natural bicoloring (Z'^+, Z'^-) such that $\rho(Z'^+, Z'^-) = p+q-1$. □

THEOREM 2.3. *See Figure 4. Suppose Δ_1 is a DS-diagram with a symbolic representation, see [6],*

$$\{\cdots B \cdots, \cdots B \cdots, \cdots B \cdots, \cdots\cdots\},$$

where B is a distributed 1-label. Suppose Δ_2 is a DS-diagram with a symbolic representation

$$\{\partial\zeta, \partial\eta, \partial\lambda, \partial\mu, \partial\nu, \partial\gamma_m, \partial\beta_1, \partial\beta_2, \cdots\cdots\},$$

where

$$\partial\zeta = B_* P_1 P_2 B_* U V, \quad \partial\eta = B_* Q_1 Q_2,$$
$$\partial\lambda = T^{-1} S^{-1}, \quad \partial\mu = P_2^{-1} T Q_2, \quad \partial\nu = Q_1 S P_1^{-1},$$
$$\partial\gamma_m = \cdots B_1 V Q_2^{-1} S P_2 V^{-1} B_2 \cdots,$$
$$\partial\beta_1 = \cdots B_1 U^{-1} P_1 T Q_1^{-1} U B_2 \cdots,$$
$$\partial\beta_2 = \cdots B_2^{-1} B_1^{-1} \cdots.$$

Then Δ_1 and Δ_2 are DS-isomorphic to each other.

PROOF. Applying II$^-$-deformation $\Psi^{-1}(\eta)$ to Δ_2, we obtain a DS-diagram Δ_3 with a symbolic representation:

$$\{P_1 P_2 U V, T^{-1} S^{-1} W_*^{-1}, T P_2^{-1}, S P_1^{-1}; \cdots B_1 V S P_2 W_* V^{-1} B_2 \cdots,$$
$$\cdots B_1 U^{-1} W_* P_1 T U B_2 \cdots, \cdots B_2^{-1} B_1^{-1} \cdots\}.$$

Then applying 2-gon collapsing $\delta(SP_1^{-1})$ to Δ_3, we obtain a DS-diagram Δ_4 with a symbolic representation:

$$\{W_{**}^{-1}, U V_*; \cdots B_1 V_* W_{**} V_*^{-1} B_2 \cdots, \cdots B_1 U^{-1} W_{**} U B_2 \cdots, \cdots B_2^{-1} B_1^{-1} \cdots\}. \quad □$$

Again applying 2-gon collapsing $\delta(UV_*)$ to Δ_4, we obtain Δ_1. Hence Δ_1 and Δ_2 are DS-isomorphic to each other.

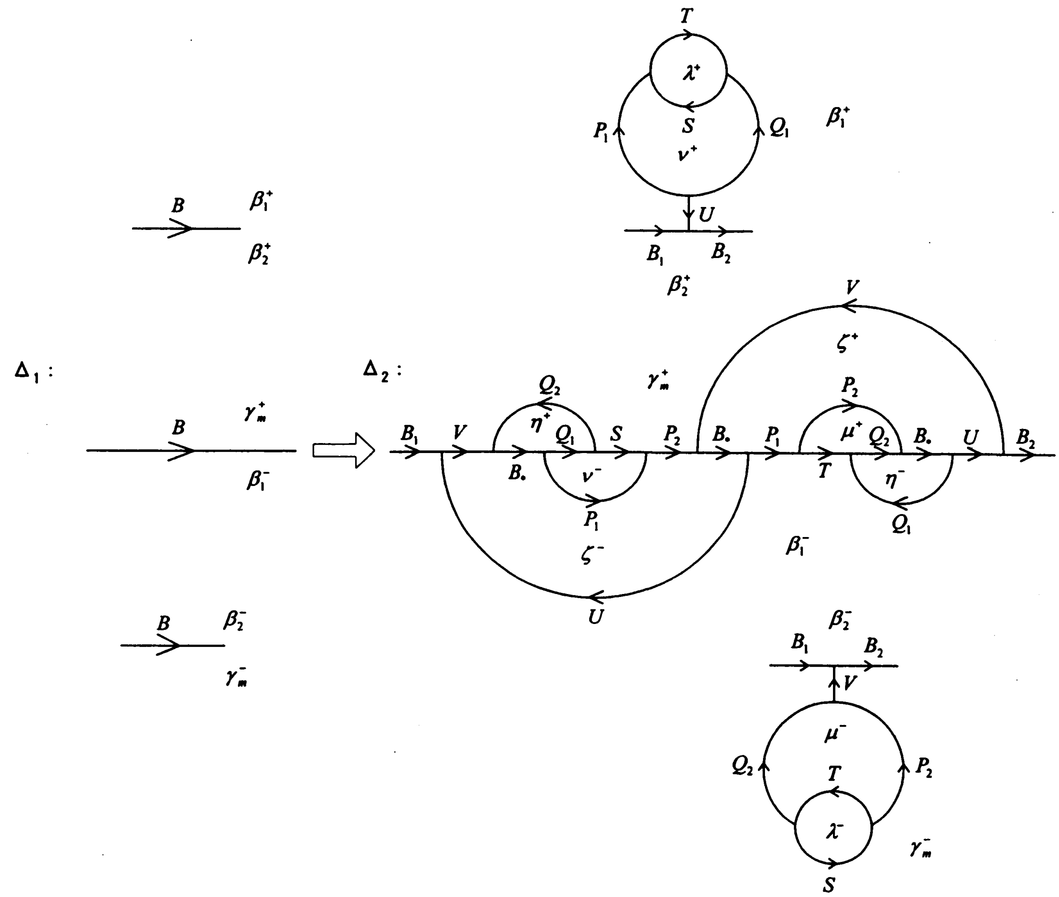


FIGURE 4. Deformation to replacing a distributed 1-label by a bordered one.

PROOF OF THEOREM 2.1. We will establish our proof by the induction on $n = \rho(Z^+, Z^-)$. If $n = 2$, then Δ is already a splittable DS-diagram. Hence we assume $n \geq 3$. And we show that there is a DS-diagram Δ' with a bicoloring (Z'^+, Z'^-) such that Δ' is DS-siomorphic to Δ and $\rho(Z'^+, Z'^-) = n - 1$. It is enough to consider the situation that

(1) $\{R_1, R_2, \cdots, R_p\}$ and $\{S_1, S_2, \cdots, S_q\}$ are connected components of Z^+ and Z^-, and $p + q = n$,

(2) S_q is a 2-disk, and R_p is the positive region which is adjacent to both of S_q and S_{q-1}.

We will attempt to replace S_q and S_{q-1} by a new negative region S'_{q-1} through DS-deformation on Δ.

Step 1. In this step, we show that we can assume there is a neutral 1-cell in $S_q \cap R_p$ with a distributed 1-label A. If there is no such 1-cell in $S_q \cap R_p$, we claim that we can change Δ to Δ_* with a bicoloring (Z_*^+, Z_*^-) having the regions $\{R_{*1}, R_{*2}, \cdots, R_{*p}\}$ and $\{S_{*1}, S_{*2}, \cdots, S_{*q}\}$ such that

(1) S_{*q} is a 2-disk and R_{*p} is adjacent to S_{*q},

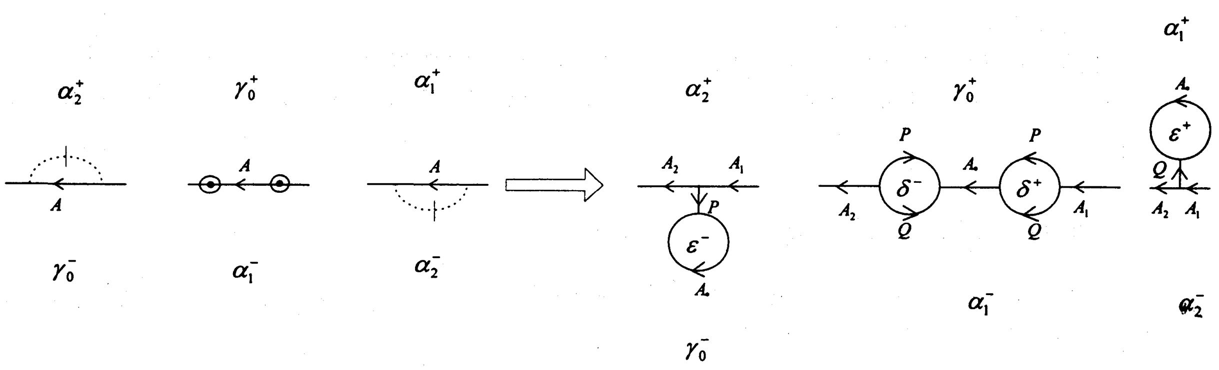


FIGURE 5. Deformation replacing a bordered 1-label by a distributed one.

(2) $S_{*q} \cap R_{*p}$ contains a neutral 1-cell with a distributed 1-label, say A_*.

We can construct Δ_* as follows. Choose one of the 1-cells, say σ, in $S_q \cap R_p$ and suppose the 1-label $A = f(\sigma)$ of σ is bordered. Let $\alpha_1, \alpha_2, \gamma_0$ be 2-labels of Δ such that $A \subset \partial\alpha_1, \partial\alpha_2, \partial\gamma_0$. Some of $\alpha_1, \alpha_2, \gamma_0$ may possibly be coincide together. Since A is bordered, we obtain the left of Figure 5, especially we may assume $\alpha_1^- \subset S_q$.

Suppose ℓ is a proper arc in α_2 joining two points on A. Carrying out the piping $L_0(A) : \Delta \to \Delta_*$ along ℓ, we obtain

$$\Delta_* = \{A_*, PQ^{-1}; A_1^+ PA_*P^{-1}A_2^-, A_1^+ QA_*Q^{-1}A_2^-, A_1^+A_2^-\}$$

which is DS-isomorphic to Δ. This DS-diagram Δ_* has a natural bicoloring (Z_*^+, Z_*^-) such that $\rho(Z_*^+, Z_*^-) = p + q$ and $\alpha_1^- \cup \delta^- \subset S_q$, where δ^- is a 2-cell with 2-label δ so that $\partial\delta = PQ^{-1}$. A 1-label A_* is a distributed one on a new $S_{*q} \cap R_{*p}$ with respect to (Z_*^+, Z_*^-).

Step 2. Suppose Δ is a DS-diagram with a bicoloring (Z^+, Z^-) having the regions $\{R_1, R_2, \cdots, R_p\}$ and $\{S_1, S_2, \cdots, S_q\}$, $p + q = n$, such that

(1) S_q is a 2-disk and R_p is adjacent to S_q,

(2) $S_q \cap R_p$ contains a neutral 1-cell with a distributed 1-label, and

(3) S_{q-1} is adjacent to R_p.

We want to find a neutral 1-cell in $R_p \cap S_{q-1}$ with bordered 1-label. Suppose there is no such 1-cell in $R_p \cap S_{q-1}$. Choose an arbitrary 1-cell, say τ, in $R_p \cap S_{q-1}$. Then the 1-label $B = f(\tau)$ of τ does not appear in $S_q \cap R_p$ since B is distributed and $\tau \subset R_p \cap S_{q-1}$ is neutral. Applying the DS-deformation in Theorem 2.3 to the 1-label B, we obtain the required DS-diagram Δ_* and a new 1-label B_*.

Step 3. From the argument of Step 1 and Step 2, if necessary, we can seek for a DS-diagram Δ_* with a bicoloring (Z_*^+, Z_*^-) having the regions $\{R_{*1}, R_{*2}, \cdots, R_{*p}\}$ and $\{S_{*1}, S_{*2}, \cdots, S_{*q}\}$, $p + q = n$, such that

(1) S_{*q} is a 2-disk and R_{*p} is adjacent to S_{*q},

(2) $S_{*q} \cap R_{*p}$ contains a neutral 1-cell σ_* with a distributed 1-label,

(3) $S_{*,q-1}$ is adjacent to R_p,

(4) $R_{*p} \cap S_{*,q-1}$ contains a neutral 1-cell τ_* with a bordered 1-label,
(5) Δ_* is DS-isomorphic to Δ.

To Δ_* applying the digging along a proper arc in R_{*p} joining σ_* and τ_*, we will obtain a DS-diagram Δ' with a bicoloring (Z'^+, Z'^-) such that $\rho(Z'^+, Z'^-) = n-1$. This completes the proof. □

3. Remodeling a splittable DS-diagram.

In this section, let Δ be a splittable DS-diagram with (Z^+, Z^-), and Λ a splitting cycle of Δ associated with (Z^+, Z^-).

PROPOSITION 3.1. *Suppose A is an arbitrary 1-label of Δ. Then A appears on Λ exactly once if A is distributed, and exactly three if A is bordered.*

PROOF. It is an obvious since a 1-cell σ with 1-label A is contained in $Z^+ \cap Z^-$ if and only if σ is neutral. □

PROPOSITION 3.2. *If $\Lambda = \cdots AA \cdots$, then we have $\Lambda = \cdots AAA \cdots$.*

PROOF. It is easy to see that a DS-diagram contains a path AAA if it contains a path AA. And further, A is bordered since A appears on Λ at least twice as AA. Therefore the third edge with 1-label A is also contained in Λ. □

DEFINITION 3.1. A splitting cycle $\Lambda = A_1A_2\cdots A_m$ is called a *good cycle* if $A_i \neq A_{i+1}$ for each i (mod m).

LEMMA 3.1. *There is a splittable DS-diagram Δ_* with a good cycle such that Δ_* is DS-isomorphic to Δ.*

PROOF. If a splitting cycle Λ does not contain any adjacent 1-cells with the same 1-label, Λ is already a good cycle. Suppose not. Then there is a 1-label A so that $\Lambda = w_1AAAw_2$. In this case, there are two kind of surroundings of A (Figure 6-a):

$$\Sigma_1(A) = \{X^+AAY^-, Y^-AX^+, Y^-X^+\}, \quad \text{and}$$
$$\Sigma_2(A) = \{Y^-AAX^+, X^+AY^-, Y^-X^+\}.$$

Note that $X^\pm \neq A$ and $Y^\pm \neq A$, but possibly $Y = X^\pm$. For each case, we claim that we can obtain a DS-diagram Δ_* with a splitting cycle $\Lambda_* = w_1XAPRAQSAYw_2$. For $\Sigma_1(A)$, if we apply an elementary deformation $\Phi = \Phi(X^+A^-, A^+Y^-)$ of type I^+, we obtain

$$\Phi(\Sigma_1(A)) = \{PQ^{-1}, RS^{-1}, X^+AQRAY^-, Y^-QSAPRX^+, Y^-PSX^+\},$$

(see Figure 6-b). It is similar to $\Sigma_2(A)$. By repeating this operation, we can obtain a splittable DS-diagram Δ_* with a good cycle. □

As we saw in §2, any DS-diagram is DS-isomorphic to a splittable one with a splitting cycle Λ. Furthermore, by the previous lemma, we may assume Λ is a good cycle, that is, Λ does not contain AA, where A is a 1-label. We will start with this situation. Note that Λ consists of distributed 1-labels (each of them appears exactly once on Λ) and arc-type

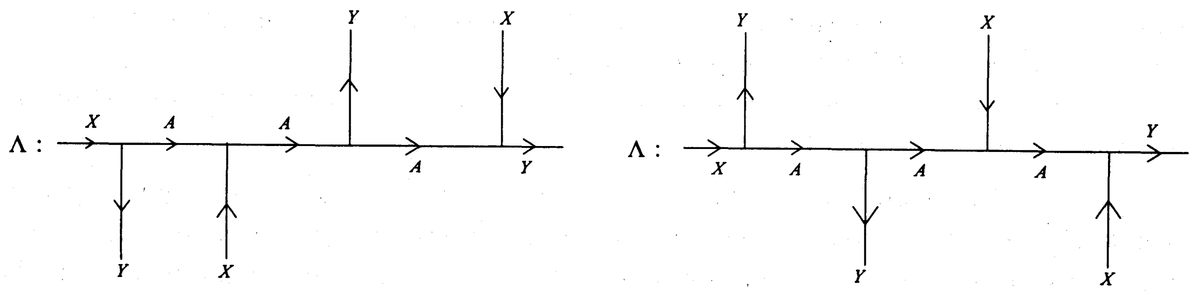


FIGURE 6-a. Surroundings of a 1-label A with AA on Λ.

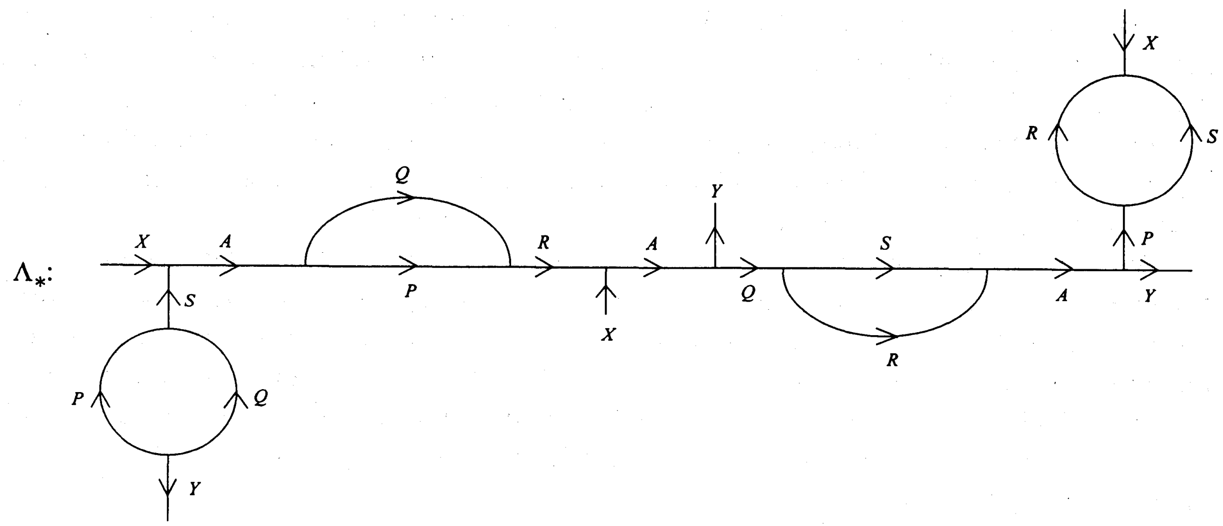


FIGURE 6-b. $\Phi(\Sigma_1(A))$.

bordered 1-labels (each of them appears just three times on Λ), see [6]. We will eliminate these bordered 1-labels from Λ step by step. Then, a resulting splitting cycle Λ_* will not contain any bordered 1-label, and hence Λ_* will be automatically an E-cycle of a DS-diagram which is DS-isomorphic to the original DS-diagram.

PROPOSITION 3.3. *Suppose τ is a neutral 0-cell in Δ and $\sigma_1, \sigma_2, \sigma_3$ are 1-cells incident to τ. Then either $\sigma_1, \sigma_2, \sigma_3 \in Z^+$ or $\sigma_1, \sigma_2, \sigma_3 \in Z^-$ holds; τ is said to be positive handed if $\sigma_1, \sigma_2, \sigma_3 \in Z^+$, and negative handed otherwise.*

PROOF. Since τ is neutral, $\tau \subset Z^+ \cap Z^-$, and two of $\sigma_1, \sigma_2, \sigma_3$ are also on $Z^+ \cap Z^-$. Hence either $\sigma_1, \sigma_2, \sigma_3 \in Z^+$ or $\sigma_1, \sigma_2, \sigma_3 \in Z^-$ holds. □

Remember $\nu(\alpha)$ means the number of the neutral cells with the label α.

PROPOSITION 3.4. *Let v be a 0-label of Δ. Then $2 \leq \nu(v) \leq 4$.*

(1) *Suppose $\nu(v) = 2$. Then one of the neutral 0-cells with label v is positive handed and the other is negative handed.*

(2) *Suppose $\nu(v) = 3$. Then three neutral 0-cells with label v are either positive handed all or negative handed all. If these neutral 0-cells are positive (negative) handed, the non-neutral 0-cells with label v is negative (positive).*

(3) *Suppose $\nu(v) = 4$. Then two of the 0-cells with label v are positive handed and the other two are negative handed.*

PROOF. For an arbitrary 0-label v in Δ, there are four 0-cells with the same 0-label v. Each 0-cell has three 1-cells as hands, and has three corners, see [6]. Hence there are twelve angles corresponding to six corners for a 0-label v. Six of these angles are in Z^+ and the others are in Z^-. This fact leads us to the proof. □

Let Λ be a good cycle. Let A be a bordered 1-label with the initial 0-label x and the terminal 0-label y. $x \neq y$ and that three 1-cells with label A are all on the good cycle Λ. Let σ be a 1-cell with label A and p, q be the initial and the terminal 0-cell of σ; that is, $f(\sigma) = A$, $f(p) = x$ and $f(q) = y$. Then there are four types of σ on Λ (Figure 7). If p and q are both positive handed (negative handed), then we say σ is of type U^+ (type U^-). If p is negative handed (positive handed) and q is positive handed (negative handed), then we say σ is of type N^+ (type N^-, respectively).

We will consider patterns of the intersection of good cycle Λ and surroundings around A. Note that $3 \leq \nu(x), \nu(y) \leq 4$. Hence $(\nu(x), \nu(y))$ is one of (3, 3), (3, 4), (4, 3), (4, 4). Note that (3, 4) and (4, 3) are essentially of the same type. Hence we get typical eight patterns (see Figure 8). They are denoted by $(\nu(x), \nu(y) : N(A), U(A))$; where $N(A)$ is the number of 1-cells, with label A, of type N^+ and N^-, and $U(A)$ the number of 1-cells with type U^+ and U^-. The list is: (3, 3 : 0, 3), (3, 3 : 3, 0), (4, 3 : 1, 2), (4, 3 : 2, 1), (4, 4 : 0, 3), (4, 4 : 1, 2), (4, 4 : 2, 1), (4, 4 : 3, 0).

DEFINITION 3.2. A good cycle Λ is called a *better cycle* if $(\nu(x), \nu(y) : N(A), U(A))$ = (3, 3 : 0, 3) for each bordered 1-label A.

$b(\Delta)$ means the number of bordered 1-labels of Δ.

THEOREM 3.1. *Let Λ be a good cycle and $b(\Delta) = k$. If there is a bordered 1-label whose pattern in not (3, 3 : 0, 3), we can reform Δ to a DS-diagram Δ_* with a good cycle and $b(\Delta_*) = k - 1$.*

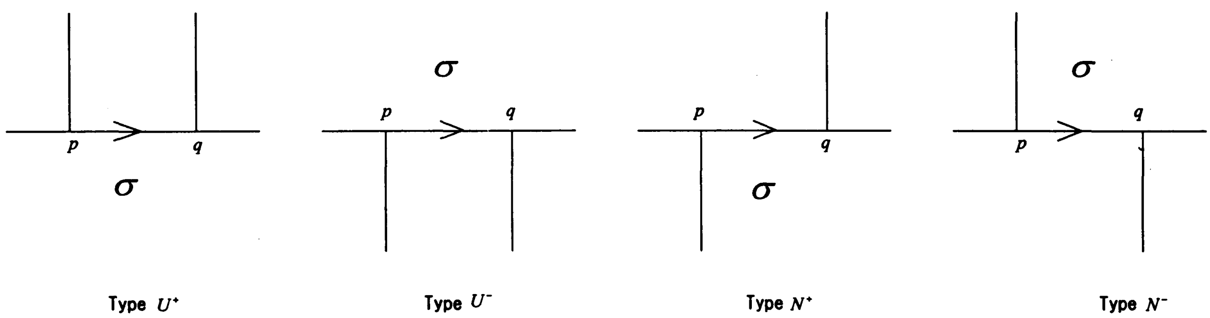
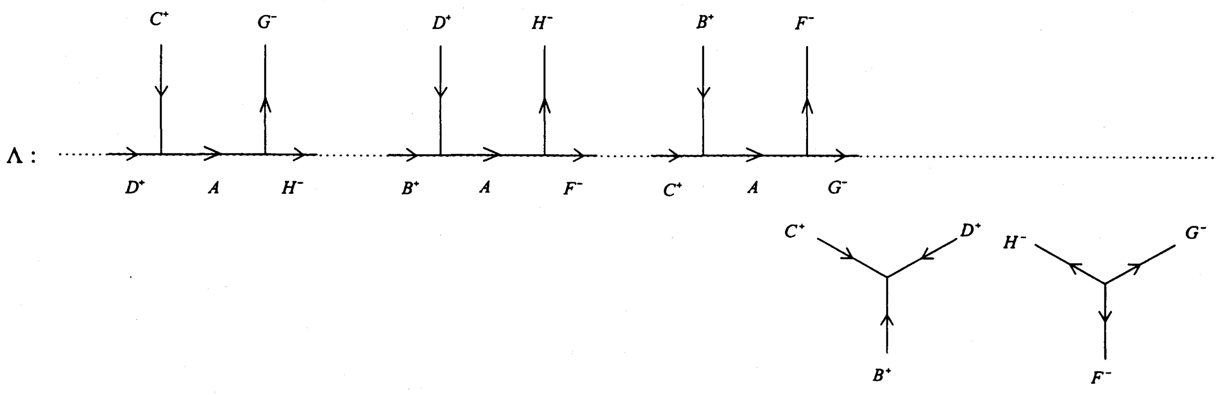
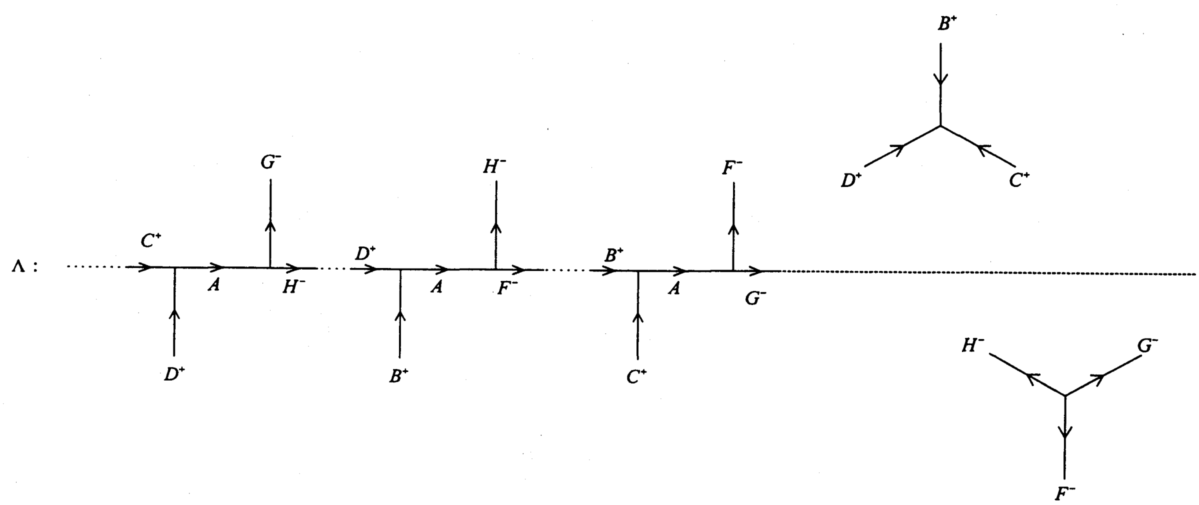


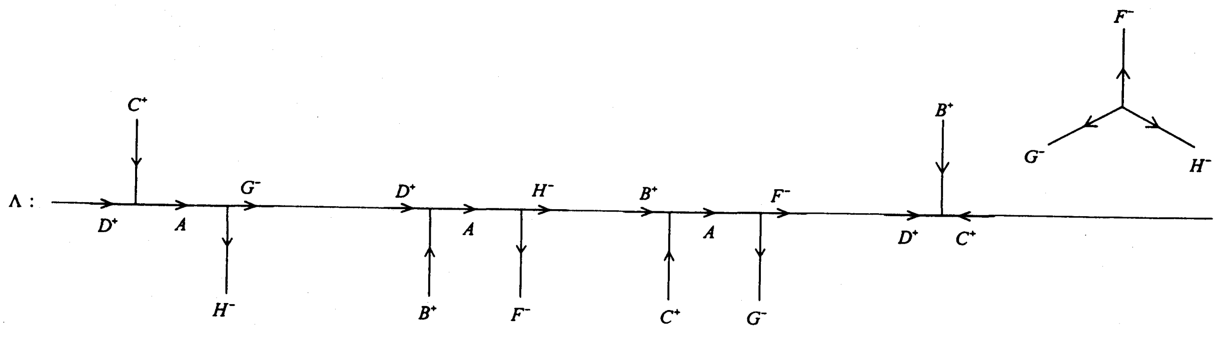
FIGURE 7. Type of 1-cell on Λ.



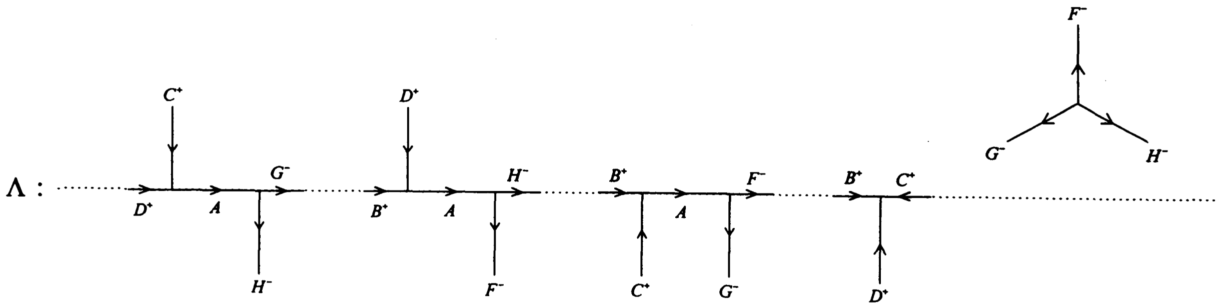
Pattern 0: (3, 3 : 0, 3).



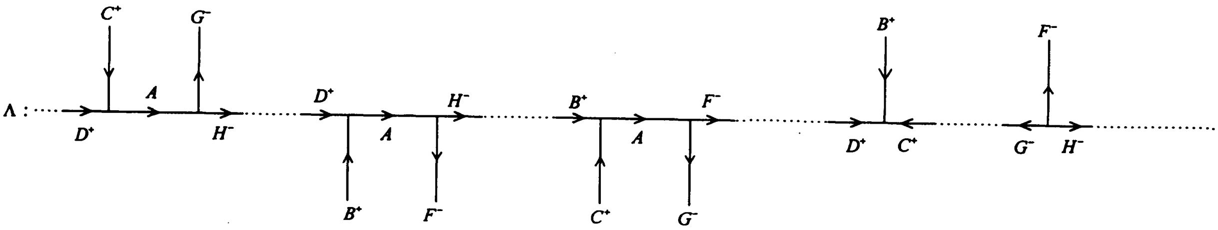
Pattern 1: (3, 3 : 3, 0).



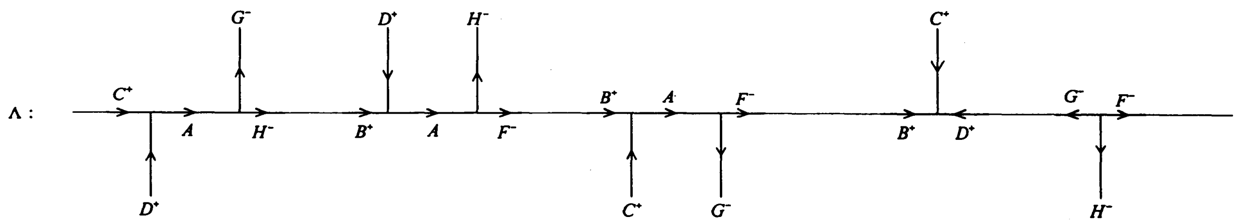
Pattern 2: (4, 3 : 1, 2).



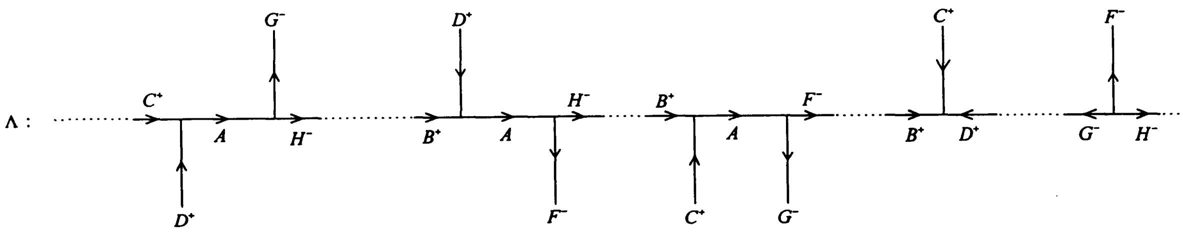
Pattern 3: (4, 3 : 2, 1).



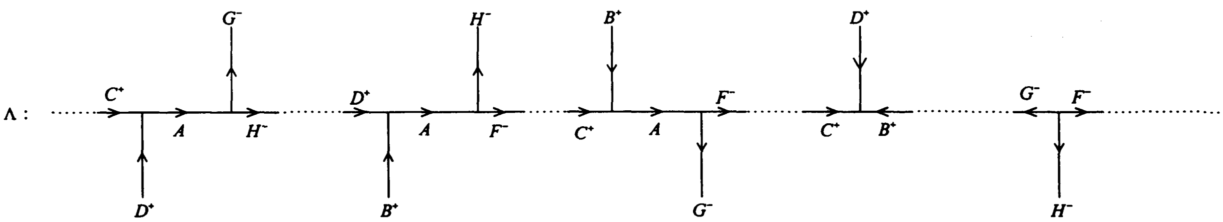
Pattern 4: (4, 4 : 0, 3).



Pattern 5: (4, 4 : 1, 2).



Pattern 6: (4, 4 : 2, 1).



Pattern 7: (4, 4 : 3, 0).

FIGURE 8. Eight patterns of bordered 1-labels on a good cycle Λ.

PROOF. Suppose A is a bordered 1-label whose pattern is not $(3, 3 : 0, 3)$. For each pattern, we can show the above statement by applying elementary deforamtion of type II^+ once or twice. Suppose $\Sigma(A)$ is the surroundings of A. We may describe it as

$$\Sigma(A) = \{B^+AF^-, C^+AG^-, D^+AH^-, B^+C^+, C^+D^+, D^+B^+, F^-G^-, G^-H^-, H^-F^-\}.$$

Suppose x and y are an initial and a terminal 0-label of A. Since $x \neq y$, we can apply an elementary deformation $\Psi(A)$ of type II^+ to our 1-label A. Then we obtain

$$\Sigma_* = \{PQR; B^+F^-, C^+G^-, D^+H^-, B^+RC^+, C^+PD^+, D^+QB^+, F^-RG^-, G^-PH^-, H^-QF^-\}.$$

Let Δ_* be the DS-diagram associated with Σ_*.

Pattern 1. Suppose A is of $(3, 3 : 3, 0)$. We may assume a good cycle Λ of Δ is

$$\Lambda : \cdots C^+AH^- \cdots D^+AF^- \cdots B^+AG^- \cdots.$$

Then

$$\Lambda_* : \cdots C^+PH^- \cdots D^+QF^- \cdots B^+RG^- \cdots$$

is obviously a good cycle of Δ_*, where $b(\Delta_*) = k - 1$ (Figure 9).

Pattern 2. Suppose A is of $(4, 3 : 1, 2)$. By $\Psi(A)$, a good cycle

$$\Lambda : \cdots D^+AG^- \cdots D^+AH^- \cdots B^-AF^- \cdots D^+C^+ \cdots$$

of Δ is changes to

$$\Lambda_* : \cdots D^+P^{-1}G^- \cdots D^+H^- \cdots B^+F^- \cdots D^+QRC^+ \cdots$$

which is a good cycle of Δ_* with $b(\Delta_*) = k - 1$ (Figure 10).

By the similar arguments, we can deal with patterns $(4, 3 : 2, 1)$, $(4, 4 : 0, 3)$, $(4, 4 : 1, 2)$. For these patterns, we list here only the good cycles Λ and Λ_* of Δ and Δ_*, where $b(\Delta) = k$ and $b(\Delta_*) = k - 1$.

Pattern 3. $(4, 3 : 2, 1)$.

$$\Lambda : \cdots D^+AG^- \cdots B^+AH^- \cdots B^+AF^- \cdots B^+C^+ \cdots,$$

$$\Lambda_* : \cdots D^+PG^- \cdots B^+QH^- \cdots B^+F^- \cdots B^+Q^{-1}C^+ \cdots.$$

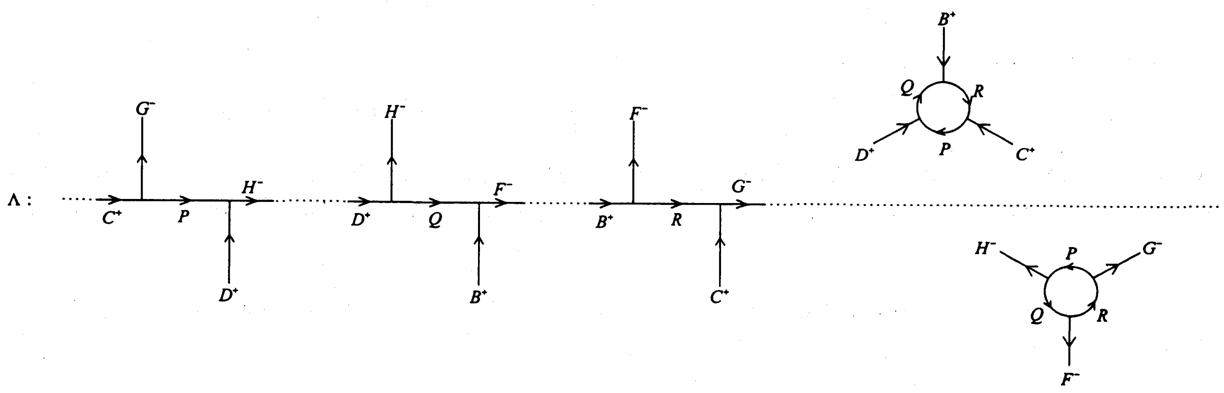


FIGURE 9. The case of pattern $(3, 3 : 3, 0)$.

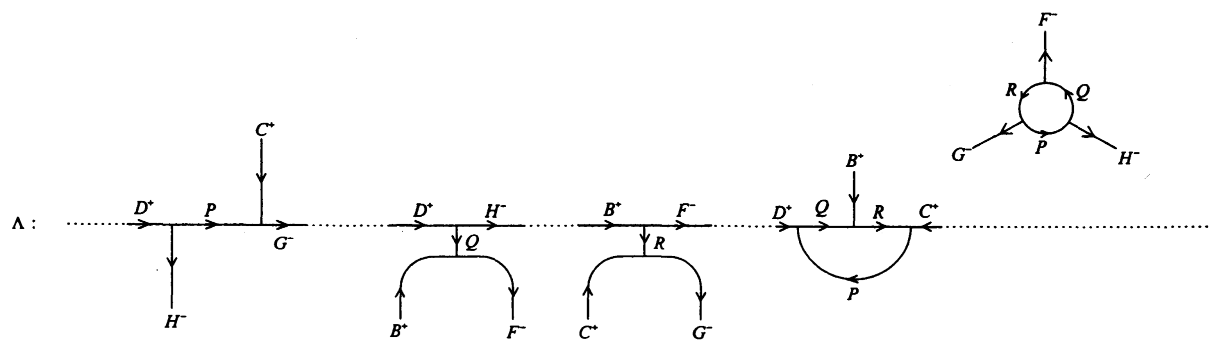


FIGURE 10. The case of pattern (4, 3 : 1, 2).

Pattern 4. (4, 4 : 0, 3).

$$\Lambda : \cdots D^+AH^- \cdots D^+AH^- \cdots B^+AF^- \cdots D^+C^+ \cdots G^-H^- \cdots ,$$

$$\Lambda_* : \cdots D^+H^- \cdots D^+H^- \cdots B^+F^- \cdots D^+P^{-1}C^+ \cdots G^-R^{-1}Q^{-1}H^- \cdots .$$

Pattern 5. (4, 4 : 1, 2).

$$\Lambda : \cdots C^+AH^- \cdots B^+AF^- \cdots B^+AF^- \cdots B^+D^+ \cdots G^-F^- \cdots ,$$

$$\Lambda_* : \cdots C^+PH^- \cdots B^+F^- \cdots B^+F^- \cdots B^+QD^+ \cdots G^-RF^- \cdots .$$

Now we will deal with the remaining two patterns (4, 4 : 2, 1) and (4, 4 : 3, 0).

Pattern 6. Suppose A is of (4, 4 : 2, 1). Λ is written

$$\Lambda : \cdots C^+AH^- \cdots B^+AH^- \cdots B^+AF^- \cdots B^+D^+ \cdots G^-H^- \cdots .$$

Applying $\Psi(A)$, we obtain a good cycle

$$\Lambda' : \cdots C^+PH^- \cdots B^+Q^{-1}H^- \cdots B^+F^- \cdots B^+RPD^+ \cdots G^-PH^- \cdots$$

of Δ' (Figure 11-a); still $b(\Delta') = k$. But we can find a new bordered 1-label P of pattern (3, 4 : 2, 1) in Λ'. Hence we can apply $\Psi(P)$ to Δ' again, and obtain Δ_* with $b(\Delta_*) = k - 1$ (Figure 10-b) which has a good cycle Λ_*.

Pattern 7. Suppose A is of (4, 4 : 3, 0). Λ is written

$$\Lambda : \cdots C^+AH^- \cdots D^+AF^- \cdots C^+AF^- \cdots C^+B^+ \cdots G^-F^- \cdots .$$

Applying $\Psi(A)$ to Λ, we obtain a good cycle

$$\Lambda' : \cdots C^+PH^- \cdots D^+QF^- \cdots C^+R^{-1}F^- \cdots C^+R^{-1}B^+ \cdots G^-R^{-1}F^- \cdots$$

of another DS-diagram Δ' such that $b(\Delta') = k$. R is a new bordered 1-label of a pattern of (4, 4 : 1, 2). Again we apply $\Psi(R)$ to Δ'. Then we obtain Δ_* with a good cycle Λ_* so that $b(\Delta_*) = k - 1$. □

By the induction on the number of the bordered 1-labels of seven patterns, we obtain the following corollary.

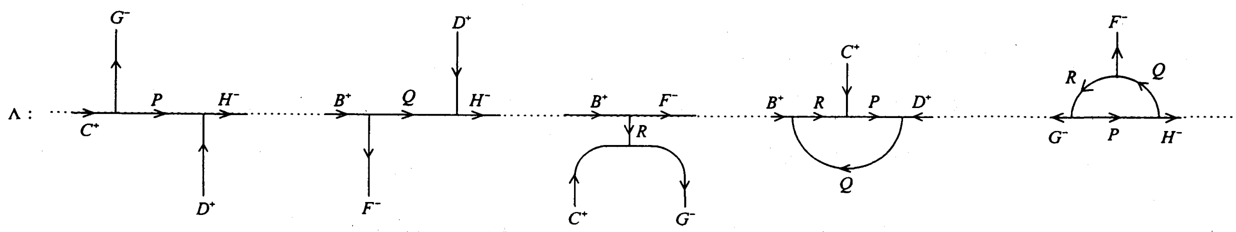


FIGURE 11-a. The case of pattern (4, 4 : 2, 1); first step.

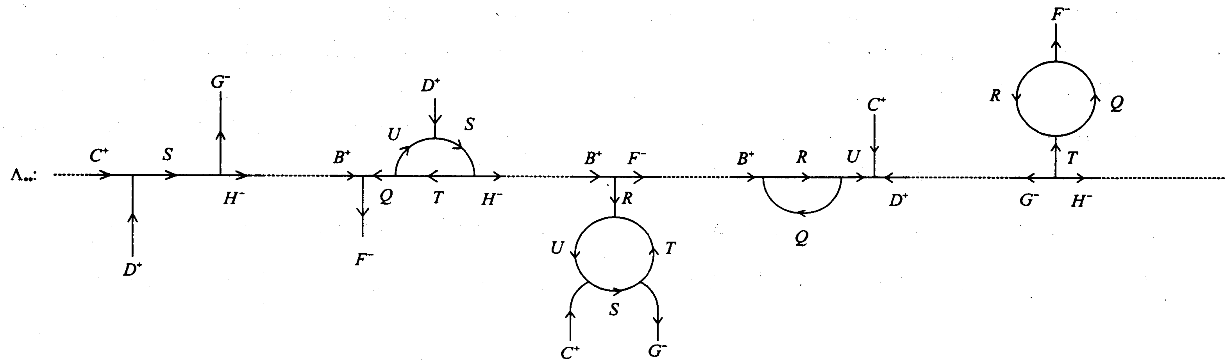


FIGURE 11-b. The case of pattern (4, 4 : 2, 1); second step.

COROLLARY 3.1. *For any DS-diagram Δ, there is a splittable DS-diagram Δ' with a better cycle such that Δ' is DS-isomorphic to Δ.*

4. Remodeling into a DS-diagram with E-cycle.

In the previous section, we saw that any DS-diagram can be remodeled into a splittable DS-diagram with a better cycle. In this section we will consider exclusively a splittable DS-diagram Δ with a better cycle Λ.

Remember that $b(\Delta)$ is the number of the bordered 1-labels of Δ. The following theorem was found by Prof. Dr. Ippei Ishii.

THEOREM 4.1. *The number $b(\Delta)$ is even, and the number of the bordered 1-labels of type (U^+, U^+, U^+) is equal to that of type (U^-, U^-, U^-).*

PROOF. Let (Z^+, Z^-) be the bicoloring of Δ. We will denote the number of positive (or negative) i-cell of Δ by $\nu^+(i)$ (or $\nu^-(i)$). By $\nu^\circ(i)$ we mean the number of the neutral i-cells. Then obviously $\nu^+(2) = \nu^-(2)$, $\nu^\circ(2) = 0$. And it holds that $\nu^+(1) = \nu^-(1)$ since 1-cells with a bordered 1-label are all neutral and three 1-cells with a distributed 1-label consists of one positive, one negative and one neutral 1-cell. Since both Z^+ and Z^- are 2-disks, $\chi(Z^+) = \chi(Z^-) = 1$, where $\chi(Z^\pm)$ is the Euler number of $Z^\pm$. That is,

$$\chi(Z^\pm) = \{\nu^\circ(0) + \nu^\pm(0)\} - \{\nu^\circ(1) + \nu^\pm(1)\} + \nu^\pm(2)\,.$$

Hence $\nu^+(0) = \nu^-(0)$ follows from

$$\begin{aligned} 0 &= \chi(Z^+) - \chi(Z^-) \\ &= \{\nu^+(0) - \nu^-(0)\} - \{\nu^+(1) - \nu^-(1)\} + \{\nu^+(2) - \nu^-(2)\} \\ &= \nu^+(0) - \nu^-(0) . \end{aligned}$$

Suppose m and n are the number of the bordered 1-labels of type (U^+, U^+, U^+) and of type (U^-, U^-, U^-) respectively. Suppose x is a 0-label. Let $\nu^+(x)$, $\nu^-(x)$ and $\nu^\circ(x)$ be the number of positive, negative, and neutral 0-cells with the 0-label x, respectively. If each 1-label incident with x is distributed, $\nu^+(x) = \nu^-(x) = 1$ and $\nu^\circ(x) = 2$. If there is a bordered 1-label, say A, incident with x, then there are no other bordered 1-labels incident with x. If A is of type (U^+, U^+, U^+), then $\nu^\circ(x) = 3$, $\nu^+(x) = 0$ and $\nu^-(x) = 1$ holds. If A is of type (U^-, U^-, U^-), then $\nu^\circ(x) = 3$, $\nu^+(x) = 1$ and $\nu^-(x) = 0$ holds. Therefore $\nu^+(0) - \nu^-(0) = n - m$. Hence $m = n$. □

Showing the following theorem, we complete the proof of our main Theorem 1.1.

THEOREM 4.2. *There is a DS-diagram Δ_* with E-cycle such that Δ_* is DS-isomorphic to Δ.*

PROOF. Suppose $b(\Delta) = 2k$. By the previous theorem, there is a pair of bordered 1-labels X and Y such that X is of type (U^+, U^+, U^+) and Y is of type (U^-, U^-, U^-). We can assume

$$\begin{aligned} \Delta = \{ & C_X^+ X G_X^-, D_X^+ X H_X^-, B_X^+ X F_X^-, \\ & C_X^+ D_X^+, D_X^+ B_X^+, B_X^+ C_X^+, G_X^- H_X^-, H_X^- F_X^-, F_X^- G_X^-, \\ & C_Y^+ Y G_Y^-, D_Y^+ Y H_Y^-, B_Y^+ Y F_Y^-, \\ & C_Y^+ D_Y^+, D_Y^+ B_Y^+, B_Y^+ C_Y^+, G_Y^- H_Y^-, H_Y^- F_Y^-, F_Y^- G_Y^-, \cdots\cdots \}, \end{aligned}$$

(see Figure 12-a), and

$$\begin{aligned} \Lambda = & \cdots D_X^+ X H_X^- \cdots B_X^+ X F_X^- \cdots C_X^+ X G_X^- \cdots \\ & \cdots C_Y^+ Y G_Y^- \cdots D_Y^+ Y H_Y^- \cdots B_Y^+ Y F_Y^- \cdots . \end{aligned}$$

Applying an elementary deformation $\Psi(Y)$ of type II$^+$ on Δ, we can obtain a DS-diagram Δ_1 (see Figure 12-b) such that

$$\begin{aligned} \Delta_1 = \{ & PQR; C_X^+ X G_X^-, D_X^+ X H_X^-, B_X^+ X F_X^-, \\ & C_X^+ D_X^+, D_X^+ B_X^+, B_X^+ C_X^+, G_X^- H_X^-, H_X^- F_X^-, F_X^- G_X^-, \\ & C_Y^+ P D_Y^+, D_Y^+ Q B_Y^+, B_Y^+ R C_Y^+, G_Y^- P H_Y^-, H_Y^- Q F_Y^-, F_Y^- R G_Y^-, \\ & C_Y^+ G_Y^-, D_Y^+ H_Y^-, B_Y^+ F_Y^-, \cdots \}. \end{aligned}$$

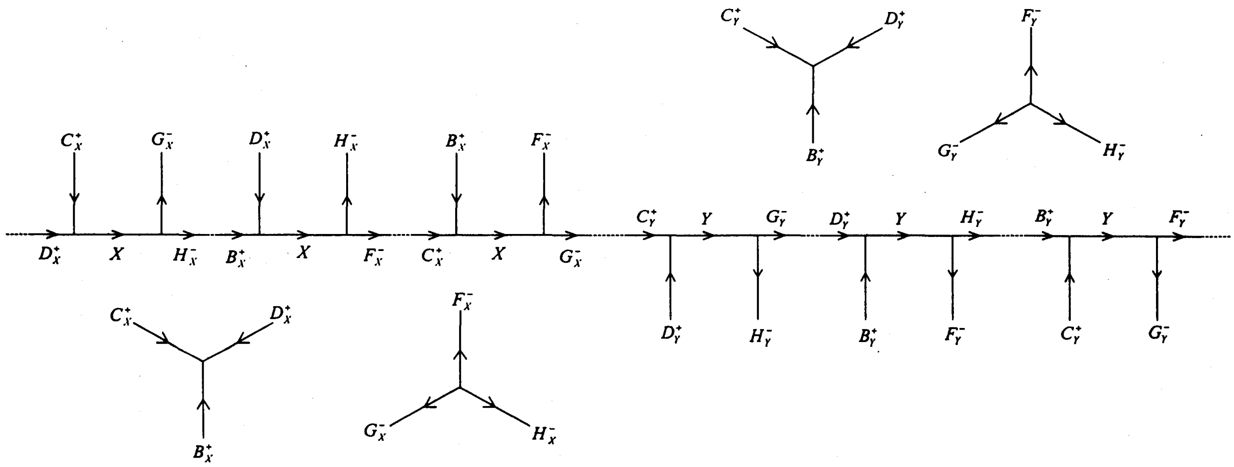


FIGURE 12-a. Pair of 1-labels X and Y; X is of type (U^+, U^+, U^+) and Y is of type (U^-, U^-, U^-).

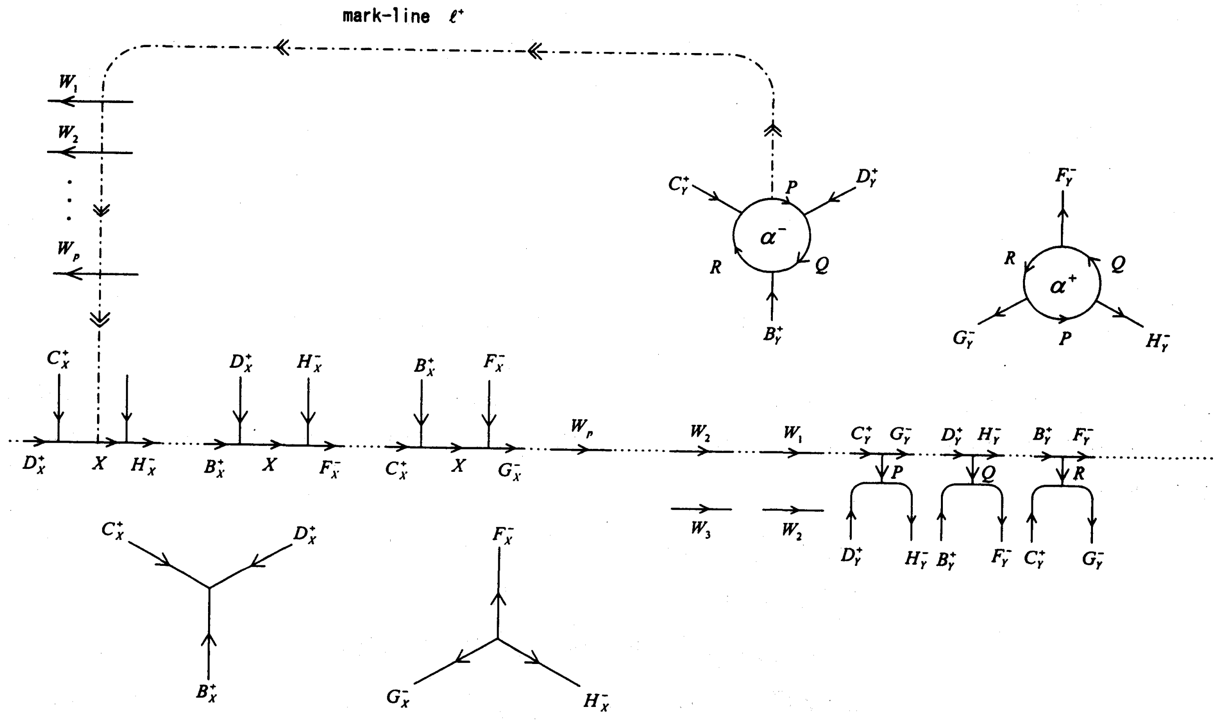


FIGURE 12-b. After applying an elementary deformation $\Psi(Y)$.

Note that

$$\begin{aligned} \Lambda_1 = &\cdots D_X^+ X H_X^- \cdots B_X^+ X F_X^- \cdots C_X^+ X G_X^- \cdots \\ &\cdots C_Y^+ G_Y^- \cdots D_Y^+ H_Y^- \cdots B_Y^+ F_Y^- \cdots \end{aligned}$$

is no longer a splitting cycle of Λ_1. But Z^+ is still connected, a new 1-label $P \subset Z^+$ is distributed and a 1-label $X \subset Z^+$ is still bordered. We can find a path ℓ^+ from P to X as

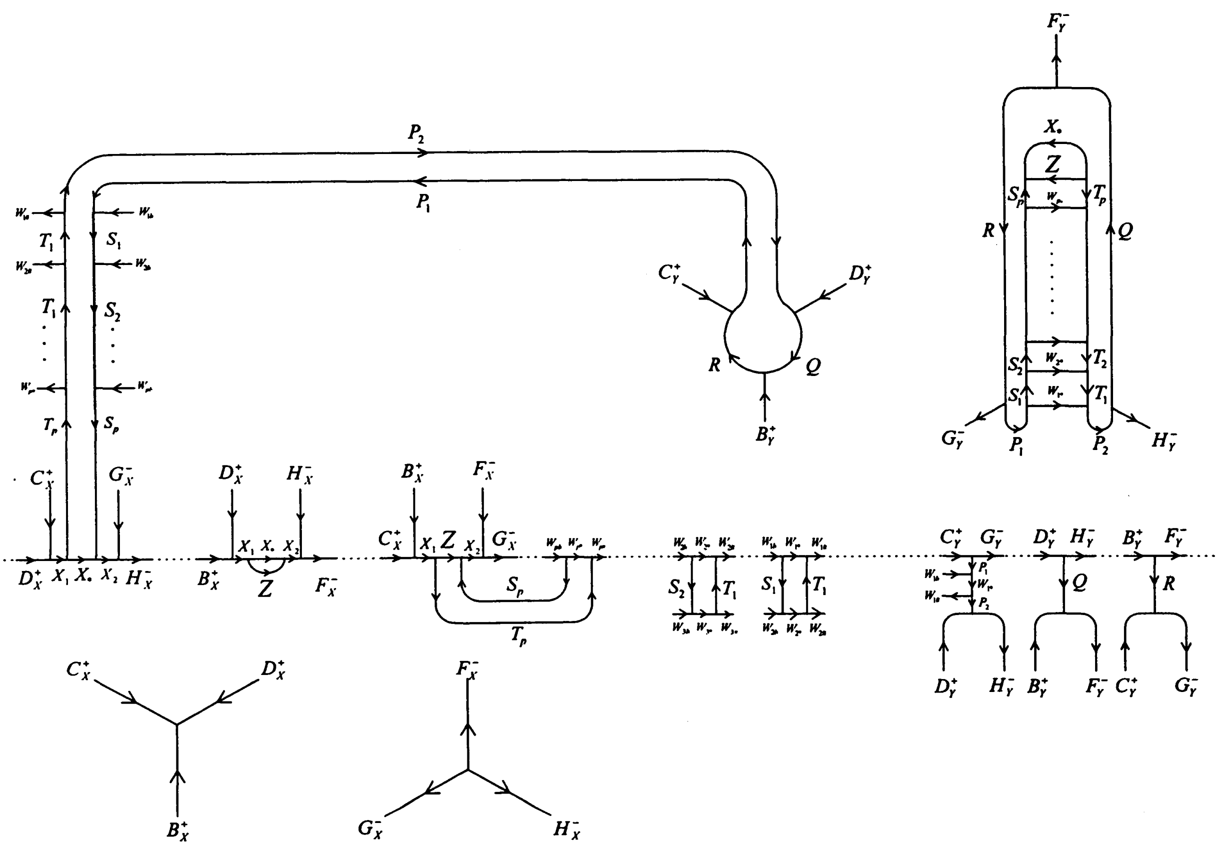


FIGURE 12-c. After applying a digging $d(\ell^+)$.

a mark line. ℓ^+ may cut across some 1-cells with 1-labels, say $W_1, W_2, \cdots, W_p$. Applying digging $d(\ell^+)$ along ℓ^+, we can obtain a splittable DS-diagram Δ_2 (see Figure 12-c) with a good cycle

$$\begin{aligned}\Lambda_2 = &\cdots D_X^+ X_1 T_p \cdots T_2 T_1 P_2 Q R P_1 S_1 S_2 \cdots S_p X_2 H_X^- \cdots \\ &\cdots B_X^+ X_1 X_* X_2 F_X^- \cdots C_X^+ X_1 Z X_2 G_X^- \cdots \\ &\cdots W_{p\flat} W_{p*} W_{p\sharp} \cdots W_{2\flat} W_{2*} W_{2\sharp} W_{1\flat} W_{1*} W_{1\sharp} \cdots \\ &\cdots C_Y^+ Y G_Y^- \cdots D_Y^+ Y H_Y^- \cdots B_Y^+ Y F_Y^- \cdots .\end{aligned}$$

Note that both of X_1 and X_2 on Λ_2 are of pattern 2: (3, 3 : 3, 0). Hence by applying $\Psi(X_1)$ and $\Psi(X_2)$, we can obtain a splittable DS-diagram Δ_3 with a better cycle

$$\begin{aligned}\Lambda_3 = &\cdots D_X^+ J_1^{-1} T_p \cdots T_2 T_1 P_2 Q R P_1 S_1 S_2 \cdots S_p J_2 H_X^- \cdots \\ &\cdots B_X^+ K_1^{-1} X_* K_2 F_X^- \cdots C_X^+ L_1^{-1} Z L_2 G_X^- \cdots \\ &\cdots W_{p\flat} W_{p*} W_{p\sharp} \cdots W_{2\flat} W_{2*} W_{2\sharp} W_{1\flat} W_{1*} W_{1\sharp} \cdots \\ &\cdots C_Y^+ Y G_Y^- \cdots D_Y^+ Y H_Y^- \cdots B_Y^+ Y F_Y^- \cdots .\end{aligned}$$

It is an easy observation that $b(\Delta_3) = 2k - 2$. This argument shows that there is an algorithm to obtain a DS-diagram with E-cycle which is DS-isomorphic to the original Δ. $\square$

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