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On Additive Volume Invariants of Riemannian Manifolds

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Introduction.

Let (M, g) be an n-dimensional Riemannian C^ω manifold and $p \in M$ a point. For small $r > 0$, we denote by $V_p(r)$ the volume of the geodesic ball of radius r with the center p. It is known that $V_p(r)$ is given by a power series expansion

$$V_p(r) = V_0(r)(1 + B_2(p)r^2 + B_4(p)r^4 + \cdots + B_{2k}(p)r^{2k} + \cdots)$$

where $V_0(r)$ is the volume of an n-dimensional Euclidean ball of the same radius and B_2, $B_4, \cdots, B_{2k}, \cdots$ are *the volume invariants*, which are analytic functions of p, or, more specifically, scalar curvature invariants of order $2, 4, \cdots, 2k, \cdots$ respectively (see e.g. [G1]). If (M, g) is flat, $B_{2k} \equiv 0$ for all $k \in \mathbf{N}$; we have the following conjecture:

VOLUME CONJECTURE [Gray and Vanhecke, 1979]. *Assume that $V_p(r) = V_0(r)$ for any $p \in M$ and small $r > 0$ i.e. $B_{2k} \equiv 0$ for any $k \in \mathbf{N}$. Then (M, g) is flat.*

In general case this conjecture is open. A. Gray and L. Vanhecke [GV] has proved that, under some assumptions on the dimension n and/or the curvature, the conjecture is true by calculating the first three invariants B_2, B_4, B_6 explicitly in terms of the curvature tensor, the Ricci tensor, the scalar curvature and their covariant derivative. Moreover, they constructed an example of a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^8))$ for each $p \in M$ ([GV]). O. Kowalski [K] defined *the additive volume invariants* of a Riemannian manifold (cf. the next section) and proved the following theorem by using them.

THEOREM [K]. *There exists a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{16}))$ for each $p \in M$.*

This paper is concerned with the following question: *For given $k \in \mathbf{N}$, does there exist a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{2k+2}))$*? We will prove the following.

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THEOREM. *For each $k_0 \leq 100$, there exists a non-flat homogeneous Riemannian manifold M such that $V_p(r) = V_0(r)(1 + O(r^{2k_0+2}))$ for each $p \in M$.*

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1. Additive volume invariants.

In this section we recall *the additive volume invariants* and their properties briefly. The readers should refer to [K] for details.

Let $x_0^{-1}, x_1, x_2, \cdots, x_n, \cdots$ be independent variables and $\mathbf{Q}[x_0^{-1}, x_1, x_2, \cdots, x_n, \cdots]$ the corresponding ring of polynomials over rational numbers. We shall write briefly x_0^{-k} instead of $(x_0^{-1})^k$. Let us define a derivation D in $\mathbf{Q}[x_0^{-1}, x_1, x_2, \cdots, x_n, \cdots]$ as follows: $D(r) = 0$ for $r \in \mathbf{Q}$, $D(x_0^{-1}) = -x_1 x_0^{-2}$, $D(x_i) = x_{i+1}$ for $i \geq 1$. We also define formally

$$D(\log x_0) = x_1 x_0^{-1} .$$

As we see easily by the induction, for every $k \geq 1$, the k-iteration $D^{(k)}(\log x_0) \in \mathbf{Q}[x_0^{-1}, x_1, x_2, \cdots, x_n, \cdots]$ has the form

$$D^{(k)}(\log x_0) = \sum_{\substack{i_1+\cdots+i_l=k \\ i_1 \geq i_2 \geq \cdots \geq i_l > 0}} c_{i_1 \cdots i_l} x_{i_1} \cdots x_{i_l} x_0^{-l}, \quad c_{i_1 \cdots i_l} \in \mathbf{Q}. \tag{1.0}$$

Here each $c_{i_1 \cdots i_l} \in \mathbf{Q}$ is uniquely determined.

We shall call the polynomial $D^{(k)}(\log x_0)$ *the logarithmic operator form of order k*, and we denote it by L_k.

If X is a linear differential operator on a smooth manifold M, and if f is a smooth function on M, then we can consider a non-linear differential operator $L_k(X)$ on M defined by the following formula:

$$L_k(X)(f) = \sum_{\substack{i_1+\cdots+i_l=k \\ i_1 \geq i_2 \geq \cdots \geq i_l > 0}} c_{i_1 \cdots i_l} (X^{(i_1)} f) \cdots (X^{(i_l)} f) f^{-l} .$$

An informal definition of $L_k(X)$ is the following: consider the arbitrary function $F(t)$ of one variable (of class C^ω) and calculate the expression $d^k/(dt)^k(\log F(t))$. Then substituting $F \to f$, $F' \to Xf, \cdots, F^{(k)} \to X^{(k)} f$ everywhere, we obtain the value of $L_k(X)$ on f.

Let N be another smooth manifold, g a smooth function on N, and Y a linear differential operator on N. We shall consider the product manifold $M \times N$ with the projections $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$. The function $(f \circ \pi_1)(g \circ \pi_2)$ on $M \times N$ will be denoted briefly by fg, and the linear differential operator $\pi_1^* X + \pi_2^* Y$ on $M \times N$ will be denoted by $X + Y$. Now we have the following proposition.

PROPOSITION 1.1. *Let M, N, f, g, X, Y be as above. Then*

$$L_k(X + Y)(fg) = L_k(X)(f) + L_k(Y)(g), \quad k \in \mathbf{N}.$$

We also define the *reduced logarithmic operator form* $\hat{L}_k$ *of order* k by substituting $x_0^{-1} = 1$ in (1.0). For the reduced operator forms we have the following:

COROLLARY 1.2. *Let M, N be smooth manifolds, $(p, q) \in M \times N$ a fixed point, f, g smooth functions on M, N respectively such that $f(p) = g(q) = 1$, and X, Y linear differential operators on M, N respectively. Then*

$$\hat{L}_k(X + Y)_{(p,q)}(fg) = \hat{L}_k(X)_p(f) + \hat{L}_k(Y)_q(g). \tag{1.2}$$

Let us remark that (1.2) has a local character; the linear operators X, Y and the functions f, g are to be defined only in some neighborhoods of the points p, q respectively.

We shall now recall concept and results on the volume of a geodesic ball in a Riemannian manifold.

Let (M, g) be an n-dimensional analytic Riemannian manifold and $p \in M$. If $(x_1, \cdots, x_n)$ is any system of normal coordinates at p, then *Euclidean Laplacian* $\tilde{\Delta}_p$ and normal volume function θ_p are defined by the following formulas:

$$\tilde{\Delta}_p = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$
$$\theta_p = \omega\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right),$$

where ω is a volume element of (M, g) near p (such that $\theta_p > 0$). The definitions $\tilde{\Delta}_p$ and θ_p are independent of the choice of normal coordinates at p (here "independent" means in the sense of germs). $\tilde{\Delta}_p$ is a local linear differential operator on (M, g).

If we denote the volume of a small geodesic ball with center $p \in M$ and radius r by $V_p(r)$, we have the following power series expansion of $V_p(r)$;

$$V_p(r) = V_0(r)(1 + B_2(p)r^2 + \cdots + B_{2k}(p)r^{2k} + \cdots)$$

where $V_0(r)$ is the volume of the n-dimensional Euclidean ball of the same radius r and the coefficients $B_2, \cdots, B_{2k}, \cdots$, or "volume invariants" are curvature invariants of order $2, 4, \cdots$ respectively. The coefficients $\{B_{2k}\}_{k\in\mathbf{N}}$, $\tilde{\Delta}_p$ and θ_p satisfy the following formula;

$$\tilde{\Delta}_p^k(\theta_p)(p) = 2^k k!(n + 2k)(n + 2k - 2) \cdots (n + 2)B_{2k}(p). \tag{1.3}$$

If (M_i, g_i), $i = 1, 2$, are two analytic Riemannian manifolds and $(p_1, p_2) \in M_1 \times M_2$, we can consider an adapted normal coordinate system $(x_1, \cdots, x_{n_1}, x_{n_1+1}, \cdots, x_{n_1+n_2})$ defined in a "rectangular" normal neighborhood $U_{p_1} \times U_{p_2}$ in $M_1 \times M_2$. With respect to these adapted normal coordinates, we can see easily that

$$\tilde{\Delta}_{(p_1,p_2)} = \tilde{\Delta}_{1,p_1} + \tilde{\Delta}_{2,p_2}$$
$$\theta_{(p_1,p_2)} = \theta_{1,p_1}\theta_{2,p_2}$$

for the corresponding Euclidean Laplacians and normal volume functions (via the corresponding projections π_1, π_2). Moreover, $\theta_{1,p_1}(p_1) = \theta_{2,p_2}(p_2) = 1$.

On formula (1.2) now implies

$$\hat{L}_k(\tilde{\Delta}_{(p_1,p_2)})_{(p_1,p_2)}(\theta_{(p_1,p_2)}) = \hat{L}_k(\tilde{\Delta}_{1,p_1})_{p_1}(\theta_{1,p_1}) + \hat{L}_k(\tilde{\Delta}_{2,p_2})_{p_2}(\theta_{2,p_2})\,.$$

Now we are ready to define *the additive volume invariants.*

DEFINITION 1.3. Let (M, g) be an n-dimensional analytic Riemannian manifold and B_{2k}^M be the $2k$th coefficient of the power series expansion for the volume of a geodesic ball $V_p(r)$ with center $p \in M$ and small radius r. *The additive volume invariants of order* $2k$ are functions $A_{2k} : M \to \mathbf{R}$ defined by

$$A_{2k}(p) = \hat{L}_k(\tilde{\Delta}_p)_p(\theta_p) \quad (p \in M)\,.$$

We make the following *conventions*:

(a) For a given Riemannian manifold M, the corresponding invariant A_{2k} will be written as A_{2k}^M.

(b) For a homogeneous Riemannian manifold M the invariants A_{2k}^M are constant functions and denoted $A_{2k}(M)$ in this case.

THEOREM 1.4 [K]. (i) *For each $k \in \mathbf{N}$, there exists a countable set of polynomials with rational coefficients $\{P_{n,k} \mid P_{n,k} \in \mathbf{Q}[t_1, \cdots, t_k]\}_{n\in\mathbf{N}}$ with the following property: for each analytic Riemannian manifold (M, g) of dimension n we have*

$$A_{2k}^M = P_{n,k}(B_2^M, \cdots, B_{2k}^M)\,.$$

(ii) $B_2^M = B_4^M = \cdots = B_{2k}^M = 0$ *if and only if* $A_2^M = A_4^M = \cdots = A_{2k}^M = 0$.

(iii) *If (M_1, g_1), (M_2, g_2) are two analytic Riemannian manifolds, then on the product manifold $(M_1 \times M_2, g_1 \times g_2)$*

$$A_{2k}^{M_1\times M_2} = A_{2k}^{M_1} \circ \pi_1 + A_{2k}^{M_2} \circ \pi_2 \quad (k \in \mathbf{N})$$

where $\pi_i : M_1 \times M_2 \to M_i$ are projections $(i = 1, 2)$.

In particular, if (M_i, g_i) $(i = 1, 2)$ are homogeneous, then

$$A_{2k}(M_1 \times M_2) = A_{2k}(M_1) + A_{2k}(M_2)\,.$$

(iv) *On a Riemannian manifold (M, g),*

$$A_2^M = -\frac{\tau}{3}\,, \quad A_4^M = \frac{-3\|R\|^2 + 8\|\rho\|^2 - 18\Delta\tau}{45}$$

where τ, $\|R\|$, $\|\rho\|$ denote the scalar curvature, the length of the Riemannian curvature tensor and the length of the Ricci tensor respectively.

(v) *We define a formal power series $h_p(s)$* by

$$h_p(s) = 1 + \sum_{i=1}^{\infty} \frac{\tilde{\Delta}_p^i(\theta_p)(p)}{i!} s^i\,,$$

that is,

$$h_p(s) = 1 + \sum_{i=1}^{\infty} 2^i(n+2)(n+4)\cdots(n+2i)B_{2i}^M(p)s^i \tag{1.4.1}$$

by (1.3). *Then we have*

$$A_{2k}(p) = \frac{d^k}{ds^k} \log h_p(s) \bigg|_{s=0} \quad (p \in M), \tag{1.4.2}$$

PROPOSITION 1.5 [K]. *Let (M, g) an analytic Riemannian manifold and λ a positive real number. Let $M(\lambda)$ denote the manifold $(M, \lambda^{-1} g)$. Then*

$$A_{2k}^{M(\lambda)} = \lambda^k A_{2k}^M \quad (k \in \mathbf{N}).$$

Let $\mathfrak{S}(r)$ denote the volume of a geodesic sphere with small radius r in the n-sphere S^n with constant sectional curvature 1. Then

$$\mathfrak{S}(r) = \mathfrak{S}_0(1) \sin^{n-1} r \tag{1.5}$$

where $\mathfrak{S}_0(1)$ denotes the volume of the unit sphere in the n-dimensional Euclidean space. The corresponding formula for the n-dimensional hyperbolic space H^n with the constant sectional curvature -1 is given by substituting sinh for sin. (see e.g. [G1]) Kowalski proved the next propositions by using these facts.

PROPOSITION 1.6 [K]. *For the n-sphere S^n and the n-dimensional hyperbolic space H^n,*

$$A_{2k}(S^n) = (-1)^k A_{2k}(H^n) \in \mathbf{Q} \quad (k \in \mathbf{N}).$$

In particular $A_{2(2i-1)}(S^n)$ and $A_{2(2i-1)}(H^n)$ have opposite sign.

PROPOSITION 1.7 [K]. *The first 6 additive volume invariants of $(n+2)$-sphere S^{n+2} with constant sectional curvature 1 are given by the following formulas*:

$$\begin{aligned}
A_2(S^{n+2}) &= -\frac{1}{3}(n+1)(n+2), \\
A_4(S^{n+2}) &= \frac{2}{45}(n+1)(n+2)(4n+1), \\
A_6(S^{n+2}) &= \frac{8}{945}(n+1)(n+2)(-16n^2+15n+1), \\
A_8(S^{n+2}) &= \frac{16(n+1)(n+2)}{3^3 \cdot 5^2 \cdot 7}(16n^3-209n^2+89n-1), \\
A_{10}(S^{n+2}) &= \frac{128(n+1)(n+2)}{3^4 \cdot 5 \cdot 7 \cdot 11}(64n^4-461n^3-1008n^2+484n-1), \\
A_{12}(S^{n+2}) &= \frac{256(n+1)(n+2)}{3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}(-207744n^5+195125n^4+4349166n^3 \\
&\quad -6862618n^2+3266748n+23).
\end{aligned}$$

We now consider the 3-dimensional Heisenberg group G_3, that is,

$$G_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; (x, y, z) \in \mathbf{R}^3 \right\}$$

with the left invariant metric $dx^2 + dz^2 + (dy - xdz)^2$. The vector fields $X_1 = (\partial/\partial x)$, $X_2 = (\partial/\partial y)$, $X_3 = x(\partial/\partial y) + (\partial/\partial z)$ form an orthogonal basis of the corresponding Lie algebra $(G_3)_e$ where e denotes the identify of G_3. We can determine the Levi-Civita connection ∇ in the standard way:

$$\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = -\frac{1}{2} X_3 , \quad \nabla_{X_1} X_3 = -\nabla_{X_3} X_1 = \frac{1}{2} X_2 ,$$
$$\nabla_{X_2} X_3 = -\nabla_{X_3} X_2 = \frac{1}{2} X_1 .$$

By calculating the exponential map $Exp : (u, v, w) \in T_e G_3 \mapsto (x, y, z) \in G_3$ explicitly, we get the normal volume function θ as follows (see e.g. [K]);

$$\theta = \frac{(u^2 + w^2)(2 - 2\cos v - v \sin v)}{v^4} + \frac{2(1 - \cos v)}{v^2} . \tag{1.8.0}$$

Using (1.8.0), Kowalski proved the following proposition.

PROPOSITION 1.8 [K]. *On* G_3, $A_{2k}(G_3) \in \mathbf{Q}$ $(k \in \mathbf{N})$, *and*

$$A_2(G_3) = \frac{1}{6} , \quad A_4(G_3) = -\frac{1}{20} , \quad A_6(G_3) = \frac{26}{945} , \quad A_8(G_3) = -\frac{811}{37800} = -0.21455 .$$

2. Propositions and our main theorem.

O. Kowalski proved that there exists a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{16}))$ for each $p \in M$ using additive volume invariants for the direct product of non-flat homogeneous Riemannian manifolds and the existence of a positive rational solution of a system of certain linear equations with 7 unknowns. But if we apply the same method as Kowalski [K] in order to annihilate B_{2k} for $k > 7$, we will need more unknowns and have some difficulty to show the existence of a positive rational solution of the system of the corresponding linear equations.

We begin with defining some concepts.

DEFINITION 2.1. A family $\{M_1, \cdots, M_l\}$ of homogeneous Riemannian manifolds is said to be *k-splitting* if (1) all values $A_{2i}(M_j)$, $i = 1, \cdots, k$; $j = 1, \cdots, l$ are rational numbers and (2) for each $i = 1, \cdots, k$, the invariant A_{2i} is negative for some M_α and it is positive for some M_β.

DEFINITION 2.2. A family $\{M_1, \cdots, M_l\}$ of homogeneous Riemannian manifolds is said to be *d-flat* $(k - d)$*-splitting* if (1) $A_{2i}(M_j) = 0$ for $i = 1, \cdots, d$; $j = 1, \cdots, l$ (2) all values $A_{2i}(M_j)$, $i = d + 1, \cdots, k$; $j = 1, \cdots, l$ are rational numbers and (3) for each $i = d + 1, \cdots, k$, the invariant A_{2i} is negative for some M_α and it is positive for some M_β.

Note that a k-splitting family can be regarded as d-flat $(k - d)$-splitting with $d = 0$.

LEMMA 2.3. *For* $d = 0, 1, \cdots, k-2$, *if* $\mathcal{L} = \{L_1, \cdots, L_l\}$ *is a d-flat* $(k-d)$*-splitting family of homogeneous Riemannian manifolds, then there exists a family* $\{K_1, \cdots, K_l\}$ *of homogeneous Riemannian manifolds such that* $A_{2k}(K_j) = 0$ *for* $j = 1, \cdots, d + 1$; $j =$

$1, \cdots, l$ *and for* $i = d+2, \cdots, k$, $A_{2i}(K_j)$ *has the same sign as* $A_{2i}(L_j)$ *for* $j = 1, \cdots, l$. *In particular, the family* $\{K_1, \cdots, K_l\}$ *is* $(d+1)$*-flat* $(k-d-1)$*-splitting of homogeneous Riemannian manifolds for* $d+1 < k$.

PROOF. Put $\mathcal{L}_+ = \{L \in \mathcal{L} \mid A_{2(d+1)}(L) > 0\}$, $\mathcal{L}_- = \{L \in \mathcal{L} \mid A_{2(d+1)}(L) < 0\}$ and $\mathcal{L}_0 = \{L \in \mathcal{L} \mid A_{2(d+1)}(L) = 0\}$. Then

$$\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_- \cup \mathcal{L}_0$$

is a disjoint union and $\mathcal{L}_+ \neq \phi$, $\mathcal{L}_- \neq \phi$.

Consider $L_{\beta 0} \in \mathcal{L}_+$ and fix it. Take an element $L_\alpha \in \mathcal{L}_-$. Then for $L_{\beta 0}$ and L_α, there are positive rational numbers x and y satisfying $xA_{2(d+1)}(L_\alpha) + yA_{2(d+1)}(L_{\beta 0}) = 0$. Consider Riemannian manifolds $L_\alpha(\lambda)$ and $L_{\beta 0}(\mu)$. Then we have

$$rA_{2i}(L_\alpha(\lambda)) + sA_{2i}(L_{\beta 0}(\mu)) = r\lambda^i A_{2i}(L_\alpha) + s\mu^i A_{2i}(L_{\beta 0}).$$

Now, by taking positive rational numbers λ sufficiently large and μ (sufficiently small), we see that for $i = d+2, \cdots, k$, $x\lambda^{i-d-1}A_{2i}(L_\alpha) + y\mu^{i-d-1}A_{2i}(L_{\beta 0})$ has the same sign as $A_{2i}(L_\alpha)$, if $A_{2i}(L_\alpha) \neq 0$, and let positive rational numbers q_1 and q_2 be $x = q_1\lambda^{d+1}$ and $y = q_2\mu^{d+1}$. Now write q_1 and q_2 as $q_1 = n_1/m_1$ and $q_2 = n_2/m_2$, where pairs of integers (n_1, m_1), (n_2, m_2) are relatively prime. Put $K_\alpha = [L_\alpha(\lambda)]^{n_1 m_2} \times [L_{\beta 0}(\mu)]^{n_2 m_1}$. Then $A_{2i}(K_\alpha) = 0$ for $i = 1, \cdots, d+1$ and or $i = d+2, \cdots, k$, $A_{2i}(K_\alpha)$ has the same sign as $A_{2i}(L_\alpha)$, if $A_{2i}(L_\alpha) \neq 0$.

Now consider $L_{\alpha 0} \in \mathcal{L}_-$ and fix it. Then, for each element $L_\beta \in \mathcal{L}_+$, we can associate K_β with the property that $A_{2i}(K_\beta) = 0$ for $i = 1, \cdots, d+1$ and that, for $i = d+2, \cdots, k$, $A_{2i}(K_\beta)$ has the same sign as $A_{2i}(K_\beta)$, if $A_{2i}(K_\beta) \neq 0$, by the same argument as above. For each element $L_\gamma \in \mathcal{L}_0$, put $K_\gamma = L_\gamma$. Then the family $\{K_1, \cdots, K_l\}$ satisfies the conditions of Lemma 2.3. □

Now, using Lemma 2.3, we have the following proposition.

THEOREM 2.4. *For given* $k \in \mathbf{N}$, *suppose that there exists a finite* k*-splitting family of homogeneous Riemannian manifolds* $\mathcal{M} = \{M_1, M_2, \cdots, M_l\}$. *Then there exists a non-flat homogeneous manifold* M *such that* $V_p(r) = V_0(r)(1 + O(r^{2k+2}))$.

PROOF. Applying Lemma 2.3 for $\mathcal{M}$ as $d = 0$, we get a 1-flat $(k-1)$-splitting family $\mathcal{N}_1$. Now we get a d-flat $(d-1)$-splitting family $\mathcal{N}_d$ for $d = 2, \cdots, k-1$ by Lemma 2.3, inductively. Put $\mathcal{N}_+ = \{N \in \mathcal{N}_{k-1} \mid A_{2k}(N) > 0\}$ and $\mathcal{N}_- = \{N \in \mathcal{N}_{k-1} \mid A_{2k}(N) < 0\}$. Then $\mathcal{N}_+ \neq \phi$ and $\mathcal{N}_- \neq \phi$.

Take elements $N_\beta \in \mathcal{N}_+$ and $N_\alpha \in \mathcal{N}_-$. Then there are positive rational numbers x and y satisfying $xA_{2k}(N_\alpha) + yA_{2k}(N_\beta) = 0$. Now write x and y as $x = n'_1/m'_1$ and $y = n'_2/m'_2$, where pairs of integers (n'_1, m'_1), (n'_2, m'_2) are relatively prime. Put $M = [N_\alpha]^{n'_1 m'_2} \times [N_\beta]^{n'_2 m'_1}$. Then we see that $A_{2i}(M) = 0$ for $i = 1, \cdots, k$ i.e. $V_p(r) = V_0(r)(1 + O(r^{2k+2}))$. □

COROLLARY 2.5. *There exists a non-flat homogeneous Riemannian manifold such that* $V_p(r) = V_0(r)(1 + O(r^8))$.

PROOF. Put $\mathcal{M} = \{S^3, H^2, G_3\}$, then $\mathcal{M}$ is 3-splitting from Propositions 1.6, 1.7 and 1.8. So there exists a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^8))$ from Theorem 2.4. □

COROLLARY 2.6. *There exists a non-flat homogeneous Riemannian manifold such that* $V_p(r) = V_0(r)(1 + O(r^{16}))$.

PROOF. We put $\mathcal{M} = \{S^2, S^{15}, H^2, G_3\}$. Then $\mathcal{M}$ is 7-splitting from Propositions 1.6, 1.7 and 1.8. Applying Theorem 2.4 to $\mathcal{M}$, we have a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{16}))$. □

REMARK. Gray and Vanhecke have constructed a non-flat homogeneous Riemannian manifold satisfying $V_p(r) = V_0(r)(1 + O(r^8))$ from G_3, H^2 and S^3 in [GV]. Kowalski [K] used a family of homogeneous Riemannian manifolds $\{S^2, H^2, M_1, M_2, M_3\}$ where M_1, M_2, M_3 satisfy $A_4(M_1) < 0$, $A_8(M_2) > 0$, $A_{12}(M_3) < 0$ (e.g. $M_1 = G_3$, $M_2 = S^{15}$, $M_3 = S^6$) to prove the existence of a non-flat homogeneous Riemannian manifold such that $V_p(r) = V_0(r)(1 + O(r^{16}))$.

For n-sphere S^n with the constant curvature 1, it is not easy to calculate $A_{2k}(S^n)$ for all k explicitly in general. But, for $n = 3$ we obtain the following proposition.

PROPOSITION 2.7. *For 3-sphere* S^3, $A_{2k}(S^3) \in \mathbf{Q}$ $(k \in \mathbf{N})$ ([K]) *and*

$$A_2(S^3) < 0\,, \quad A_{2(4i-2)}(S^3) > 0\,,$$

$$A_{2\cdot 4i}(S^3) < 0\,, \quad A_{2(4i+1)}(S^3) = A_{2(4i-1)}(S^3) = 0 \quad (i \in \mathbf{N})\,.$$

PROOF. First we recall the Bernoulli numbers β_i defined by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} - \sum_{i=1}^{\infty} \frac{(-1)^i \beta_i z^{2i}}{(2i)!}\,.$$

It is known that all β_i are positive rational numbers and

$$\frac{x}{\sinh x} = 1 + 2\sum_{i=1}^{\infty} \frac{(2^{2i-1} - 1)(-1)^i}{(2i)!} \beta_i x^{2i} \tag{2.7.1}$$

(see e.g. [L]). Next we calculate $h_p(s)$ in Theorem 1.4 for S^3. By (1.5), the volume of a small geodesic sphere $\mathfrak{S}(r)$ in S^3 is

$$\begin{aligned}\mathfrak{S}(r) &= 4\pi \sin^2 r \\ &= 4\pi r^2 \left(1 + \sum_{i=1}^{\infty} \tilde{B}_{2i} r^{2i}\right)\,,\end{aligned}$$

where

$$\tilde{B}_{2i} = \frac{2^{2i+1}(-1)^i}{(2i+2)!}\,. \tag{2.7.2}$$

Noting that $\mathfrak{S}(r) = \frac{d}{dr} V_p(r)$ for any dimension n, we have

$$B_{2i} = \frac{n}{n+2i} \tilde{B}_{2i} \,. \tag{2.7.3}$$

By (1.4.1), (2.7.2), (2.7.3) and $n = 3$, we get

$$h_p(s) = \sum_{i=0}^{\infty} \frac{(-4s)^i}{(i+1)!} \tag{2.7.4}$$

$$= \frac{1 - e^{4s}}{4s} \,. \tag{2.7.5}$$

Put $f(s) = \log h_p(s)$. Then we have $f^{(k)}(0) = A_{2k}(S^3)$ from (1.4.2), so we investigate the kth derivative of the $f(s)$. It follows that $f'(0) = A_2(S^3) = -2 < 0$ from $f'(s) = h'_p(s)/h_p(s)$ and (2.7.4). By (2.7.5), it holds that

$$f''(s) = \frac{1}{s^2} \left\{ 1 - \left(\frac{2 \cdot 2s}{e^{2s} - e^{-2s}} \right)^2 \right\} .$$

By using (2.7.1),

$$\frac{2 \cdot 2s}{e^{2s} - e^{-2s}} = 1 + \sum_{i=1}^{\infty} (-1)^i a_i (2s)^{2i}$$

where $a_i = 2(2^{2i-1} - 1)\beta_i/(2i)!$. Note that a_i are also positive rational ones.

Define $\tilde{a}_j$ by

$$\tilde{a}_j = \sum_{j_1 + j_2 = j+1} a_{j_1} a_{j_2} \,.$$

By using $\{\tilde{a}_i\}$ we can write $f''(s)$ as follows.

$$f''(s) = \sum_{i=1}^{\infty} (-1)^{i-1} (8a_i + 4\tilde{a}_{i-1})(4s^2)^{i-1} \,,$$

where we put $\tilde{a}_0 = 0$ formally. Now we have

$$A_{2 \cdot 4i}(S^3) < 0 \,,$$

$$A_{2 \cdot (4i-2)}(S^3) > 0 \,,$$

$$A_{2(4i+1)}(S^3) = A_{2(4i-1)}(S^3) = 0 \quad (i \in \mathbf{N}) \,. \quad \square$$

Now we consider 3-Heisenberg group G_3.

PROPOSITION 2.8. *On* G_3,

$$A_{2(2i-1)}(G_3) > 0 \,, \quad A_{2(2i)}(G_3) < 0 \quad (i \in \mathbf{N}) \,.$$

PROOF. By using (1.8.0), we calculate $\tilde{\Delta}^k(\theta)$. Put

$$g_1(v) = \frac{2 - 2\cos v - v\sin v}{v^4} = \sum_{i=0}^{\infty} \frac{(-1)^i(2i+2)}{(2i+4)!} v^{2i},$$

$$g_2(v) = \frac{2 - 2\cos v}{v^2} = \sum_{i=0}^{\infty} \frac{2(-1)^i}{(2i+2)!} v^{2i}.$$

Then

$$\begin{aligned}\tilde{\Delta}^k(\theta) &= \tilde{\Delta}^k\{(u^2+w^2)g_1(v) + g_2(v)\} \\ &= (u^2+w^2)g_1^{(2k)}(v) + g_2^{(2k)}(v) + 4kg_1^{(2k-2)}(v).\end{aligned}$$

Hence we have

$$\tilde{\Delta}^k(\theta)\,\big|_e = \frac{2(-1)^{k+1}}{(2k+1)(2k+2)(2k-1)} \quad (k \in \mathbf{N}).$$

For simplicity we put

$$c_j = \frac{2}{(j+1)!(2j+3)(2j+4)(2j+1)} > 0,$$

then we have from (1.4.1)

$$\begin{aligned}\log h_p(s) &= \log\left(1 + \sum_{j=0}^{\infty}(-1)^j c_j s^{j+1}\right) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1}\left\{\sum_{j=0}^{\infty}(-1)^j c_j s^{j+1}\right\}^{i+1} \\ &= \sum_{i=0}^{\infty}\left\{\sum_{j=0}^{\infty}(-1)^{ij+i+j}\frac{G_{i,j}}{i+1}s^{ij+i+j+1}\right\} \\ &= \sum_{k=0}^{\infty}(-1)^k \hat{C}_k s^{k+1},\end{aligned}$$

where

$$C_{i,j} = \sum_{j_1+\cdots+j_{i+1}=ij+i+j+1} c_{j_1}\cdots c_{j_{i+1}} \quad \text{and} \quad \hat{C}_k = \sum_{i+j=k}\frac{C_{i,j}}{i+1}.$$

Since $C_{i,j}$ are positive rational numbers, $\hat{C}_k$ are also positive rational ones. From (1.4.2), we obtain

$$\begin{aligned}\log h_p(s) &= \sum_{k=1}^{\infty}\frac{A_{2k}(G_3)}{k!}s^k \\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\hat{C}_{k-1}s^k.\end{aligned}$$

Thus $A_{2(2i-1)}(G_3) > 0$, $A_{2(2i)}(G_3) < 0$ $(i \in \mathbf{N})$. □

Kowalski proved $A_8(S^n) > 0$ for $n \geq 15$ (see Proposition 1.7). So using (1.4.1) and (1.4.2) we show the following.

PROPOSITION 2.9. *For each* i $(i \leq 25)$, *there exists* $n \in \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ *such that* $A_{8i}(S^n) > 0$.

PROCEDURE OF PROOF. From (1.5) and (2.7.3), we can get the coefficients $\tilde{B}_2, \cdots, \tilde{B}_{2k}$ of the expansion of $\mathfrak{S}(r)$ around $r = 0$ by using the first k coefficients of the expansion of $\sin r$ around $r = 0$.

From (1.4.1) we obtain the first k coefficients of the expansion of $h_p(s)$ around $s = 0$. Using them, we will calculate the expansion of $\log h_p(s)$ around $s = 0$ by kth order. Put

$$h(s) = \sum_{l=1}^{k} 2^l n(n+2)(n+4)\cdots(n+2l-2)\tilde{B}_{2l}s^l,$$

$$h_1(s) = h(s), \quad h_j(s) = -h(s) \cdot h_{j-1}(s) \pmod{s^k}$$

and

$$f(s) = \sum_{i=1}^{\infty} \frac{h_i(s)}{i},$$

then the first k terms of $f(s)$ as a polynomial of s agree with those of the expansion of $\log h_p(s)$ around $s = 0$. By (1.4.2) we obtain $A_2, \cdots, A_{2k}$. Now we can compute A_{2k} of S^n $(n = 2, 4, 5, \cdots, 14, 15)$ for $k \leq 100$. The following program for *Mathematica*® 2.2. is used to calculate A_{2k} of S^n.

```
n=?; (* the dimension of the sphere *)
d=101;
b[r,1]=Normal[Series[(Sin[r]/r),{r,0,2*d}]];
b[r_,i_]:=b[r,i]=Normal[Series[b[r,i-1]*b[r,1],{r,0,2*d}]];
c[i_]:=c[i]
=(2^i)*Product[(n+2*(j-1)),{j,1,i}]*Coefficient[b[r,n-1],
r,2*i];
h1[s_]:=h1[s]=Sum[c[i]*s^i,{i,1,d-1}];
h[s,1]=h1[s];
h[s_,k_]:=h[s,k]
=Normal[Series[(-1)*h[s,k-1]*h[s,1],{s,0,d-1}]];
f[s]=Sum[h[s,i]/i,{i,1,d-1}];
add[i_]:=Coefficient[f[s],s,i]*i!
Do[Print[add[i]],{i,1,d-1}]
```

We omit the precise values of $A_{8i}(S^n)$ $(n = 2, 4, 5, \cdots, 14, 15)$ but list their signatures.

The signatures of $A_{8i}(S^n)$ $(n = 2, 4, 5, \cdots, 14, 15)$

	S^2	S^4	S^5	S^6	S^7	S^8	S^9	S^{10}	S^{11}	S^{12}	S^{13}	S^{14}	S^{15}
A_8	-	-	-	-	-	-	-	-	-	-	-	-	+
A_{16}	+	-	+	+	+	+	+	+	+	+	+	+	+
A_{24}	+	+	+	+	+	+	-	-	-	-	-	-	-
A_{32}	+	+	+	-	-	-	-	-	-	-	-	-	-
A_{40}	-	+	-	-	-	-	+	+	+	+	+	+	+
A_{48}	-	+	-	-	+	+	+	+	+	-	-	-	-
A_{56}	-	+	-	+	+	+	-	-	-	-	-	-	-
A_{64}	+	-	+	+	-	-	-	-	+	+	+	+	+
A_{72}	+	-	+	-	-	-	+	+	+	+	+	-	-
A_{80}	+	-	+	-	-	+	+	+	-	-	-	-	-
A_{88}	-	-	+	-	+	+	-	-	-	-	+	+	+
A_{96}	-	-	-	+	+	-	-	+	+	+	+	+	-
A_{104}	-	+	-	+	-	-	+	+	+	-	-	-	-
A_{112}	+	+	-	+	-	+	+	-	-	-	-	+	+
A_{120}	+	+	+	-	+	+	-	-	-	+	+	+	+
A_{128}	+	+	+	-	+	-	-	+	+	+	-	-	-
A_{136}	-	+	+	+	-	-	+	+	-	-	-	-	+
A_{144}	-	-	-	+	-	+	+	-	-	+	+	+	+
A_{152}	-	-	-	+	-	+	-	-	+	+	+	-	-
A_{160}	+	-	-	-	+	-	-	+	+	-	-	-	+
A_{168}	+	-	+	-	+	-	+	+	-	-	+	+	+
A_{176}	+	-	+	+	-	+	+	-	-	+	+	-	-
A_{184}	-	+	+	+	-	+	-	+	+	+	-	-	-
A_{192}	-	+	-	+	+	-	-	+	-	-	-	+	+
A_{200}	-	+	-	-	+	-	+	-	-	+	+	+	-

Now we can prove the main theorem.

THEOREM. *For each k ($k \leq 100$), there exists a non-flat homogeneous manifold M such that $V_p(r) = V_0(r)(1 + O(r^{2k+2}))$.*

PROOF. Put $\mathcal{M} = \{G_3, S^2, S^3, S^4, S^5, S^6, S^{15}, H^2, H^3, H^4, H^5, H^6, H^{15}\}$ then $\mathcal{M}$ is k-splitting by Propositions 1.6, 2.7, 2.8 and 2.9. Applying Theorem 2.4 to $\mathcal{M}$, we obtain a non-flat homogeneous manifold M such that $V_p(r) = V_0(r)(1 + O(r^{2k+2}))$. □

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