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Equivariant Cutting and Pasting of G Manifolds

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Abstract. Let G be a finite abelian group and let $SK_*^G(pt, pt)$ be a cutting and pasting group (an SK group) based on G manifolds with boundary. In this paper, we first obtain a basis for a **Z** module $\mathcal{T}_*^G$ consisting of all homomorphisms (G-SK invariants) $T : SK_*^G(pt, pt) \to \mathbf{Z}$. Let SK_*^G be the SK group based on closed G manifolds. We next study a relation between the theories SK_*^G and $SK_*^G(pt, pt)$ by performing equivariant cuttings and pastings of G manifolds, and characterize a class of multiplicative invariants which are related to χ^G.

0. Introduction.

Throughout this paper G is a finite abelian group and a G manifold M means an unoriented compact smooth manifold M, which may have boundary ∂M, with smooth action of G on it. Let T be a map for G manifolds which takes values in the ring **Z** of rational integers and is additive with respect to the disjoint union of G manifolds. Such T is said to be a G-SK *invariant* if it is invariant under equivariant cuttings and pastings (G-SK processes) ([4]). Let $\bar{\chi}(M) = \chi(M) - \chi(\partial M)$, where χ is the Euler characteristic, and K a subgroup of G. Then $\bar{\chi}^K(M) = \bar{\chi}(M^K)$ (or $\chi^K(M) = \chi(M^K)$) is a G-SK invariant, where $M^K = \{x \in M; kx = x \text{ for any } k \in K\}$. In [3], H. Koshikawa and the author have studied an SK group $SK_*^G(pt, pt)$ resulting from equivariant cuttings and pastings of G manifolds. An invariant T is considered to be an additive homomorphism $T : SK_*^G(pt, pt) \to \mathbf{Z}$.

The main object of this paper is to characterize these invariants, and we also remark a relation between our theory and the SK theory SK_*^G of closed G manifolds ([5], [8]).

In Section 1, we explain the basic properties of G slice types, especially a total ordering on the family $St(G)$ of all slice types, and define the theory $SK_*^G(pt, pt)$, whose structure is known by using the ordering (Proposition 1.6).

Let $\mathcal{T}_*^G$ be the set consisting of all G-SK invariants. In Section 2, we first obtain a basis for $\mathcal{T}_*^G$ as a free **Z** module. As a result, we show that a set $\{\bar{\chi}(M_\sigma)\}$ of integers determines a class $[M]$ in $SK_*^G(pt, pt)$, where M_σ is the invariant submanifold of M which consists of those points with slice type containing $\sigma \in St(G)$ (Theorem 2.6 and Corollary 2.11).

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The theories $SK_*^G \otimes R_2$ and $SK_*^G(pt, pt) \otimes R_2$ are closely related to each other, where R_2 is the subring of the rationals given by $R_2 = \mathbf{Z}[\frac{1}{2}]$. In Section 3, we first obtain an exact sequence consisting of these theories. Using this, we show that the set $\{\chi(M_\sigma)\}$ determines a class $[M]$ in SK_*^G modulo torsion (Propositions 3.1 and 3.3). There is a basis for $SK_*^G(pt, pt) \otimes R_2$ which comes from the one for $SK_*^G \otimes R_2$ (Corollary 3.5). Finally we study a relation between such basis and the original one in Proposition 1.6, and give some related examples by using G-SK processes (Theorem 3.11, Examples 3.13 and 3.15).

1. Preliminaries.

Let M be an m-dimensional G manifolds. Let $(L, \partial L) \subset (M, \partial M)$ be a G invariant codimension one submanifold which satisfies the condition that the normal bundle of $(L, \partial L)$ in $(M, \partial M)$ is G equivalent to the trivial bundle $(L, \partial L) \times \mathbf{R}$ with trivial action of G on the set $\mathbf{R}$ of real numbers. We admit the case in which $\partial L = \emptyset$ and $\partial M \neq \emptyset$.

We further assume that L separates M, that is, $M = N_1 \cup N_2$ (pasting along the common parts L) for some G invariant submanifolds N_i of codimension zero, and denote this decomposition simply by $M = N_1 \cup N_2$.

DEFINITION 1.1. Let M_1 and M_2 be m-dimensional G manifolds. Then M_1 and M_2 are said to be obtained from each other by an equivariant cutting and pasting (a G-SK *process*) if M_2 has been obtained from M_1 by the process as mentioned above, that is, $M_1 = N_1 \cup_\varphi N_2$ and $M_2 = N_1 \cup_\psi N_2$ pasting along the common parts $L \subset M_i$ for some G diffeomorphisms φ and $\psi : L \to L$. Moreover we say that M_1 and M_2 are G-SK *equivalent*, in symbols $M_1 \sim M_2$, if there is a G manifold K, which may be $\emptyset$, such that the disjoint union $M_2 + K$ can be obtained from $M_1 + K$ by a finite sequence of equivariant cuttings and pastings.

The SK equivalence $\sim$ is an equivalent relation on the set of m-dimensional G manifolds. Denote by $[M]$ the equivalence class containing a G manifold M. The set $\Gamma_m^G(pt, pt)$ of all these classes forms a cancellative abelian semigroup if we use disjoint union as addition, and has a zero $[\emptyset]$. We define by $SK_m^G(pt, pt)$ the Grothendieck group of this semigroup. By defining $SK_*^G(pt, pt) = \bigoplus_{m\geq 0} SK_m^G(pt, pt)$ we have a graded SK_* module with multiplication given by the cartesian product of manifolds, where SK_* is the SK ring of closed manifolds.

REMARK 1.2. $SK_* = \sum_{m\geq 0} SK_m$ is a polynomial ring $\mathbf{Z}[\alpha]$ with a generator α represented by the real projective plane $\mathbf{R}P^2$, and a class $[M]$ in SK_m is determined by the value $\chi(M) \in \mathbf{Z}$. Let $SK_*(pt, pt) = SK_*^{\{1\}}(pt, pt)$ ($\{1\}$; the trivial group), which is a free SK_* module with basis $\{[D^0], [D^1]\}$, where D^m is the m-disk in general. Moreover a class $[M]$ in $SK_m(pt, pt)$ is determined by the value $\bar{\chi}(M) \in \mathbf{Z}$. In particular, $\alpha^n = [D^{2n}]$ and hence the inclusion map $i_* : SK_* \to SK_*(pt, pt)$ is injective (cf. [5], [7; Theorem 1.2] and Example 3.9). Note that $\bar{\chi}(M) = (-1)^{\dim(M)}\chi(M)$ by applying χ to the double $\mathcal{D}M = M \cup M$ of M.

Let H be a subgroup of G, then an H module means a finite dimensional real vector space together with a linear action of H on it. If M is a G manifold and $x \in M$, then the slice theorem tells us that there is a G_x module U_x which is equivariantly diffeomorphic to a G_x neighbourhood of x, where $G_x = \{g \in G;\, gx = x\}$ is the isotropy subgroup at x. This U_x decomposes as $U_x = \mathbf{R}^p \oplus V_x$, where G_x acts trivially on $\mathbf{R}^p$ and $V_x^{G_x} = \{0\}$. The pair $[G_x; V_x]$ is called the *slice type* of x. By a slice type in general, we mean a pair $[H; V]$ of a subgroup H and an H module V such that $V^H = \{0\}$.

DEFINITION 1.3. Suppose that J is a subgroup of H such that the quotient H/J is cyclic. Then an irreducible H module $V(J, j)$ $\left(1 \le j < \left(\frac{1}{2}\right)\phi(|H/J|) + 1\right)$, where $|H/J|$ denotes the number of elements in H/J and ϕ is the Euler phi-function, is defined as follows.

(1) If $|H/J| = 1$, then $V(J, 1) = \mathbf{R}$ with trivial H action.

(2) If $|H/J| = 2$, then $V(J, 1) = \mathbf{R}$ with action of $h \in H$ given by multiplication by $+1$ if $h \in J$ or -1 if $h \in H \setminus J$.

(3) If $|H/J| = d > 2$, then $V(J, j_k)$ is the set $\mathbf{C}$ of complex numbers with a generator h of $H/J \cong \mathbf{Z}_d$ acting by multiplication by $\exp(2\pi i j_k/d)$, where $\{j_k\}$ is a complete set of integers such that $0 < j_1 < \cdots < j_{\phi(d)} < d$ and each j_k is prime to d (cf. [8; 1.6]).

The G slice types are therefore of the form $\sigma = [H; V]$, where V is a product of non-trivial irreducible H modules $V(J, j_k)$ as in (2) or (3). We denote by σ_{-1} the slice type $[\{1\}; \{0\}]$. For any positive divisor k of $|G|$, let $L(k)$ be the set consisting of all subgroups H of G with $|H| = k$. Now consider a total ordering on $L(k)$, then the ordering gives the one of all subgroups of G naturally, preserving inclusion of subgroups, that is, if $H \subseteq K$ then $H \le K$. Moreover, for any H this ordering leads to the one of the non-trivial irreducible H modules: $V(J_1, j_1) < V(J_2, j_2)$ if $J_2 < J_1$ or $J_1 = J_2$ and $j_1 < j_2$. We then order the set $St(G)$ of all G slice types as follows.

(1) $[H; V] < [K; W]$ if $\dim(V) < \dim(W)$.

(2) Suppose that $\dim(V) = \dim(W)$, then $[H; V] < [K; W]$ if $H < K$.

(3) Suppose that $\dim(V) = \dim(W)$ and $H = K$, then $[H; V] < [H; W]$ if $V < W$ in the ordering of H modules induced lexicographically from the one of irreducible H modules.

DEFINITION 1.4. Let W be a K module and H a subgroup of K, then denote by W_H and H module W induced from $H \subseteq K$. Let $\{W_j\}$ be the set of non-trivial irreducible K modules. If $\tau = [K; W]$, $W = \prod_j W_j^{a(j)}$ $(a(j) \ge 0)$ is a slice type, we define a slice type τ_H by $\tau_H = [H; V]$ where V is the non-trivial part of the H module $\prod_j (W_j)_H^{a(j)}$. Note that $\tau_{\{1\}} = \sigma_{-1}$ for any τ.

REMARK 1.5. (i) More precisely, let $W_j = V(L, l)$ for some $L \subset K$ with K/L cyclic and $(l, a) = 1$ $(a = |K/L|)$. Then $(W_j)_H = V(L \cap H, l')$ for some l' with $(l', b) = 1$ $(b = |H/L \cap H|)$. The integer l' is determined by the action of $H/L \cap H \simeq LH/L$ on $(W_j)_H$ induced from the one of K/L on W_j. We see that $(W_j)_H$ is the trivial H module only if $H \subseteq L$, and $(W_j)_H = \mathbf{R}$ if $a = 2$ or $\mathbf{R}^2$ if $a > 2$. Moreover it follows that $|\tau| - |\tau_H|$ is the sum of $\dim((W_j)_H)$ such that $(W_j)_H = \mathbf{R}$ or $\mathbf{R}^2$, where $|\tau| = \dim(W)$.

(ii) $W_H = \mathbf{R}^{|\tau| - |\tau_H|} \times V$ as an H module, and a K invariant subspace $W^H = \mathbf{R}^{|\tau| - |\tau_H|} \times \{0\}$ of W has slice types τ_U with $H \subseteq U \subseteq K$. We note that $\tau_U \le \tau$ because $|\tau_U| \le |\tau|$.

Now rename the G slice types: $\sigma_{-1} = \rho_0 < \rho_1 < \cdots < \rho_k < \cdots$ by using the total ordering on $St(G)$. Then it follows from Remark 1.5(ii) that $\mathcal{F}_k = \{\rho_i ; 0 \le i \le k\}$ is a family of G slice types in the sense of [8; 1.2] ($k \ge 0$). Using these families, we have the following proposition.

PROPOSITION 1.6 (cf. [3; Proposition 1.13]). *$SK_*^G(pt, pt)$ is a free SK_* module with basis $\mathcal{B} = \{x_\sigma, \hat{x}_\sigma ; \sigma \in St(G)\}$, where $x_\sigma = [G \times_H D(V)]$ and $\hat{x}_\sigma = [G \times_H D(V \times \mathbf{R})]$ for $\sigma = [H; V]$ ($D(V)$; the associated H disk).*

2. G-SK invariants.

DEFINITION 2.1. If $\sigma = [H; V]$ and M is a G manifold, then define M_σ to be the set consisting of those points $x \in M$ whose slice types σ_x satisfy the condition $(\sigma_x)_H = \sigma$ in the sense of Definition 1.4.

REMARK 2.2. (i) Let M_H be M with the induced H action, then M_σ is precisely the set $(M_H)_\sigma = \{x \in M_H ; \sigma_x = \sigma\}$. Thus the maximality of σ in the family $\mathcal{F}(M_H) = \{\sigma_x ; x \in M_H\}$ implies that M_σ is a G invariant submanifold of M with $\dim(M_\sigma) = \dim(M) - |\sigma|$ via the slice theorem. It is easy to see that $\partial(M_\sigma) = (\partial M)_\sigma$ by using a G collar. Note that $M_{\sigma_{-1}} = M$. Further, if $\sigma = [H; V]$ and $\sigma' = [H; V']$ with $\sigma \neq \sigma'$, then $M_\sigma \cap M_{\sigma'} = \emptyset$. Thus $M^H \doteq \coprod_\sigma M_\sigma$ summing over all σ with H as isotropy subgroup. It follows from the definition that if $M_1 \sim M_2$ then $(M_1)_\sigma \sim (M_2)_\sigma$ naturally (cf. [5; Chapter 3]).

(ii) Let $M \times N$ be the cartesian product of G manifolds M and N straightening the angle. Then, for any $\sigma = [H; V]$ we have $(M \times N)_\sigma = \coprod_{(\sigma', \sigma'')} M_{\sigma'} \times N_{\sigma''}$ by using the decomposition of $(M \times N)^H$ as mentioned above, where the sum is taken over all paires (σ', σ'') of slice types with H as isotropy subgroup such that $\sigma' \times \sigma'' = \sigma$.

EXAMPLE 2.3. Let $M = G \times_K D(W)$ for $\tau = [K; W]$, then $M_\sigma = G \times_K D(W^H)$ if $H \subseteq K$ and $\sigma = \tau_H$, or $\emptyset$ otherwise. Therefore it follows from Remarks 1.2 and 1.5(ii) that

$$[M_{\tau_H}] = |G/K|[D^{|\tau| - |\tau_H|}]$$
$$= \begin{cases} |G/K|\alpha^{\frac{1}{2}(|\tau| - |\tau_H|)}[D^0] & \text{if } |\tau| \equiv |\tau_H| \pmod 2, \\ |G/K|\alpha^{\frac{1}{2}(|\tau| - |\tau_H| - 1)}[D^1] & \text{if } |\tau| \equiv |\tau_H| + 1 \pmod 2 \end{cases}$$

in $SK_*(pt, pt)$.

DEFINITION 2.4. Let T be a map for m-dimensional G manifolds, which is assumed to take values in $\mathbf{Z}$ and to be additive with respect to disjoint union $+$, that is, if $M = M_1 + M_2$ then $T(M) = T(M_1) + T(M_2)$. We call T a G-SK *invariant* if $T(N_1 \cup_\varphi N_2) = T(N_1 \cup_\psi N_2)$ for any G diffeomorphisms φ and $\psi : L \to L$ in Definition 1.1. If $M_1 \sim M_2$, then $T(M_1) = T(M_2)$. Thus the map T induces an additive homomorphism $T : SK_m^G(pt, pt) \to \mathbf{Z}$. The set $\mathcal{T}_m^G$ consisting of all these invariants is a $\mathbf{Z}$ module under the natural addition.

We sometimes write $T(M)$ instead of $T(x)$ for $x = [M]$ if no confusion can arise.

EXAMPLE 2.5. (i) For any $\sigma \in St(G)$, a map $\bar{\chi}_\sigma$ (or χ_σ) defined by $\bar{\chi}_\sigma(M) = \bar{\chi}(M_\sigma)$ (or $\chi_\sigma(M) = \chi(M_\sigma)$) is a G-SK invariant respectively. We see that $\bar{\chi}_\sigma = (-1)^{m-|\sigma|}\chi_\sigma$ as an element of $\mathcal{T}_m^G$. Further $\bar{\chi}^K$ (or χ^K) in Introduction is also an invariant because $\bar{\chi}^K = \sum_\tau \bar{\chi}_\tau$ (or $\chi^K = \sum_\tau \chi_\tau$) summing over all τ with K as isotropy subgroup respectively. Note that $\bar{\chi}^{\{1\}} = \bar{\chi}_{\sigma_{-1}} = \bar{\chi}$ (cf. Remarks 1.2 and 2.2(i)). See Example 2.9 for another related invariant.

(ii) Suppose that an invariant T is defined for all G manifolds. In general, T is determined by the values $T(\alpha^u x_\tau)$ and $T(\alpha^v \hat{x}_\tau)$ for $\tau \in St(G)$ and u, v (≥ 0) (cf. Proposition 1.6). We now consider the case in which T is a sum of χ^K. First note that $T(\alpha^u x_\tau) = T(\alpha^v \hat{x}_\tau)$, and it follows from Example 2.3 that the value $T(x_\tau)$ for $\tau = [K; W]$ does not depend on a specific slice type τ but depends on K. Hence we may write $T(x_\tau) = \lambda_K$. An invariant T is then of the form:

$$T = |G|^{-1} \sum_K \xi_K \chi^K ,$$

where

$$\xi_K = \sum_{H \subseteq K} |H| \lambda_H \mu(H, K)$$

and μ is the Möbius function which is, in our case, defined inductively for $H \subseteq K$ by $\mu(H, H) = 1$ and $\mu(H, K) = -\sum_{H \subseteq U \subset K} \mu(H, U)$ if $H \subset K$ (Here $H \subset K$ means that $H \subseteq K$ and $H \neq K$.). In fact, if we write T as $T = |G|^{-1} \sum_U c_U \chi^U$ ($|G|^{-1} c_U \in \mathbf{Z}$), then $|K|^{-1} \sum_{U \subseteq K} c_U = \lambda_K$ ($= T(x_\tau)$) from Example 2.3. On the other hand, we have

$$\begin{aligned}
\sum_{U \subseteq K} \xi_U &= \sum_{H \subseteq U \subseteq K} |H| \lambda_H \mu(H, U) \\
&= |K| \lambda_K + \sum_{H \subset K} |H| \lambda_H \left(\sum_{H \subseteq U \subseteq K} \mu(H, U) \right) \\
&= |K| \lambda_K .
\end{aligned}$$

Hence $c_K = \xi_K$ by the uniqueness of the class $\{c_K\}$, showing that T is of the desired form. For example, let $G_r = \mathbf{Z}_{p^r}$ (p; a prime number and $r \geq 1$). Then $\mu(G_s, G_t) = 1$ if $s = t$, -1 if $s = t - 1$ or 0 if $s < t - 1$, so T is of the form

$$T = (1/p)^r \left\{ \lambda_0 \chi + \sum_{0 < t \leq r} p^{t-1} (p\lambda_t - \lambda_{t-1}) \chi^{G_t} \right\}$$

where $\lambda_t = \lambda_{G_t}$. In particular, we have

$$\chi + \sum_{0 < t \leq r} (p^t - p^{t-1}) \chi^{G_t} \equiv 0 \pmod{p^r}$$

by setting $\lambda_t = 1$ ($0 \leq t \leq r$) (cf. [6; Corollary 5.20]).

Let H be a subgroup of G. Then define inductively an integer $n_H(K)$ for (totally ordered) subgroups K with $H \subseteq K \subseteq G$ by $n_H(K) = |K/H| - \sum_{H \subseteq L \subset K} n_H(L)$ and $n_H(H) = 1$

(cf. [6; Definition 5.3]). On the other hand, for $\sigma = [H; V]$ and a subgroup K with $H \subseteq K$, denote by $\mathcal{S}_K(\sigma)$ the set consisting of those slice types $\tau = [K; W]$ such that $\tau_H = \sigma$.

THEOREM 2.6. *For $\sigma = [H; V] \in St(G)$, define $\bar{\theta}_\sigma$ by*

$$\bar{\theta}_\sigma = |G/H|^{-1}\left\{\bar{\chi}_\sigma + \sum_{H \subset K \subseteq G} n_H(K)\left(\sum_{\tau \in \mathcal{S}_K(\sigma)} (-1)^{|\tau|-|\sigma|}\bar{\chi}_\tau\right)\right\}.$$

Then the class $\{\bar{\theta}_\sigma; |\sigma| \le m\}$ provides a basis for $\mathcal{T}_m^G$ as a free $\mathbf{Z}$ module.

PROOF. For $\sigma = [H; V]$, let $g_\sigma : SK_*^G(pt, pt) \to SK_{*-|\sigma|}(pt, pt)$ be a map given by $g_\sigma([M]) = [M_\sigma]$ and define a map f_σ by

$$f_\sigma = |G/H|^{-1}\left\{g_\sigma + \sum_{H \subset K \subseteq G} n_H(K)\left(\sum_{\tau \in \mathcal{S}_K(\sigma)} [D^{|\tau|-|\sigma|}]g_\tau\right)\right\}.$$

Now look at the basis elements of $\mathcal{B}$ in Proposition 1.6. Then the value $f_\sigma(x_\tau)$ for $\tau = [K; W]$ which does not vanish is

$$f_{\tau_L}(x_\tau) = \begin{cases} \alpha^{\frac{1}{2}(|\tau|-|\tau_L|)}[D^0] & \text{if } |\tau| \equiv |\tau_L| \pmod 2, \\ \alpha^{\frac{1}{2}(|\tau|-|\tau_L|-1)}[D^1] & \text{if } |\tau| \equiv |\tau_L| + 1 \pmod 2 \end{cases} \tag{2.1}$$

for $L \subseteq K$. In fact

$$\begin{aligned} f_{\tau_L}(x_\tau) &= |G/L|^{-1}\left\{g_{\tau_L}(x_\tau) + \sum_{L \subset U \subseteq K} n_L(U)[D^{|\tau_U|-|\tau_L|}]g_{\tau_U}(x_\tau)\right\} \\ &= |K/L|^{-1}\left(\sum_{L \subseteq U \subseteq K} n_L(U)\right)[D^{|\tau|-|\tau_L|}] \\ &= [D^{|\tau|-|\tau_L|}] \end{aligned}$$

by Example 2.3 and the identity $\sum_{L \subseteq U \subseteq K} n_L(U) = |K/L|$. The values on the elements $\hat{x}_\tau$ are given by $f_\sigma(\hat{x}_\tau) = [D^1]f_\sigma(x_\tau)$. Thus each f_σ induces a map $f_\sigma : SK_*^G(pt, pt) \to SK_{*-|\sigma|}(pt, pt)$ of degree $-|\sigma|$. Now define an SK_* homomorphism f_* by

$$f_* = \oplus_k f_{\rho_k} : SK_*^G(pt, pt) \to A = \oplus_k SK_{*-|\rho_k|}(pt, pt), \tag{2.2}$$

where $St(G) = \{\rho_k; k \ge 0\}$ is totally ordered as mentioned in §1. Let $\mathcal{B}'_k = \{[D^0], [D^1]\}$ be the basis for the k-th copy of $SK_*(pt, pt)$ in A (cf. Remark 1.2). We can totally order the basis elements of $\mathcal{B}$ and $\mathcal{B}' = \cup_k \mathcal{B}'_k$ (for A) naturally by using the ordering of $St(G)$, that is, for $\mathcal{B}$ first $x_{\rho_i} < x_{\rho_j}$ if $i < j$ and then $x_{\rho_i} < \hat{x}_{\rho_i}$ for any i. It follows from (2.1) that f_* is an isomorphism because the matrix relative to the ordered bases $\mathcal{B}$ and $\mathcal{B}'$ is triangular with components 1 on the diagonal. Now let T be an element of $\mathcal{T}_m^G$, then there is a factorization:

$$T : SK_m^G(pt, pt) \overset{f_*}{\cong} \oplus_k SK_{m-|\rho_k|}(pt, pt) \overset{\oplus_k \bar{\chi}}{\cong} \oplus_k \mathbf{Z} \xrightarrow{T'} \mathbf{Z} \tag{2.3}$$

for some T', where the direct sum is taken over all k with $|\rho_k| \leq m$ (cf. Remark 1.2). This implies that $T = \sum_k T'(1_k)\bar{\theta}_{\rho_k}$ where $\bar{\theta}_{\rho_k} = \bar{\chi} \circ f_{\rho_k}$ and $1_k = 1$ in the k-th copy of $\mathbf{Z}$ in $\oplus_k \mathbf{Z}$. Thus we have the theorem. q.e.d.

EXAMPLE 2.7. Suppose that $G = \mathbf{Z}_n$ ($n \geq 2$). Then, for $\sigma = [\mathbf{Z}_s; V]$ with $s \mid n$, we have

$$\bar{\theta}_\sigma = (n/s)^{-1} \left\{ \bar{\chi}_\sigma + \sum_{s<t\leq n, s|t|n} \phi(t/s) \left(\sum_{\tau \in \mathcal{S}_{\mathbf{Z}_t}(\sigma)} (-1)^{|\tau|-|\sigma|} \bar{\chi}_\tau \right) \right\}$$

because $n_{\mathbf{Z}_s}(\mathbf{Z}_t) = \phi(t/s)$ by definition.

REMARK 2.8. Consider

$$\theta_\sigma = |G/H|^{-1} \left\{ \chi_\sigma + \sum_{H \subset K \subseteq G} n_H(K) \left(\sum_{\tau \in \mathcal{S}_K(\sigma)} \chi_\tau \right) \right\}$$

for $\sigma = [H; V]$. Then $\theta_\sigma = (-1)^{m-|\sigma|}\bar{\theta}_\sigma$ as an element of $\mathcal{T}_m^G$ (cf. Example 2.5(i)). Thus the class $\{\theta_\sigma; \sigma \in St(G)\}$ is also a basis for $\mathcal{T}_m^G$.

EXAMPLE 2.9. If M is a G manifold and $\sigma = [H; V]$ a slice type, then define $M_{[\sigma]} = \{x \in M; \sigma_x = \sigma\}$. The set $M_{[\sigma]}$ is a codimension zero open invariant submanifold of M_σ because $M_{[\sigma]}$ is given locally by $G \times_H (\mathbf{R}^p \oplus \{0\})$ in $G \times_H U_x$ with $p = \dim(M) - |\sigma|$, where $U_x = \mathbf{R}^p \oplus V$ is the H equivariant neighbourhood of $x \in M_{[\sigma]}$ in M. Since the triad $(M_{[\sigma]}, (N_1)_{[\sigma]}, (N_2)_{[\sigma]})$ is excisive for a decomposition $M = N_1 \cup_\varphi N_2$, a map $\chi_{[\sigma]}$ defined by $\chi_{[\sigma]}(M) = \chi(M_{[\sigma]})$ is a G-SK invariant as in the case of χ_σ (Remark. The set $M_{[\sigma]}$ coincides with the one $M_{[H,\sigma]}$ in [5; Chapter 3], where $[H, \sigma] = \sigma$.). The above remark tells us that $\chi_{[\sigma]}$ is a linear combination of the invariants θ_τ. For example, let $G_r = \mathbf{Z}_{p^r}$ (p; a prime number and $r \geq 1$). In this case, if $\tau = [G_t; W]$ then

$$\theta_\tau = (1/p)^{r-t} \left\{ \chi_\tau + \sum_{t<u\leq r} p^{u-t-1} \left(\sum_{\mu \in \mathcal{S}_{G_u}(\tau)} \chi_\mu \right) \right\}$$

because $\phi(p^{u-t}) = p^{u-t-1}$ as mentioned in Example 2.7. Since $\chi_{[\sigma]}(\alpha^u x_\tau) = \chi_{[\sigma]}(\alpha^v \hat{x}_\tau) = \chi_{[\sigma]}(x_\tau)$ for the basis elements of $\mathcal{B}$, the invariant $\chi_{[\sigma]}$ is determined by the values $\{\chi_{[\sigma]}(x_\tau); \tau \in St(G)\}$. Now let $\sigma = [G_s; V]$ and denote $M(\tau) = G_r \times_{G_t} D(W)$ the representative of x_τ. Then it follows from Remark 1.5(ii) that

$$M(\tau)_{[\sigma]} = \begin{cases} G_r \times_{G_t} \{D(W)^{G_s} \setminus D(W)^{G_{s+1}}\} & \text{if } \sigma = \tau_s \text{ with } 0 \leq s \leq t\,, \\ \emptyset & \text{otherwise}\,, \end{cases} \tag{2.4}$$

where $D(W)^{G_s} = D^{|\tau|-|\tau_s|}$ ($\tau_s = \tau_{G_s}$) and $D(W)^{G_{t+1}} = \emptyset$. First we suppose that $p = 2$. Note that $|\tau_s|$ is even if $0 \leq s < t$ (cf. Remark 1.5(i)). Then $\chi(D(W)^{G_s} \setminus D(W)^{G_{s+1}}) = 1$ if $s = t$, $\chi(S^{|\tau|-|\tau_{t-1}|-1})$ if $s = t-1$, $|\tau| > |\tau_{t-1}|$ or 0 if $s = t-1$, $|\tau| = |\tau_{t-1}|$ or $s \leq t-2$. Thus, in case $|\sigma|$ is even, $\chi_{[\sigma]}(M(\tau)) = 2^{r-s}$ if $\tau = \sigma$ or τ is of the form $[G_{s+1}; W]$ with $\tau_s = \sigma$ and $|\tau|$ odd, or 0 otherwise. In case $|\sigma|$ is odd, $\chi_{[\sigma]}(M(\tau)) = 2^{r-s}$ if $\tau = \sigma$ or 0 otherwise. On the other hand, $\theta_\sigma(M(\tau)) = (-1)^{|\tau|-|\sigma|}\bar{\theta}_\sigma(M(\tau)) = 1$ if $\tau_s = \sigma$ $(0 \le s \le t)$ or 0 otherwise by (2.1) and Remark 2.8. We therefore have the following:

$$\chi_{[\sigma]} = 2^{r-s}\left(\theta_\sigma - \sum_{\mu}\theta_\mu\right), \tag{2.5}$$

where the sum is taken over all $\mu = [G_{s+1}; W]$ with $\mu_s = \sigma$ and $|\mu|$ even. Next suppose that p is odd prime. In this case, $|\tau_s|$ is always even $(0 \le s \le t)$ (cf. Definition 1.3(3) and Remark 1.5(i)), so $\chi(D(W)^{G_s} \setminus D(W)^{G_{s+1}}) = 1$ if $s = t$ or 0 otherwise. This means that $\chi_{[\sigma]}(M(\tau)) = p^{r-s}$ if $\tau = \sigma$ or 0 otherwise from (2.4). Thus we also have the equality (2.5) with 2^{r-s} replaced by p^{r-s}.

REMARK 2.10. Let $e_\sigma = (1/p)^{r-s}\chi_{[\sigma]}$, then it follows from (2.5) that the class $\{e_\sigma; \sigma \in Sr(G_r)\}$ is also a basis for $\mathcal{T}_m^{G_r}$.

Now go back to the basis $\{\bar{\theta}_\sigma\}$ in Theorem 2.6. We then have the following corollary by using the isomorphism $\oplus_k\bar{\theta}_{\rho_k}$ in (2.3) and the equality $\chi_\sigma = (-1)^{m-|\sigma|}\bar{\chi}_\sigma$.

COROLLARY 2.11. *Let M_1 and M_2 be m-dimensional G manifolds. Then $[M_1] = [M_2]$ in $SK_m^G(pt, pt)$ if and only if $\bar{\chi}_\sigma(M_1) = \bar{\chi}_\sigma(M_2)$ (or $\chi_\sigma(M_1) = \chi_\sigma(M_2)$) for all $\sigma \in St(G)$ with $|\sigma| \le m$.*

3. G-SK processes.

Let SK_*^G be the SK theory resulting from equivariant cuttings and pastings of closed G manifolds (cf. [5] and [8]). In this section, we study a relation between the theory $SK_*^G \otimes R_2$ and the one $SK_*^G(pt, pt) \otimes R_2$, and give some related examples by performing G-SK processes.

PROPOSITION 3.1. *The sequence*

$$0 \longrightarrow SK_*^G \otimes R_2 \xrightarrow{i_*} SK_*^G(pt, pt) \otimes R_2 \xrightarrow{\partial_*} SK_*^G \otimes R_2 \longrightarrow 0$$

is split exact, where i_* is the inclusion map and ∂_* is defined by $\partial_*([M] \otimes 1) = [\partial M] \otimes 1$. A splitting map D_* (or d_*) to i_* (or ∂_*) is given by $D_*([M]\otimes 1) = [\mathcal{D}M]\otimes \frac{1}{2}$ (or $d_*([M]\otimes 1) = [M \times D^1] \otimes 1/2$) respectively. Here $\mathcal{D}M$ is the double of M if $\partial M \ne \emptyset$ or $2M$ if $\partial M = \emptyset$.

PROOF. We see that the kernel of the inclusion map $i_{0,*} : SK_*^G \to SK_*^G(pt, pt)$ consists of elements of order 2. This implies that i_* in the above exact sequence is injective because $SK_*^G(pt, pt)$ has no torsion from Proposition 1.6 or (2.3). To show the above, suppose that $i_{0,*}(x) = 0$ for $x = [M_1] - [M_2]$ in SK_*^G, then $[M_1 \times D^1] = [M_2 \times D^1]$ in $SK_*^G(pt, pt)$ naturally. By applying the map ∂_* to this, we have $2[M_1] = 2[M_2]$ in SK_*^G, that is, x is of order 2. Clearly $\partial \circ i_* = 0$. Now, for any G manifold M we have

$$2[M] = [\mathcal{D}M] + [\partial M \times D^1] \tag{3.1}$$

in $SK_*^G(pt, pt)$ (cf. [7; Lemma 4.9]). Using this, we can prove that $\mathrm{Ker}(\partial_*) \subseteq \mathrm{Im}(i_*)$ as follows. If $\partial_*(x) = 0$ for $x = ([M_1]-[M_2])\otimes(1/2)^k$, then $x = ([\mathcal{D}M_1]-[\mathcal{D}M_2])\otimes(1/2)^{k+1}$

from (3.1), showing that x is in the image of i_*. By definition, d_* is a splitting map to ∂_*. On the other hand, consider a map D_*. If $[M_1] = [M_2]$ in $SK_*^G(pt, pt) \otimes R_2$ with $\partial M_1 \neq \emptyset$ and $\partial M_2 = \emptyset$, then $[\mathcal{D}M_1] = 2[M_2]$ in $SK_*^G(pt, pt) \otimes R_2$ from (3.1) and hence in $SK_*^G \otimes R_2$ via the injection i_*. Thus D_* is well-defined and a splitting map to i_*. q.e.d.

We see that the torsion subgroup of SK_*^G is a 2-group by using Triviality 2.10, Theorem 3.1 and Lemma 3.2 in [5]. More precisely, we have the following from the beginning of the above proof.

COROLLARY 3.2. *A possible torsion element of SK_*^G is of order* 2.

Now the following result is immediate from Corollary 2.11 and Proposition 3.1.

PROPOSITION 3.3. *Let M_1 and M_2 be m-dimensional closed G manifolds. Then $[M_1] = [M_2]$ in $SK_m^G \otimes R_2$ if and only if $\chi_\sigma(M_1) = \chi_\sigma(M_2)$ for all $\sigma \in St(G)$ with $|\sigma| \leq m$ and $|\sigma| \equiv m$* (mod 2). *In particular, the set $\{\chi_\sigma(M)\}$ determines a class $[M]$ in SK_*^G modulo torsion.*

In the above, note that $\chi_\sigma(M) = 0$ if $|\sigma| \equiv m+1$ (mod 2) because $\dim(M_\sigma) = m - |\sigma| \equiv 1$ (mod 2) and M_σ is closed (cf. Remark 2.2(i)).

REMARK 3.4. Let M be a closed G manifold and $\sigma = [H; V]$ a slice type of M. Then the manifold $M_{[\sigma]}$ fibers over the orbit space $M_{[\sigma]}/G$ with fiber G/H. We see that $M_{[\sigma]}/G$ is usually non-compact manifold given locally by $\mathbf{R}^p \oplus \{0\} \subset U_x$ of $x \in M_{[\sigma]}$ (cf. Example 2.9). A G-SK process on M induces the one on $M_{[\sigma]}/G$ naturally, so we can define a G-SK invariant e_σ by $e_\sigma(M) = \chi(M_{[\sigma]}/G)$. Now let $\mathcal{T}(R_2)_m^G$ be the set consisting of all G-SK invariants $T' : SK_m^G \otimes R_2 \to R_2$. Then we see that the set $\{e_\sigma \otimes id\}$ with $|\sigma| \leq m$, $|\sigma| \equiv m$ (mod 2) provides a basis for $\mathcal{T}(R_2)_m^G$, and it gives us another set $\{e_\sigma(M)\}$ of integers which also determines a class $[M]$ in SK_*^G modulo torsion (cf. [5; Corollary 3.3]).

We know that $SK_*^G \otimes R_2$ is a free $SK_* \otimes R_2 \cong R_2[\alpha]$ module with basis $\mathcal{C} = \{y_\sigma; \sigma \in St(G)\}$, where $y_\sigma = [G \times_H S(V \times \mathbf{R})]$ and $S(V \times \mathbf{R})$ is the associated H sphere (cf. [5; p. 40]). Using this, we have the following from Proposition 3.1.

COROLLARY 3.5. *$SK_*^G(pt, pt) \otimes R_2$ is a free $SK_* \otimes R_2$ module with basis $\mathcal{B}_0 = \{y_\sigma, \hat{y}_\sigma; \sigma \in St(G)\}$, where $\hat{y}_\sigma = [G \times_H S(V \times \mathbf{R}) \times D^1]$ for $y_\sigma = [G \times_H S(V \times \mathbf{R})]$.*

On the other hand, the set $\mathcal{B}$ in Proposition 1.6 also provides a basis for $SK_*^G(pt, pt) \otimes R_2$. The rest of this section will be devoted to studying a relation between these bases $\mathcal{B}$ and $\mathcal{B}_0$. To proceed our argument, we first need the following Lemmas 3.6 and 3.7.

Let us consider a multiplication on $SK_*^G(pt, pt)$ induced by the cartesian product of G manifolds.

LEMMA 3.6. *The multiplicative relations on the basis elements of $\mathcal{B}$ are given by the following* (i) *and* (ii):

(i) $[G \times_H D(V)] \cdot [G \times_K D(W)] = a(H, K)[D^b][G \times_L D(V_L \times W_L)]$ *for any* $\sigma = [H; V]$ *and* $\tau = [K; W]$, *where* $L = H \cap K, a(H, K) = |G||L|/|H||K|$ *and* $b = |\sigma| + |\tau| - (|\sigma_L| + |\tau_L|)$.

(ii) $\hat{x} \cdot y = x \cdot \hat{y} = \widehat{x \cdot y}$ *and* $\widehat{(\hat{x})} = \alpha x$ *for any x and y, where* $\hat{x} = [D^1]x$ *in general.*

PROOF. The both sides of (i) have the slice types $\{\sigma_{L'} \times \tau_{L'}; L' \subseteq L\}$ in the sense of Definition 1.4 (cf. Remark 1.5(ii)), and it follows from Example 2.3 that each side has the data:

$$\begin{aligned}\bar{\chi}_{\sigma_{L'} \times \tau_{L'}} &= |G/H|(-1)^{|\sigma|-|\sigma_{L'}|} \cdot |G/K|(-1)^{|\tau|-|\tau_{L'}|} \\ &= a(H, K)(-1)^b \cdot |G/L|(-1)^{|\sigma_L \times \tau_L| - |\sigma_{L'} \times \tau_{L'}|}\end{aligned}$$

and $\bar{\chi}_\nu = 0$ if $\nu \notin \{\sigma_{L'} \times \tau_{L'}\}$. Thus (i) is obtained from Corollary 2.11. The last equality of (ii) follows from the one $\alpha = [D^2]$ in Remark 1.2. q.e.d.

Let M_1 and M_2 be $H \times \mathbf{Z}_2$ manifolds such that $\mathbf{Z}_2$ acts freely on them. If there is an $H \times \mathbf{Z}_2$-SK equivalence between them in the sense of Definition 1.1, then we write it as $M_1 \overset{\mathbf{Z}_2}{\sim} M_2$. This induces the one $M_1 \times_{\mathbf{Z}_2} P \sim M_2 \times_{\mathbf{Z}_2} P$ naturally for any $\mathbf{Z}_2$ manifold P, where an H action on $M_i \times_{\mathbf{Z}_2} P$ is given by that on M_i. In particular, $\bar{M}_1 \sim \bar{M}_2$ where $\bar{M}_i = M_i/\mathbf{Z}_2$ is the orbit space of M_i.

LEMMA 3.7. *Let* W_i $(i = 1, 2)$ *be H modules, then*

$$S(W_1 \times W_2) + [0, 1] \times S(W_1) \times S(W_2) \overset{\mathbf{Z}_2}{\sim} S(W_1) \times D(W_2) + D(W_1) \times S(W_2) \quad (3.2)$$

where $\mathbf{Z}_2$ *acts on* $S(Y)$, $Y = W_1 \times W_2$, W_i *or* $D(W_i)$ *by multiplication by* -1, *and H (or* $\mathbf{Z}_2$) *acts trivially on the interval* $[0, 1]$.

PROOF. Let $N_i = A_i + B_i$ $(i = 1, 2)$, where $A_1 = S(W_1) \times D(W_2)$, $A_2 = D(W_1) \times S(W_2)$, $B_1 = [0, 1/2] \times S(W_1) \times S(W_2)$ and $B_2 = [1/2, 1] \times S(W_1) \times S(W_2)$. Further let $L = L' + L''$ where $L' = S(W_1) \times S(W_2) = \partial A_i$ and $L'' = \{1/2\} \times S(W_1) \times S(W_2)$. By pasting N_1 to N_2 in two ways by ($H \times \mathbf{Z}_2$-equivariant) natural identifications φ and $\psi : L \to L$, we obtain

$$\begin{aligned} N_1 \cup_\varphi N_2 &= S(W_1 \times W_2) + [0, 1] \times S(W_1) \times S(W_2)\,, \\ N_1 \cup_\psi N_2 &= S(W_1) \times D(W_2) + D(W_1) \times S(W_2)\,. \end{aligned}$$

This implies (3.2). q.e.d.

To obtain Theorem 3.11, we represent the class $[S(W)]$ of an H sphere as a sum of basis elements $[H \times_J D(Z)]$ and $[H \times_J D(Z \times \mathbf{R})]$ for $SK_*^H(pt, pt)$ in the following Lemmas 3.8 and 3.10.

LEMMA 3.8. *Let* $W = W' \cdot W'' \times \mathbf{R}^{2n}$, *where* $W' = \prod_k V_0(K_k)^{2s(k)}$, $W'' = \prod_l V(L_l, m_l)^{t(l)}$ *and* $V_0(K_k) = V(K_k, 1)$ *(or* $V(L_l, m_l)$) *are irreducible, inequivalent H modules in Definition* 1.3(2) *(or* (3)) *respectively. Then we have* $S(W) \overset{\mathbf{Z}_2}{\sim} \emptyset$.

PROOF. We first consider the case $W = W''$ and suppose that $S(W_0'') \overset{\mathbf{Z}_2}{\sim} \emptyset$ for any W_0'' with $W_0'' \subset W''$ and $\dim(W_0'') < \dim(W'')$. Then we have $S(W'') \overset{\mathbf{Z}_2}{\sim} \emptyset$ by considering the $H \times \mathbf{Z}_2$-SK equivalence (3.2) when $(W_1, W_2) = (W_0'', V(L, m))$, where $W'' = W_0'' \times V(L, m)$. Hence, to complete the proof in this case, we must show that $S(V) \overset{\mathbf{Z}_2}{\sim} \emptyset$ if $V = V(L, m)$. Now divide the circle $S(V) = S^1$ into four parts $A_u = \mathbf{Z}_{d_0}\{\exp(2\pi i t); (u-1)/4d_0 \le t \le u/4d_0\}$ $(1 \le u \le 4)$, where $i = \sqrt{-1}$, $H/L \cong \mathbf{Z}_d$ for some $d\ (> 2)$, $d_0 = d$ if d is even or $2d$ if d is odd, and $\mathbf{Z}_{d_0}\{\cdots\}$ means the union of the orbits $\mathbf{Z}_{d_0}(x)$ of $x \in \{\cdots\}$. These A_u are $H \times \mathbf{Z}_2$ invariant naturally. Let $N_1 = A_1 + A_3$ and $N_2 = A_2 + A_4$, and set $\partial N_1 = \{a_j\}$ and $\partial N_2 = \{b_j\}$ where $a_j = b_j = \exp(2\pi i j/4d_0)$ $(0 \le j < 4d_0)$. Further define an $H \times \mathbf{Z}_2$-equivariant identification φ or $\psi : \partial N_1 \to \partial N_2$ by $\varphi(a_j) = b_j$ or $\psi(a_{2j}) = b_{2j}$, $\psi(a_{2j+1}) = b_{2j+3}$ $(b_{4d_0+1} = b_1)$ respectively. Then $N_1 \cup_\varphi N_2 = S(V)$ and $N_1 \cup_\psi N_2 = 2S(V)$. This implies that $S(V) \overset{\mathbf{Z}_2}{\sim} 2S(V)$ and hence $S(V) \overset{\mathbf{Z}_2}{\sim} \emptyset$. Next consider the case $W = W'$ or $\mathbf{R}^{2n}$. If $W_1 = V_0(K_k)^2$ or $\mathbf{R}^2$, then $S(W_1) = S^1$ (via $W_1 \subset \mathbf{C}$) is the union of four $H \times \mathbf{Z}_2$ invariant parts $A_u = \{\exp(2\pi i t); (u-1)/4 \le t \le u/4\}$ $(1 \le u \le 4)$. The same argument as above implies that $S(W_1) \overset{\mathbf{Z}_2}{\sim} \emptyset$ and hence $S(W) \overset{\mathbf{Z}_2}{\sim} \emptyset$ by induction. Consequently, we have the result for a general $W = W' \cdot W'' \times \mathbf{R}^{2n}$ by using (3.2). q.e.d.

EXAMPLE 3.9. Let $\mathbf{R}P(W \times \mathbf{R})$ be the associated projective space for the above H module W, then we have $[G \times_H \mathbf{R}P(W \times \mathbf{R})] = [G \times_H D(W)]$ in $SK_*^G(pt, pt)$. To show this, consider the $H \times \mathbf{Z}_2$-SK equivalence (3.2) when $(W_1, W_2) = (W, \mathbf{R})$. Since $S(W) \overset{\mathbf{Z}_2}{\sim} \emptyset$ as stated above, there exists the induced SK equivalence

$$[\mathbf{R}P(W \times \mathbf{R})] = [S(W) \times_{\mathbf{Z}_2} D^1] + [D(W)] - [D^1 \times S(W)] = [D(W)]. \tag{3.3}$$

Thus if W^* is another H^* module as W, then it follows from Remark 1.5(i) and Lemma 3.6(i) that

$$\begin{aligned}[G \times_H \mathbf{R}P(W \times \mathbf{R})] \cdot [G \times_{H^*} \mathbf{R}P(W^* \times \mathbf{R})] \\ = a(H, H^*)\alpha^{b/2}[G \times_L D(W_L \times W_L^*)] \\ = a(H, H^*)\alpha^{b/2}[G \times_L \mathbf{R}P(W_L \times W_L^* \times \mathbf{R})]\end{aligned}$$

in $SK_*^G(pt, pt)$. We note that this equality holds in SK_*^G modulo torsion via the injection $i_{0,*}$ in the proof of Proposition 3.1. Hence, if $G_r = \mathbf{Z}_{2^r}$ or $(\mathbf{Z}_2)^r$ for example, then the both sides actually coincide because $SK_*^{G_r}$ has no torsion (cf. [8; Theorems 5.5.1 and 5.6]). Now let us consider $\{1\} \times \mathbf{Z}_2$-SK equivalence (3.2) when $(W_1, W_2) = (\mathbf{R}^{2n}, \mathbf{R})$. Then it gives that $[\mathbf{R}P^{2n}] = [D^{2n}]$ in Remark 1.2 by the same way as in (3.3). By applying the double $\mathcal{D}$ in Proposition 3.1 to this equality or (3.3), we have $2[\mathbf{R}P^{2n}] = [S^{2n}]$ in SK_* or more generally $2[\mathbf{R}P(W \times \mathbf{R})] = [S(W \times \mathbf{R})]$ in $SK_*^{G_r}$ respectively. We will study such relations in detail in case $G_2 = (\mathbf{Z}_2)^2$ (See Example 3.13.).

In general, if $M_1 + M_3 \sim M_2$ for G manifolds M_i, then we write it by $M_1 \sim M_2 - M_3$. Now we recall that $V_0(J_r \cap J_I)$ in the following lemma is the J_I module $V_0(J_r)_{J_I}$ in Definition 1.3(1) or (2) induced from an H module $V_0(J_r)$ (cf. Remark 1.5(i)).

LEMMA 3.10. *Let* $\{V_0(J_i); 1 \le i \le p\}$ *be a set of one-dimensional irreducible, inequivalent* H *modules as in Lemma* 3.8. *Then* $S(V_0(J_1)^{2j(1)+1} \cdots V_0(J_p)^{2j(p)+1}) \sim \sum_{1 \le t \le p} (-1)^{t-1} D^{t-1} \times Y(t)$, *where*

$$Y(t) = \sum_I (\mathbf{R}P^2)^{|j(I)|} a(J_I) \cdot H \times_{J_I} D\left(\prod_{r \notin I} V_0(J_r \cap J_I)^{2j(r)+1}\right)$$

summing over all t-tuples $I = (i_1, \cdots, i_t)$ *in* $\{1, \cdots, p\}$ *such that* $i_1 < \cdots < i_t$; $|j(I)| = j(i_1) + \cdots + j(i_t)$, $J_I = J_{i_1} \cap \cdots \cap J_{i_t}$ *and* $a(J_I) = |H|^{t-1}|J_I| / \prod_u |J_{i_u}| = 2^t(|H|/|J_I|)^{-1}$. *If* $t = p$, *then the disk* $D(-)$ *in* $Y(p)$ *means that* $D(\{0\}) = D^0$.

PROOF. First suppose that $p = 1$, and put $V_1 = V_0(J_1)$. Since $S(V_1^{2j(1)+1})$ fibers equivariantly over $(\mathbf{R}P^2)^{j(1)} = S(V_1^{2j(1)+1})/\mathbf{Z}_2$, we have

$$S(V_1^{2j(1)+1}) \sim (\mathbf{R}P^2)^{j(1)} \cdot \mathbf{Z}_2 \simeq (\mathbf{R}P^2)^{j(1)} H \times_{J_1} D(\{0\}) = Y(1) \tag{3.4}$$

from [8; Theorem 2.4.1(iii)]. In general, we consider (3.2) when $W_1 = \prod_{0 \le i < p} V_0(J_i)^{2j(i)+1}$ and $W_2 = V_0(J_p)^{2j(p)+1}$. Then the result follows straightforwardly by induction on p. Here we use the following equalities in Lemma 3.6(i):

$$\begin{aligned} &H \times_{J_I} D\left(\prod_{r \notin I} V_0(J_r \cap J_I)^{2j(r)+1}\right) \times D(V_0(J_p)^{2j(p)+1}) \\ &\qquad \sim H \times_{J_I} D\left(\prod_{r \notin I} V_0(J_r \cap J_I)^{2j(r)+1} V_0(J_p \cap J_I)^{2j(p)+1}\right), \\ &a(J_I) \cdot a(J_I, J_p) = a(J_{I \cup \{p\}}) \\ &\qquad\qquad (I = (i_1, \cdots, i_t);\ t\text{-tuple in } \{1, \cdots, p-1\}). \end{aligned}$$

q.e.d.

THEOREM 3.11. *An element* $y_\sigma = [G \times_H S(V \times \mathbf{R})]$ *(of* $\mathcal{B}_0$*) is represented as a sum of elements of* $\mathcal{B}$ *over* SK_* *by using the following equalities* (i) *and* (ii):

Let $V = V_0(J_1)^{2j(1)+1} \cdots V_0(J_p)^{2j(p)+1} \cdot W$, *where* $W = W' \cdot W''$ *as in Lemma* 3.8. *Then*

(i)

$$S(V \times \mathbf{R}) \sim 2D(V) - \sum_{1 \le t \le p} (-1)^{t-1} D^t \times Y(t)',$$

where

$$Y(t)' = \sum_I (\mathbf{R}P^2)^{|j(I)|} a(J_I) \cdot H \times_{J_I} D\left(\prod_{r \notin I} V_0(J_r \cap J_I)^{2j(r)+1} W_{J_I}\right),$$
$$W_{J_I} = \prod_k V_0(K_k \cap J_I)^{2s(k)} \cdot \prod_l V(L_l \cap J_I, m_l')^{t(l)}$$
$$(Y(t)' = \emptyset \text{ if } V = W).$$

(ii) *If* $J_I \subseteq J_r$ *or* K_k *then* $V_0(J_r \cap J_I)$ *or* $V_0(K_k \cap J_I) = \mathbf{R}$ *respectively, while if* $J_I \subseteq L_l$ *then* $V(L_l \cap J_I) = \mathbf{R}^2$. *Further* $D^2 \sim \mathbf{R}P^2$, *where* $[\mathbf{R}P^2] = \alpha$ *(cf. Remark* 1.5(i)).

PROOF. It follows from (3.2) (when $W_1 = V$ and $W_2 = \mathbf{R}$) that $S(V \times \mathbf{R}) \sim 2D(V) - D^1 \times S(V)$. Moreover we have

$$S(V) \sim S(V_0(J_1)^{2j(1)+1} \cdots V_0(J_p)^{2j(p)+1}) \times D(W) \tag{3.5}$$

similarly from Lemma 3.8. Thus the result follows immediately from Lemmas 3.6(i) and 3.10. q.e.d.

REMARK 3.12. We see that $G \times_H D(V) \sim \frac{1}{2}(G \times_H S(V \times \mathbf{R}) + z)$ from the above (i), where z is a sum of elements $G \times_J D(Z)$ (or $G \times_J D(Z \times \mathbf{R})$) such that $Z < V$. If $\sigma = \sigma_{-1}$, then $G \times_{\{1\}} D(\{0\}) = \frac{1}{2} G \times_{\{1\}} S(\{0\} \times \mathbf{R})$. Hence, by induction on the ordering, any element of $\mathcal{B}$ is also represented as a sum of elements of $\mathcal{B}_0$ in $SK_*^G(pt, pt) \otimes R_2$.

EXAMPLE 3.13. Let L_* be an SK_* submodule of SK_*^G generated by the class $\mathcal{C} = \{y_\sigma; \sigma \in St(G)\}$. Since $\mathcal{C}$ is a basis for $SK_*^G \otimes R_2$ as an $SK_* \otimes R_2$ module and the inclusion map $SK_* \to SK_* \otimes R_2$ is injective, we see that the class $\mathcal{C}$ is linearly independent over SK_* and for any $x \in SK_*^G$ there is an integer u (≥ 0) such that $2^u x \in L_*$. We now give an example related to this fact. Let $G = (\mathbf{Z}_2)^2$ with generators $\{g_1, g_2\}$ in particular. The non-trivial irreducible G modules are $V_i = V_0(J_i)$ $(1 \leq i \leq 3)$, where $J_1 = \langle g_1 \rangle$, $J_2 = \langle g_2 \rangle$ and $J_3 = \langle g_1 + g_2 \rangle$. Note that if $i \neq j$ then $V_0(J_i \cap J_j) = \tilde{\mathbf{R}}$ with a generator of J_j acting by multiplication by -1, while if $i = j$ then $V_0(J_i \cap J_j) = \mathbf{R}$. Given a triple $A = (a(1), a(2), a(3))$ of non-negative integers, denote by $\sigma(A)$ a slice type $[G; V^A]$ where $V^A = V_1^{a(1)} V_2^{a(2)} V_3^{a(3)}$. We show that $2^u[\mathbf{R}P(V^A \times \mathbf{R})] \in L_*$ ($u = 1$ or 2) for any A. First suppose that $A_1 = (2a(1)+1, 2a(2)+1, 2a(3)+1)$ and consider $G \times \mathbf{Z}_2$-SK equivalence (3.2) when $(W_1, W_2) = (V^{A_1}, \mathbf{R})$. We then have

$$\mathbf{R}P(V^{A_1} \times \mathbf{R}) \sim S(V^{A_1}) \times_{\mathbf{Z}_2} D^1 + D(V^{A_1}) - D^1 \times S(V^{A_1}). \tag{3.6}$$

Write $V^{A_1} = V_1^{2a(1)+1} V'$ and $V' = V_2^{2a(2)+1} V_3^{2a(3)+1}$. Then we further have

$$S(V^{A_1}) \times_{\mathbf{Z}_2} D^1 \sim Q_1 + Q_2 - D^1 \times Q_3$$

by (3.2) when $(W_1, W_2) = (V_1^{2a(1)+1}, V')$, where

$$Q_1 = (S(V_1^{2a(1)+1}) \times D(V')) \times_{\mathbf{Z}_2} D^1, \quad Q_2 = (D(V_1^{2a(1)+1}) \times S(V')) \times_{\mathbf{Z}_2} D^1,$$
$$Q_3 = (S(V_1^{2a(1)+1}) \times S(V')) \times_{\mathbf{Z}_2} D^1.$$

Note that Q_i ($i = 1$ or 3) fibers equivariantly over $\mathbf{R}P^{2a(1)} = S(V_1^{2a(1)+1})/\mathbf{Z}_2$ with fiber

$$F_1 = (\mathbf{Z}_2 \times D(V')) \times_{\mathbf{Z}_2} D^1 \simeq D(V_1 V_2^{2a(3)+1} V_3^{2a(2)+1}) \quad \text{or}$$
$$F_3 = (\mathbf{Z}_2 \times S(V')) \times_{\mathbf{Z}_2} D^1 \simeq D(V_1) \times S(V_2^{2a(3)+1} V_3^{2a(2)+1})$$

respectively by an obvious equivariant identification. Thus $Q_i \sim \mathbf{R}P^{2a(1)} \times F_i$ (cf. [8; Theorem 2.4.1(iii)]). On the other hand, by continuing the same SK process on $S(V')$ in Q_2, we have

$$Q_2 \sim \mathbf{R}P^{2a(2)} \times D(V_1^{2a(3)+1} V_2 V_3^{2a(1)+1}) + \mathbf{R}P^{2a(3)} \times D(V_1^{2a(2)+1} V_2^{2a(1)+1} V_3)$$
$$-\mathbf{R}P^{2a(2)} \times D^1 \times S(V_1^{2a(3)+1}) \times D(V_2 V_3^{2a(1)+1}). \tag{3.7}$$

Now it follows from Lemma 3.10 that

$$S(V_1^{2a(3)+1}) \sim (\mathbf{R}P^2)^{a(3)} G \times_{J_1} D(\{0\}),$$
$$S(V_2^{2a(3)+1} V_3^{2a(2)+1})$$
$$\sim \sum\nolimits_{(0)} (\mathbf{R}P^2)^{a(j_1)} G \times_{J_{j_2}} D(\tilde{\mathbf{R}}^{2a(j_2)+1}) - (\mathbf{R}P^2)^{a(2)+a(3)} G \times_{\{1\}} D(\{0\} \times \mathbf{R}),$$
$$S(V^{A_1})$$
$$\sim \sum\nolimits_{(1)} (\mathbf{R}P^2)^{a(j_1)} G \times_{J_{j_1}} D(\tilde{\mathbf{R}}^{2(a(j_2)+a(j_3)+1)}) - (\mathbf{R}P^2)^{(|\sigma(A_1)|-1)/2} G \times_{\{1\}} D(\{0\}),$$

where the sum $\sum_{(0)}$ (or $\sum_{(1)}$) is taken over $(j_1, j_2) \in \{(2,3), (3,2)\}$ (or $(j_1, j_2, j_3) \in \{(1,2,3), (2,1,3), (3,1,2)\}$) respectively. Take the first equality (or the second one) in (3.7) (or F_3 in Q_3) respectively. Then we have

$$S(V^{A_1}) \times_{\mathbf{Z}_2} D^1 \sim \sum\nolimits_{(1)} \alpha^{a(j_1)} [D(V_{j_1} V_{j_2}^{2a(j_3)+1} V_{j_3}^{2a(j_2)+1})] + w_1, \tag{3.8}$$

$$w_1 = -\sum\nolimits_{(1)} \alpha^{a(j_2)+a(j_3)} [G \times_{J_{j_1}} D(\tilde{\mathbf{R}}^{2(a(j_1)+1)} \times \mathbf{R})] + \alpha^{(|\sigma(A_1)|-1)/2} [G \times_{\{1\}} D(\{0\} \times \mathbf{R})]$$

by using Lemma 3.6(i). By taking the third equality and (3.8) in (3.6), we therefore have

$$[\mathbf{R}P(V^{A_1} \times \mathbf{R})] = [D(V^{A_1})] + \sum\nolimits_{(1)} \alpha^{a(j_1)} [D(V_{j_1} V_{j_2}^{2a(j_3)+1} V_{j_3}^{2a(j_2)+1})] + z_1, \tag{3.9}$$

where $z_1 = w_1 - [D^1 \times S(V^{A_1})]$. Further let $A_2 = (2a(1)+1, 2a(2)+1, 2a(3))$, $A_3 = (2a(1)+1, 2a(2), 2a(3))$ and $A_4 = (2a(1), 2a(2), 2a(3))$. In a similar manner, if $k = 2, 3$ or 4 then

$$[\mathbf{R}P(V^{A_k} \times \mathbf{R})] = \tag{3.10}$$
$$[D(V^{A_2})] + \sum\nolimits_{(2)} \alpha^{a(j_1)} [D(V_{j_1} V_{j_2}^{2a(3)} V_3^{2a(j_2)+1})] + \alpha^{|\sigma(A_2)|/2} [G \times_{\{1\}} D(\{0\})] + z_2,$$
$$[D(V^{A_3})] + \alpha^{a(1)} [D(V_1 V_2^{2a(3)} V_3^{2a(2)})] + z_3 \quad \text{or} \quad [D(V^{A_4})]$$

respectively, where

$$z_2 = -\sum\nolimits_{(2)} \alpha^{a(j_1)} [G \times_{J_{j_1}} D(\tilde{\mathbf{R}}^{2(a(j_2)+a(3))+1} \times \mathbf{R})] - \alpha^{a(1)+a(2)} [G \times_{J_3} D(\tilde{\mathbf{R}}^{2a(3)+1} \times \mathbf{R})],$$
$$z_3 = -\alpha^{a(1)} [G \times_{J_1} D(\tilde{\mathbf{R}}^{2(a(2)+a(3))} \times \mathbf{R})],$$

and the sum $\sum_{(2)}$ is taken over $(j_1, j_2) \in \{(1,2),(2,1)\}$. For our purpose, we apply the double $\mathcal{D}$ to both (3.9) and (3.10). If $z = [G \times_{J_i} D(\tilde{\mathbf{R}}^{2n+1} \times \mathbf{R})]$ in general, then $\mathcal{D}(z) = [G \times_{J_i} S(\tilde{\mathbf{R}}^{2n+1} \times \mathbf{R}^2)]$ and

$$S(\tilde{\mathbf{R}}^{2n+1} \times \mathbf{R}^2) \sim \mathbf{R}P^2 \times S(\tilde{\mathbf{R}}^{2n+1}) \sim \frac{1}{2}(\mathbf{R}P^2)^{n+1} J_i \times_{\{1\}} S(\{0\} \times \mathbf{R})$$

by using (3.2) and a similar SK equivalence as in (3.4). Thus we have

$$\mathcal{D}(z_2) = -\frac{3}{2}\alpha^{|\sigma(A_2)|/2}[G \times_{\{1\}} S(\{0\} \times \mathbf{R})].$$

On the other hand, $\mathcal{D}(z_i) = 0$ ($i = 1$ or 3) because z_i is a sum of terms of the form $t = [G \times_J D(\tilde{\mathbf{R}}^{2n} \times \mathbf{R})]$ and $\mathcal{D}(t) = [G \times_J S(\tilde{\mathbf{R}}^{2n} \times \mathbf{R}^2)] = 0$ from Lemma 3.8. Using these, we therefore obtain the desired equality: if $k = 1, 3$ or 4, then

$$\begin{aligned} 2[\mathbf{R}P(V^{A_k} \times \mathbf{R})] = \\ & [S(V^{A_1} \times \mathbf{R})] + \sum\nolimits_{(1)} \alpha^{a(j_1)}[S(V_{j_1} V_{j_2}^{2a(j_3)+1} V_{j_3}^{2a(j_2)+1} \times \mathbf{R})], \\ & [S(V^{A_3} \times \mathbf{R})] + \alpha^{a(1)}[S(V_1 V_2^{2a(3)} V_3^{2a(2)} \times \mathbf{R})] \quad \text{or} \quad [S(V^{A_4} \times \mathbf{R})] \end{aligned}$$

respectively, while

$$\begin{aligned} 4[\mathbf{R}P(V^{A_2} \times \mathbf{R})] = \\ & 2[S(V^{A_2} \times \mathbf{R})] + 2\sum\nolimits_{(2)} \alpha^{a(j_1)}[S(V_{j_1} V_{j_2}^{2a(3)} V_3^{2a(j_2)+1} \times \mathbf{R})] \\ & - \alpha^{|\sigma(A_2)|+1}[G \times_{\{1\}} S(\{0\} \times \mathbf{R})] \end{aligned}$$

in $SK_*^G \otimes R_2$. Since SK_*^G has no torsion as mentioned in Example 3.9, the above equalities actually hold in SK_*^G. Therefore $2^u[\mathbf{R}P(V^{A_k} \times \mathbf{R})] \in L_*$ ($u = 1$ or 2).

REMARK 3.14. It is seen that if $a(j) > 0$ $(1 \le j \le 4-k)$ then $[\mathbf{R}P(V^{A_k} \times \mathbf{R})] = [D(V^{A_k})]$ (mod SK_* decomposables) from (3.9) and (3.10). Hence we can replace $[D(V^{A_k})]$ (or $[D(V^{A_k} \times \mathbf{R})]$) by $[\mathbf{R}P(V^{A_k} \times \mathbf{R})]$ (or $[\mathbf{R}P(V^{A_k} \times \mathbf{R}) \times D^1]$) respectively as an basis element for $SK_*^G(pt, pt)$ (cf. Proposition 1.6).

EXAMPLE 3.15. For any finite abelian group G, let us consider a ring homomorphism $T : SK_*^G \to \mathbf{Z}$, which is called a multiplicative G-SK invariant. By the statement in the beginning of Example 3.13, such T is determined by the values $a = T(\alpha)$ and $\gamma_\sigma = T(y_\sigma)$ ($\sigma \in St(G)$). If $T([D^0]) = 0$, then T is trivial. From now on we consider that T is always non-trivial, so $T([D^0]) = 1$. We now characterize these T which also satisfy the condition that $T(y_\tau) = 0$ for all y_τ with $\tau = [H; Z]$ and $H \neq G$. For example, χ^G is one of these invariants. First we recall the expression of y_σ for $\sigma = [G; V]$ in Theorem 3.11, that is, $y_\sigma = 2[D(V)] + w_\sigma$ for some $w_\sigma \in Q_*$, where Q_* is an ideal of $SK_*^G(pt, pt)$ generated by all elements x_τ and $\hat{x}_\tau$ of $\mathcal{B}$ with $\tau = [H; Z]$ and $H \neq G$. Consider another $y_{\sigma^*} = 2[D(V^*)] + w_{\sigma^*}$ with $\sigma^* = [G; V^*]$. Then $y_\sigma \cdot y_{\sigma^*} = 2y_{\sigma \times \sigma^*} + w$ for some $w \in Q_*$, and w

is written as

$$w = (1/2)^u \left(\sum_\tau k_\tau y_\tau + \sum_\tau \hat{k}_\tau \hat{y}_\tau \right) \quad (k_\tau, \hat{k}_\tau \in SK_*, u \geq 0)$$

from Remark 3.12. Now apply the map ∂_* in Proposition 3.1 to the above. Since $\partial_*(\omega) = 0$ and $\partial_*(\hat{y}_\tau) = 2y_\tau$, we have $\hat{k}_\tau = 0$ and

$$2^u w = \sum_\tau k_\tau y_\tau + t$$

in SK_*^G, where t is a torsion (of order 2) (cf. Corollary 3.2). This implies that

$$T(y_\sigma)T(y_{\sigma^*}) = T(y_\sigma \cdot y_{\sigma^*}) = 2T(y_{\sigma \times \sigma^*}) \tag{3.11}$$

because $T(y_\tau) = T(t) = 0$ by assumption. Let $\gamma_i = T(S(V_i \times \mathbf{R})) \in \mathbf{Z}$ for any i, where $\{V_i\}$ is the set of all (non-trivial) irreducible G modules. By using (3.11) inductively, we have

$$T(y_\sigma) = (1/2)^{l(\sigma)-1} \prod_i \gamma_i^{a(i)} \tag{3.12}$$

for $\sigma = [G; V]$ with $V = \prod_i V_i^{a(i)}$, where $l(\sigma) = \sum_i a(i)$. This means that T is determined by the set $\{a, \gamma_i\}$ of integers. To get the form of T, let us consider the induced invariant $T' = T \otimes id : SK_*^G \otimes R_2 \to R_2$, which is also multiplicative naturally. It follows from Remark 3.4 that T' can be expressed as

$$T' = \sum_{n,\mu} p_{\mu,(2n+|\mu|)} e_{\mu,(2n+|\mu|)} \otimes id$$

summing over all $\mu \in St(G)$ and n (≥ 0), where $e_{\mu,(m)}$ is defined by $e_{\mu,(m)}(M) = e_\mu(M)$ if $\dim(M) = m$ or 0 if $\dim(M) \neq m$. Note that $e_{\mu,(m)}(M) = 0$ if $\dim(M) < |\mu|$ because $M_{[\mu]} = \emptyset$ by definition, so we may write $m = 2n + |\mu|$ for some n (≥ 0). Let $\tau = [H; Z]$ with $H \neq G$. Then $T'(y_\tau) = T(y_\tau) = 0$, and hence

$$0 = T'(\alpha^n y_\tau) = \sum_{L \subset H} p_{\tau_L,(2n+|\tau|)} e_{\tau_L}(y_\tau) + p_{\tau,(2n+|\tau|)} \cdot 2$$

by the definition of e_μ. This implies that $p_{\tau,(m)} = 0$ by using the induction (starting from $\tau = \sigma_{-1}$) on the totally ordered set $St(G)$, and

$$T' = \sum_{n,\sigma} p_{\sigma,(2n+|\sigma|)} \chi_{\sigma,(2n+|\sigma|)} \otimes id \tag{3.13}$$

summing over all σ of the form $[G; V]$ and n (≥ 0), where $e_\sigma = \chi_\sigma$ by definition. Since $T(\alpha^n y_\sigma) = T'(\alpha^n y_\sigma) = p_{\sigma,(2n+|\sigma|)} \cdot 2$, we have

$$p_{\sigma,(2n+|\sigma|)} = a^n (1/2)^{l(\sigma)} \prod_i \gamma_i^{a(i)}$$

by (3.12). Note that γ_i is even because $\gamma_i^2 = T(S(V_i \times \mathbf{R})^2) = 2T(S(V_i^2 \times \mathbf{R}))$ from (3.11). We thus obtain the desired form:

$$T = \sum_{n,\sigma} a^n \eta_\sigma \chi_{\sigma,(2n+|\sigma|)} \tag{3.14}$$

by setting $\eta_i = (1/2)\gamma_i \in \mathbf{Z}$, where $\eta_\sigma = \prod_i \eta_i^{a(i)}$. We see that it is in fact multiplicative by using the formula (when $H = G$) in Remark 2.2(ii). If $\sigma(0) = [G; \prod_i V_i^0]$ in particular, then $1 = T([D^0]) = a^0 \eta_{\sigma(0)} \chi(D^0) = a^0 \prod_i \eta_i^0$. This means that a^0 (or η_i^0) must be regarded as 1 if $a = 0$ (or $\eta_i = 0$) respectively. Finally we list typical examples of these invariants.

(i) First suppose that $a = 1$. Further if $\eta_i = 1$ for any i, then $T(M) = \sum_\sigma \chi_\sigma(M)$ for any closed G manifold M, where the sum is taken over all σ with $|\sigma| \equiv \dim(M) \pmod 2$. Thus $T = \chi^G$ (cf. Example 2.5(i)). On the other hand, if $\eta_i = 0$ for any i, then $T = \sum_n \chi_{\sigma(0),(2n)}$ because $\eta_{\sigma(0)} = \prod_i 0^0 = 1$. In other words, $T(M) = \chi(M_{\sigma(0)})$, where $M_{\sigma(0)}$ is, by definition, the union of all components of M^G such that $\dim(M_{\sigma(0)}) = \dim(M) - |\sigma(0)| = \dim(M)$ (cf. Remark 2.2(i)). Now let $M_{\mathbf{R}}^G$ (or $M_{\mathbf{C}}^G$) be the set consisting of those points x in M^G, each of whose slice types σ_x is a product of one-dimensional (or two-dimensional) irreducible G modules respectively. If $\eta_i = 1$ for $\dim(V_i) = 1$ and $\eta_j = 0$ for $\dim(V_j) = 2$, then $T = \chi_{\mathbf{R}}^G$, where $\chi_{\mathbf{R}}^G(M) = \chi(M_{\mathbf{R}}^G)$. If $\eta_i = 0$ and $\eta_j = 1$, then $T = \chi_{\mathbf{C}}^G$ similarly.

(ii) Next suppose that $a = 0$, then $T = \sum_\sigma \eta_\sigma \chi_{\sigma,(|\sigma|)}$ because $a^0 = 1$. Hence if $\eta_i = 1$ for any i, then $T(M) = \sum_\sigma \chi_{\sigma,(|\sigma|)}(M) = \chi(M_0^G)$, where M_0^G is the set of all isolated points of M^G. We denote such T by χ_0^G. In a similar manner, the invariant $\chi_{\mathbf{R},0}^G$ or $\chi_{\mathbf{C},0}^G$ is considered. On the other hand, if $\eta_i = 0$ for any i, then $T = \chi_{\sigma(0),(0)} = (\chi^G)_{(0)}$ where $(\chi^G)_{(0)}(M) = \chi(M^G)$ if $\dim(M) = 0$ or 0 if $\dim(M) > 0$.

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