TOKYO J. MATH.
VOL. 23, NO. 2, 2000

An Example of an Almost Kähler Manifold with Pointwise Constant Holomorphic Sectional Curvature

Takuji SATO

Kanazawa University

(Communicated by Y. Maeda)

1. Introduction.

Let $M = (M, J, g)$ be an almost Hermitian manifold and $U(M)$ the unit tangent bundle of M. Then the holomorphic sectional curvature $H = H(x)$ $(x \in U(M))$ can be regarded as a differentiable function on $U(M)$. If the function H is constant along each fiber, then M is called a space of pointwise constant holomorphic sectional curvature. Especially, if H is constant on the whole of $U(M)$, then M is called a space of constant holomorphic sectional curvature.

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if its Kähler form Ω is closed, or equivalently,

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0,$$

for all smooth vector fields X, Y, Z on M.

Concerning the integrability of the almost complex structure of an almost Kähler manifold, S. I. Goldberg [2] conjectured that the almost complex structure of a compact Einstein almost Kähler manifold is integrable (and the manifold is necessarily Kähler). In connection with this conjecture, P. Nurowski and M. Przanowski [4] recently constructed a non-compact example of a strictly almost Kähler, Ricci-flat manifold. This is also a space of pointwise *positive* constant holomorphic sectional curvature (see also [6]).

On one hand, the author [5, 6] investigated some properties on almost Kähler 4-manifolds of pointwise constant holomorphic sectional curvature.

In the present paper, we shall show that there exists an example of almost Kähler manifold of pointwise *negative* constant holomorphic sectional curvature. Our example is also weakly $*$-Einstein, but not Einstein. The example is constructed by the framework of Nurowski and Przanowski [4]. We also show that there are other examples of weakly $*$-Einstein almost Kähler manifolds. But we cannot construct a strictly almost Kähler Einstein manifold in our case.

Received March 3, 1999
Revised May 19, 1999

Our example shows that the theorem of Schur does not hold for the class of 4-dimensional almost Kähler manifolds (see [3]). And it shows that an almost Kähler manifold of pointwise constant holomorphic sectional curvature is not always Einstein. The example is 4-dimensional and non-compact. We do not know whether or not there exist arbitrary dimensional, compact almost Kähler manifolds which are of (pointwise) constant holomorphic sectional curvature.

In §2, after preparing some definitions and notations, we shall give a curvature expression for a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature by using the expressions A_{ij} introduced by J. T. Cho and K. Sekigawa [1]. §3 is devoted to the construction of an example. We begin with the Nurowski-Przanowski lemma. By calculating curvature tensors, we derive conditions for our almost Kähler manifold to be of pointwise constant holomorphic sectional curvature. Taking account of these conditions, we shall obtain an example (Theorem 3.6, Theorem 3.7).

The author wishes to express his hearty thanks to Prof. K. Sekigawa for his valuable suggestions. The author also would like to express his thanks to the referee for his valuable advice.

2. Preliminaries.

Let $M = (M, J, g)$ be an $m(= 2n)$-dimensional almost Hermitian manifold with an almost Hermitian structure (J, g). We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X, Y) = g(X, JY)$ and $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. The Nijenhuis tensor N satisfies

$$N(JX, Y) = N(X, JY) = -JN(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Further we denote by ∇, $R = (R_{ijk}{}^{l})$, $\rho = (\rho_{ij})$, τ, $\rho^* = (\rho^*_{ij})$ and τ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the Ricci $*$-tensor and the $*$-scalar curvature of M, respectively:

$$\begin{aligned} R(X, Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \\ R(X, Y, Z, W) &= g(R(X, Y)Z, W), \\ \rho(x, y) &= \text{trace of } [z \mapsto R(z, x)y], \\ \tau &= \text{trace of } \rho, \\ \rho^*(x, y) &= \text{trace of } [z \mapsto R(x, Jz)Jy], \\ \tau^* &= \text{trace of } \rho^*, \end{aligned}$$

where $X, Y, Z, W \in \mathfrak{X}(M)$, $x, y, z \in T_pM$, $p \in M$. The Ricci $*$-tensor ρ^* satisfies

$$\rho^*(JX, JY) = \rho^*(Y, X), \quad X, Y \in \mathfrak{X}(M).$$

An almost Hermitian manifold M is called a *weakly $*$-Einstein manifold* if it satisfies $\rho^* = \lambda^* g$ for some function λ^* on M. In particular, if λ^* is constant on M, then M is called a *$*$-Einstein manifold.*

Now we assume that $M = (M, J, g)$ is an almost Kähler manifold. Then we have

$$2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z)), \tag{2.1}$$

$$\tau - \tau^* = -\frac{1}{2}\|\nabla J\|^2 = -\frac{1}{8}\|N\|^2. \tag{2.2}$$

In the sequel, we adopt the following notational convention: for an orthonormal basis $\{e_i\}$ of a tangent space T_pM, we put

$$\begin{aligned} J_{ij} &= g(Je_i, e_j), & \nabla_i J_{jk} &= g((\nabla_{e_i} J)e_j, e_k), \\ N_{ijk} &= g(e_i, N(e_j, e_k)), & R_{ijkl} &= R(e_i, e_j, e_k, e_l), \\ \nabla_{\bar{i}} J_{jk} &= g((\nabla_{Je_i} J)e_j, e_k), & N_{\bar{i}jk} &= g(Je_i, N(e_j, e_k)), \\ N_{\bar{i}\bar{j}k} &= g(Je_i, N(Je_j, e_k)), & R_{ij\bar{k}\bar{l}} &= R(e_i, e_j, Je_k, Je_l), \quad \text{etc.} \end{aligned}$$

Then it is easy to see that

$$\begin{aligned} &\nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0, \\ &\nabla_{\bar{i}} J_{jk} = \nabla_i J_{\bar{j}k} = \nabla_i J_{j\bar{k}}, \\ &N_{ijk} = -2\nabla_{\bar{i}} J_{jk}, \quad 2\nabla_i J_{jk} = N_{\bar{i}jk}. \end{aligned}$$

Next, we consider a four-dimensional almost Kähler manifold M. We set

$$A_{ij} = g(e_i, (\nabla_{e_j} N)(e_1, e_3)) = \nabla_j N_{i13}, \tag{2.3}$$

for a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of T_pM, $p \in M$. We note that

$$A_{ij} - A_{ji} = -2(R_{ij13} - R_{ij24}).$$

By using these A_{ij}, J. T. Cho and K. Sekigawa obtained the following characterization of almost Kähler manifolds of pointwise constant holomorphic sectional curvature:

PROPOSITION 2.1 ([1]). *Let M be a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)$ ($p \in M$). Then*

$$\begin{aligned} R_{1212} &= R_{3434} = -c(p), \\ R_{1234} &= -\frac{c(p)}{2} - \frac{1}{16}(\tau^* - \tau), \\ R_{1324} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}), \\ R_{1423} &= \frac{c(p)}{4} + \frac{3}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}), \\ R_{1313} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{3}{8}(A_{13} - A_{31}) - \frac{1}{8}(A_{24} - A_{42}), \\ R_{1414} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) - \frac{1}{8}(A_{13} + A_{42}) - \frac{3}{8}(A_{31} + A_{24}), \end{aligned}$$

$$R_{2323} = -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} + A_{24}) + \frac{3}{8}(A_{13} + A_{42}),$$
$$R_{2424} = -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{3}{8}(A_{24} - A_{42}) + \frac{1}{8}(A_{13} - A_{31}),$$
$$R_{1334} = -R_{2434} = -\frac{1}{4}(A_{34} - A_{43}),$$
$$R_{1213} = -R_{1224} = -\frac{1}{4}(A_{12} - A_{21}),$$
$$R_{1434} = R_{2334} = -\frac{1}{4}(A_{33} + A_{44}),$$
$$R_{1214} = R_{1223} = -\frac{1}{4}(A_{11} + A_{22}),$$
$$R_{1323} = \frac{1}{8}(A_{14} + A_{41} + A_{32} - 3A_{23}),$$
$$R_{2324} = \frac{1}{8}(A_{14} + A_{41} + A_{23} - 3A_{32}),$$
$$R_{1314} = -\frac{1}{8}(A_{23} + A_{32} + A_{14} - 3A_{41}),$$
$$R_{1424} = -\frac{1}{8}(A_{23} + A_{32} + A_{41} - 3A_{14}),$$

for any unitary basis $\{e_i\}$ *of* T_pM *at each point* $p \in M$.

3. An example.

In this section, we construct an example of a 4-dimensional strictly almost Kähler manifold of pointwise constant holomorphic sectional curvature.

Let M be an open set of $\boldsymbol{R}^4$, and let (x_1, x_2, x_3, x_4) be the Euclidean coordinates on M. We put

$$z_1 = x_1 + \sqrt{-1}x_2, \quad z_2 = x_3 + \sqrt{-1}x_4.$$

Let f be a non-zero real function and h be a complex function on M. Then P. Nurowski and M. Przanowski proved the following

LEMMA 3.1 ([4]). *Let* $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ *be coordinates on M. Then for each value of the real constant* $\phi \in [0, 2\pi)$, *the metric*

$$g = 2f^2(dz_1 + hdz_2)(d\bar{z}_1 + \bar{h}d\bar{z}_2) + \frac{2}{f^2}dz_2 d\bar{z}_2$$

and the almost complex structure

$$J^+_{e^{\sqrt{-1}\phi}} = 2\mathrm{Re}\left[\sqrt{-1}e^{\sqrt{-1}\phi}\left\{f^2(dz_1 + hdz_2) \otimes \left(\frac{\partial}{\partial \bar{z}_2} - \bar{h}\frac{\partial}{\partial \bar{z}_1}\right) - \frac{1}{f^2}dz_2 \otimes \frac{\partial}{\partial \bar{z}_1}\right\}\right]$$

define an almost Kähler structure on M.

When $\phi = 0$, the Riemannian metric $g = (g_{ij})$ and the almost complex structure $J_1^+ = (J_i^j)$ in the above Lemma 3.1 are given with respect to the real coordinates (x_1, x_2, x_3, x_4) by

$$(3.1)\qquad (g_{ij}) = 2\begin{pmatrix} f^2 & 0 & f^2u & -f^2v \\ 0 & f^2 & f^2v & f^2u \\ f^2u & f^2v & f^2(u^2+v^2)+\frac{1}{f^2} & 0 \\ -f^2v & f^2u & 0 & f^2(u^2+v^2)+\frac{1}{f^2} \end{pmatrix},$$

$$(3.2)\qquad (J_i^j) = \begin{pmatrix} -f^2v & f^2u & 0 & -f^2 \\ f^2u & f^2v & -f^2 & 0 \\ 0 & f^2(u^2+v^2)+\frac{1}{f^2} & -f^2v & -f^2u \\ f^2(u^2+v^2)+\frac{1}{f^2} & 0 & -f^2u & f^2v \end{pmatrix},$$

where u and v are the real and imaginary part of the complex function h, respectively. It is easy to see that (J_1^+, g) is an almost Kähler structure.

In the present paper, for the sake of simplicity, we shall consider the case where $u = v = 0$, i.e., $h = 0$, and we put $J = J_1^+$. We define a unitary frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on M by

$$(3.3)\qquad e_1 = \frac{1}{\sqrt{2}f}\frac{\partial}{\partial x_1}, \qquad e_2 = -\frac{f}{\sqrt{2}}\frac{\partial}{\partial x_4},$$

$$e_3 = \frac{1}{\sqrt{2}f}\frac{\partial}{\partial x_2}, \qquad e_4 = -\frac{f}{\sqrt{2}}\frac{\partial}{\partial x_3}.$$

With respect to this unitary frame $\{e_i\}_{i=1,2,3,4}$, we have

(3.4)

$$\nabla_{e_1}e_1 = \frac{\partial_4 f}{\sqrt{2}}e_2 - \frac{\partial_2 f}{\sqrt{2}f^2}e_3 + \frac{\partial_3 f}{\sqrt{2}}e_4, \qquad \nabla_{e_1}e_2 = -\frac{\partial_4 f}{\sqrt{2}}e_1,$$

$$\nabla_{e_1}e_3 = \frac{\partial_2 f}{\sqrt{2}f^2}e_1, \qquad \nabla_{e_1}e_4 = -\frac{\partial_3 f}{\sqrt{2}}e_1,$$

$$\nabla_{e_2}e_1 = -\frac{\partial_1 f}{\sqrt{2}f^2}e_2, \qquad \nabla_{e_2}e_2 = \frac{\partial_1 f}{\sqrt{2}f^2}e_1 + \frac{\partial_2 f}{\sqrt{2}f^2}e_3 - \frac{\partial_3 f}{\sqrt{2}}e_4,$$

$$\nabla_{e_2}e_3 = -\frac{\partial_2 f}{\sqrt{2}f^2}e_2, \qquad \nabla_{e_2}e_4 = \frac{\partial_3 f}{\sqrt{2}}e_2,$$

$$\nabla_{e_3}e_1 = \frac{\partial_1 f}{\sqrt{2}f^2}e_3, \qquad \nabla_{e_3}e_2 = -\frac{\partial_4 f}{\sqrt{2}}e_3,$$

$$\nabla_{e_3} e_3 = -\frac{\partial_1 f}{\sqrt{2} f^2} e_1 + \frac{\partial_4 f}{\sqrt{2}} e_2 + \frac{\partial_3 f}{\sqrt{2}} e_4 , \quad \nabla_{e_3} e_4 = -\frac{\partial_3 f}{\sqrt{2}} e_3 ,$$
$$\nabla_{e_4} e_1 = -\frac{\partial_1 f}{\sqrt{2} f^2} e_4 , \quad \nabla_{e_4} e_2 = \frac{\partial_4 f}{\sqrt{2}} e_4 ,$$
$$\nabla_{e_4} e_3 = -\frac{\partial_2 f}{\sqrt{2} f^2} e_4 , \quad \nabla_{e_4} e_4 = \frac{\partial_1 f}{\sqrt{2} f^2} e_1 - \frac{\partial_4 f}{\sqrt{2}} e_2 + \frac{\partial_2 f}{\sqrt{2} f^2} e_3 ,$$

where we denote $\partial_i f = \frac{\partial f}{\partial x_i}$. From (3.4) and by a straightforward calculation, we obtain

(3.5)

$$\nabla_1 J_{13} = \frac{\partial_3 f}{\sqrt{2}} , \quad \nabla_1 J_{14} = \frac{\partial_2 f}{\sqrt{2} f^2} , \quad \nabla_1 J_{23} = \frac{\partial_2 f}{\sqrt{2} f^2} , \quad \nabla_1 J_{24} = -\frac{\partial_3 f}{\sqrt{2}} ,$$
$$\nabla_1 J_{31} = -\frac{\partial_3 f}{\sqrt{2}} , \quad \nabla_1 J_{32} = -\frac{\partial_2 f}{\sqrt{2} f^2} , \quad \nabla_1 J_{41} = -\frac{\partial_2 f}{\sqrt{2} f^2} , \quad \nabla_1 J_{42} = \frac{\partial_3 f}{\sqrt{2}} ,$$
$$\nabla_2 J_{13} = \frac{\partial_2 f}{\sqrt{2} f^2} , \quad \nabla_2 J_{14} = -\frac{\partial_3 f}{\sqrt{2}} , \quad \nabla_2 J_{23} = -\frac{\partial_3 f}{\sqrt{2}} , \quad \nabla_2 J_{24} = -\frac{\partial_2 f}{\sqrt{2} f^2} ,$$
$$\nabla_2 J_{31} = -\frac{\partial_2 f}{\sqrt{2} f^2} , \quad \nabla_2 J_{32} = \frac{\partial_3 f}{\sqrt{2}} , \quad \nabla_2 J_{41} = \frac{\partial_3 f}{\sqrt{2}} , \quad \nabla_2 J_{42} = \frac{\partial_2 f}{\sqrt{2} f^2} ,$$
$$\nabla_3 J_{13} = -\frac{\partial_4 f}{\sqrt{2}} , \quad \nabla_3 J_{14} = -\frac{\partial_1 f}{\sqrt{2} f^2} , \quad \nabla_3 J_{23} = -\frac{\partial_1 f}{\sqrt{2} f^2} , \quad \nabla_3 J_{24} = \frac{\partial_4 f}{\sqrt{2}} ,$$
$$\nabla_3 J_{31} = \frac{\partial_4 f}{\sqrt{2}} , \quad \nabla_3 J_{32} = \frac{\partial_1 f}{\sqrt{2} f^2} , \quad \nabla_3 J_{41} = \frac{\partial_1 f}{\sqrt{2} f^2} , \quad \nabla_3 J_{42} = -\frac{\partial_4 f}{\sqrt{2}} ,$$
$$\nabla_4 J_{13} = -\frac{\partial_1 f}{\sqrt{2} f^2} , \quad \nabla_4 J_{14} = \frac{\partial_4 f}{\sqrt{2}} , \quad \nabla_4 J_{23} = \frac{\partial_4 f}{\sqrt{2}} , \quad \nabla_4 J_{24} = \frac{\partial_1 f}{\sqrt{2} f^2} ,$$
$$\nabla_4 J_{31} = \frac{\partial_1 f}{\sqrt{2} f^2} , \quad \nabla_4 J_{32} = -\frac{\partial_4 f}{\sqrt{2}} , \quad \nabla_4 J_{41} = -\frac{\partial_4 f}{\sqrt{2}} , \quad \nabla_4 J_{42} = -\frac{\partial_1 f}{\sqrt{2} f^2} ,$$
$$\nabla_i J_{jk} = 0 \quad \text{(otherwise)}.$$

By (2.1) and (3.5), we then have

(3.6)

$$N_{113} = -\frac{\sqrt{2}\partial_2 f}{f^2} , \quad N_{213} = \sqrt{2}\partial_3 f , \quad N_{313} = \frac{\sqrt{2}\partial_1 f}{f^2} , \quad N_{413} = -\sqrt{2}\partial_4 f ,$$
$$N_{114} = \sqrt{2}\partial_3 f , \quad N_{214} = \frac{\sqrt{2}\partial_2 f}{f^2} , \quad N_{314} = -\sqrt{2}\partial_4 f , \quad N_{414} = -\frac{\sqrt{2}\partial_1 f}{f^2} ,$$
$$N_{123} = \sqrt{2}\partial_3 f , \quad N_{223} = \frac{\sqrt{2}\partial_2 f}{f^2} , \quad N_{323} = -\sqrt{2}\partial_4 f , \quad N_{423} = -\frac{\sqrt{2}\partial_1 f}{f^2} ,$$
$$N_{124} = \frac{\sqrt{2}\partial_2 f}{f^2} , \quad N_{224} = -\sqrt{2}\partial_3 f , \quad N_{324} = -\frac{\sqrt{2}\partial_1 f}{f^2} , \quad N_{424} = \sqrt{2}\partial_4 f ,$$
$$N_{ijk} = 0 \quad \text{(otherwise)}.$$

By (3.5), (3.6), we see that (M, J, g) is non-Kählerian if f is not constant. Furthermore, from (2.3) and (3.6), we get

$$
\begin{aligned}
(3.7)\qquad A_{11} &= -\frac{\partial_1\partial_2 f}{f^3} + \frac{3\partial_1 f\partial_2 f}{f^4} - \partial_3 f\partial_4 f\,,\\
A_{12} &= \frac{\partial_2\partial_4 f}{f} + \frac{2\partial_1 f\partial_3 f}{f^2} - \frac{2\partial_2 f\partial_4 f}{f^2}\,,\\
A_{13} &= -\frac{\partial_2\partial_2 f}{f^3} - \frac{(\partial_1 f)^2}{f^4} + \frac{2(\partial_2 f)^2}{f^4} - (\partial_3 f)^2\,,\\
A_{14} &= \frac{\partial_2\partial_3 f}{f} - \frac{\partial_1 f\partial_4 f}{f^2} - \frac{\partial_2 f\partial_3 f}{f^2}\,,\\
A_{21} &= \frac{\partial_1\partial_3 f}{f} - \frac{2\partial_2 f\partial_4 f}{f^2}\,,\\
A_{22} &= -f\partial_3\partial_4 f + \frac{\partial_1 f\partial_2 f}{f^4} - \partial_3 f\partial_4 f\,,\\
A_{23} &= \frac{\partial_2\partial_3 f}{f} + \frac{\partial_1 f\partial_4 f}{f^2} - \frac{\partial_2 f\partial_3 f}{f^2}\,,\\
A_{24} &= -f\partial_3\partial_3 f + \frac{(\partial_2 f)^2}{f^4} + (\partial_4 f)^2\,,\\
A_{31} &= \frac{\partial_1\partial_1 f}{f^3} - \frac{2(\partial_1 f)^2}{f^4} + \frac{(\partial_2 f)^2}{f^4} + (\partial_4 f)^2\,,\\
A_{32} &= -\frac{\partial_1\partial_4 f}{f} + \frac{\partial_1 f\partial_4 f}{f^2} + \frac{\partial_2 f\partial_3 f}{f^2}\,,\\
A_{33} &= \frac{\partial_1\partial_2 f}{f^3} - \frac{3\partial_1 f\partial_2 f}{f^4} + \partial_3 f\partial_4 f\,,\\
A_{34} &= -\frac{\partial_1\partial_3 f}{f} + \frac{2\partial_1 f\partial_3 f}{f^2} - \frac{2\partial_2 f\partial_4 f}{f^2}\,,\\
A_{41} &= -\frac{\partial_1\partial_4 f}{f} + \frac{\partial_1 f\partial_4 f}{f^2} - \frac{\partial_2 f\partial_3 f}{f^2}\,,\\
A_{42} &= f\partial_4\partial_4 f - \frac{(\partial_1 f)^2}{f^4} - (\partial_3 f)^2\,,\\
A_{43} &= -\frac{\partial_2\partial_4 f}{f} + \frac{2\partial_1 f\partial_3 f}{f^2}\,,\\
A_{44} &= f\partial_3\partial_4 f - \frac{\partial_1 f\partial_2 f}{f^4} + \partial_3 f\partial_4 f\,.
\end{aligned}
$$

From (3.4), the components of Riemannian curvature tensor with respect to the unitary frame $\{e_i\}_{i=1,2,3,4}$ are given by the following expressions:

$$
(3.8)\qquad R_{1212} = -\frac{\partial_1\partial_1 f}{2f^3} + \frac{f}{2}\partial_4\partial_4 f + \frac{3}{2}\frac{(\partial_1 f)^2}{f^4} - \frac{(\partial_2 f)^2}{2f^4} - \frac{(\partial_3 f)^2}{2} + \frac{(\partial_4 f)^2}{2}\,,
$$

$$
\begin{aligned}
R_{3434} &= -\frac{\partial_2\partial_2 f}{2f^3} + \frac{f}{2}\partial_3\partial_3 f - \frac{(\partial_1 f)^2}{2f^4} + \frac{3}{2}\frac{(\partial_2 f)^2}{f^4} + \frac{(\partial_3 f)^2}{2} - \frac{(\partial_4 f)^2}{2}, \\
R_{1234} &= 0, \\
R_{1324} &= 0, \\
R_{1423} &= 0, \\
R_{1313} &= \frac{\partial_1\partial_1 f}{2f^3} + \frac{\partial_2\partial_2 f}{2f^3} - \frac{(\partial_1 f)^2}{2f^4} - \frac{(\partial_2 f)^2}{2f^4} + \frac{(\partial_3 f)^2}{2} + \frac{(\partial_4 f)^2}{2}, \\
R_{1414} &= -\frac{\partial_1\partial_1 f}{2f^3} + \frac{f}{2}\partial_3\partial_3 f + \frac{3}{2}\frac{(\partial_1 f)^2}{f^4} - \frac{(\partial_2 f)^2}{2f^4} + \frac{(\partial_3 f)^2}{2} - \frac{(\partial_4 f)^2}{2}, \\
R_{2323} &= -\frac{\partial_2\partial_2 f}{2f^3} + \frac{f}{2}\partial_4\partial_4 f - \frac{(\partial_1 f)^2}{2f^4} + \frac{3}{2}\frac{(\partial_2 f)^2}{f^4} - \frac{(\partial_3 f)^2}{2} + \frac{(\partial_4 f)^2}{2}, \\
R_{2424} &= -\frac{f}{2}\partial_3\partial_3 f - \frac{f}{2}\partial_4\partial_4 f + \frac{(\partial_1 f)^2}{2f^4} + \frac{(\partial_2 f)^2}{2f^4} + \frac{(\partial_3 f)^2}{2} + \frac{(\partial_4 f)^2}{2}, \\
R_{1334} &= \frac{\partial_1\partial_3 f}{2f}, \\
R_{2434} &= \frac{\partial_2\partial_4 f}{2f} - \frac{\partial_2 f\,\partial_4 f}{f^2}, \\
R_{1213} &= -\frac{\partial_2\partial_4 f}{2f}, \\
R_{1224} &= -\frac{\partial_1\partial_3 f}{2f} + \frac{\partial_1 f\,\partial_3 f}{f^2}, \\
R_{1434} &= -\frac{\partial_1\partial_2 f}{2f^3} + 2\frac{\partial_1 f\,\partial_2 f}{f^4}, \\
R_{2334} &= -\frac{f}{2}\partial_3\partial_4 f - \partial_3 f\,\partial_4 f, \\
R_{1214} &= \frac{f}{2}\partial_3\partial_4 f + \partial_3 f\,\partial_4 f, \\
R_{1223} &= \frac{\partial_1\partial_2 f}{2f^3} - 2\frac{\partial_1 f\,\partial_2 f}{f^4}, \\
R_{1323} &= -\frac{\partial_1\partial_4 f}{2f}, \\
R_{2324} &= \frac{\partial_2\partial_3 f}{2f} - \frac{\partial_2 f\,\partial_3 f}{f^2}, \\
R_{1314} &= -\frac{\partial_2\partial_3 f}{2f}, \\
R_{1424} &= \frac{\partial_1\partial_4 f}{2f} - \frac{\partial_1 f\,\partial_4 f}{f^2}.
\end{aligned}
$$

Now, we use the following notation:

$$\Delta_1 f := \partial_1\partial_1 f + \partial_2\partial_2 f\,,\quad \Delta_2 f := \partial_3\partial_3 f + \partial_4\partial_4 f\,,$$
$$\|\mathrm{grad}_1 f\|^2 := (\partial_1 f)^2 + (\partial_2 f)^2\,,\quad \|\mathrm{grad}_2 f\|^2 := (\partial_3 f)^2 + (\partial_4 f)^2\,.$$

Then, by (3.8), we obtain easily

$$\rho_{11} = \frac{\partial_1\partial_1 f}{2f^3} - \frac{\partial_2\partial_2 f}{2f^3} - \frac{f}{2}\Delta_2 f - \frac{5}{2}\frac{(\partial_1 f)^2}{f^4} + \frac{3}{2}\frac{(\partial_2 f)^2}{f^4} - \frac{1}{2}\|\mathrm{grad}_2 f\|^2\,, \tag{3.9}$$

$$\begin{aligned}
\rho_{22} &= \frac{\Delta_1 f}{2f^3} + \frac{f}{2}(\partial_3\partial_3 f - \partial_4\partial_4 f) - \frac{3}{2}\frac{\|\mathrm{grad}_1 f\|^2}{f^4} + \frac{(\partial_3 f)^2}{2} - \frac{3(\partial_4 f)^2}{2}\,,\\
\rho_{33} &= -\frac{\partial_1\partial_1 f}{2f^3} + \frac{\partial_2\partial_2 f}{2f^3} - \frac{f}{2}\Delta_2 f + \frac{3}{2}\frac{(\partial_1 f)^2}{f^4} - \frac{5}{2}\frac{(\partial_2 f)^2}{f^4} - \frac{1}{2}\|\mathrm{grad}_2 f\|^2\,,\\
\rho_{44} &= \frac{\Delta_1 f}{2f^3} - \frac{f}{2}(\partial_3\partial_3 f - \partial_4\partial_4 f) - \frac{3}{2}\frac{\|\mathrm{grad}_1 f\|^2}{f^4} - \frac{3(\partial_3 f)^2}{2} + \frac{(\partial_4 f)^2}{2}\,,\\
\rho_{12} &= \frac{\partial_1 f\partial_4 f}{f^2}\,,\\
\rho_{13} &= \frac{\partial_1\partial_2 f}{f^3} - 4\frac{\partial_1 f\partial_2 f}{f^4}\,,\\
\rho_{14} &= \frac{\partial_1 f\partial_3 f}{f^2}\,,\\
\rho_{23} &= \frac{\partial_2 f\partial_4 f}{f^2}\,,\\
\rho_{24} &= -f\partial_3\partial_4 f - 2\partial_3 f\partial_4 f\,,\\
\rho_{34} &= \frac{\partial_2 f\partial_3 f}{f^2}\,,
\end{aligned}$$

$$\tau = \frac{\Delta_1 f}{f^3} - f\Delta_2 f - 4\frac{\|\mathrm{grad}_1 f\|^2}{f^4} - 2\|\mathrm{grad}_2 f\|^2\,, \tag{3.10}$$

$$\rho^*_{11} = \rho^*_{22} = \frac{\partial_1\partial_1 f}{2f^3} - \frac{f}{2}\partial_4\partial_4 f - \frac{3}{2}\frac{(\partial_1 f)^2}{f^4} + \frac{(\partial_2 f)^2}{2f^4} + \frac{(\partial_3 f)^2}{2} - \frac{(\partial_4 f)^2}{2}\,, \tag{3.11}$$

$$\begin{aligned}
\rho^*_{33} = \rho^*_{44} &= \frac{\partial_2\partial_2 f}{2f^3} - \frac{f}{2}\partial_3\partial_3 f + \frac{(\partial_1 f)^2}{2f^4} - \frac{3}{2}\frac{(\partial_2 f)^2}{f^4} - \frac{(\partial_3 f)^2}{2} + \frac{(\partial_4 f)^2}{2}\,,\\
\rho^*_{12} = \rho^*_{21} &= \rho^*_{34} = \rho^*_{43} = 0\,,\\
\rho^*_{13} = \rho^*_{42} &= \frac{\partial_1\partial_2 f}{2f^3} - 2\frac{\partial_1 f\partial_2 f}{f^4} - \frac{f}{2}\partial_3\partial_4 f - \partial_3 f\partial_4 f\,,
\end{aligned}$$

$$\rho^*_{14} = -\rho^*_{32} = \frac{\partial_1\partial_3 f}{2f} - \frac{\partial_2\partial_4 f}{2f},$$

$$\rho^*_{23} = -\rho^*_{41} = \frac{\partial_1\partial_3 f}{2f} - \frac{\partial_1 f\partial_3 f}{f^2} - \frac{\partial_2\partial_4 f}{2f} + \frac{\partial_2 f\partial_4 f}{f^2},$$

$$\rho^*_{24} = \rho^*_{31} = \frac{\partial_1\partial_2 f}{2f^3} - 2\frac{\partial_1 f\partial_2 f}{f^4} - \frac{f}{2}\partial_3\partial_4 f - \partial_3 f\partial_4 f,$$

$$\tau^* = \frac{\Delta_1 f}{f^3} - f\Delta_2 f - 2\frac{\|\mathrm{grad}_1 f\|^2}{f^4}. \tag{3.12}$$

By Proposition 2.1, and using (3.7), (3.8), we have the following

LEMMA 3.2. *Let*

$$M = \{(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4 \mid f(x_1, x_2, x_3, x_4) \neq 0\},$$

$$g = (g_{ij}) = 2\begin{pmatrix} f^2 & 0 & 0 & 0 \\ 0 & f^2 & 0 & 0 \\ 0 & 0 & \frac{1}{f^2} & 0 \\ 0 & 0 & 0 & \frac{1}{f^2} \end{pmatrix},$$

$$J = (J_i^j) = \begin{pmatrix} 0 & 0 & 0 & -f^2 \\ 0 & 0 & -f^2 & 0 \\ 0 & \frac{1}{f^2} & 0 & 0 \\ \frac{1}{f^2} & 0 & 0 & 0 \end{pmatrix}.$$

Then the almost Kähler manifold (M, J, g) *is of pointwise constant holomorphic sectional curvature* $c = c(p)$ *if and only if*

$$\frac{1}{2f^3}(\partial_1\partial_1 f - \partial_2\partial_2 f) + \frac{f}{2}(\partial_3\partial_3 f - \partial_4\partial_4 f) = \frac{2}{f^4}\{(\partial_1 f)^2 - (\partial_2 f)^2\} - \{(\partial_3 f)^2 - (\partial_4 f)^2\}, \tag{3.13}$$

$$c = -\frac{1}{8}(\tau^* - \tau) = -\frac{\|\mathrm{grad}_1 f\|^2}{4f^4} - \frac{1}{4}\|\mathrm{grad}_2 f\|^2, \tag{3.14}$$

$$\frac{\Delta_1 f}{f^3} - f\Delta_2 f = \frac{\|\mathrm{grad}_1 f\|^2}{f^4} - \|\mathrm{grad}_2 f\|^2, \tag{3.15}$$

$$\partial_1\partial_3 f + \partial_2\partial_4 f = 2\frac{\partial_2 f\partial_4 f}{f}, \tag{3.16}$$

$$\partial_1\partial_3 f + \partial_2\partial_4 f = 2\frac{\partial_1 f\partial_3 f}{f}, \tag{3.17}$$

$$\frac{\partial_1\partial_2 f}{f^3} - f\partial_3\partial_4 f = 4\frac{\partial_1 f\partial_2 f}{f^4} + 2\partial_3 f\partial_4 f\,, \tag{3.18}$$

$$\partial_1\partial_4 f - \partial_2\partial_3 f = \frac{1}{f}(\partial_1 f\partial_4 f - \partial_2 f\partial_3 f)\,. \tag{3.19}$$

By (3.9), we have the following

LEMMA 3.3. *The almost Kähler manifold (M, J, g) in Lemma 3.2 is Einstein if and only if*

$$\frac{\partial_2\partial_2 f}{f^3} + f\partial_3\partial_3 f = \frac{1}{f^4}\{-(\partial_1 f)^2 + 3(\partial_2 f)^2\} - (\partial_3 f)^2 + (\partial_4 f)^2\,, \tag{3.20}$$

$$\partial_1\partial_1 f - \partial_2\partial_2 f = \frac{4}{f}\{(\partial_1 f)^2 - (\partial_2 f)^2\}\,, \tag{3.21}$$

$$f(\partial_3\partial_3 f - \partial_4\partial_4 f) = -2(\partial_3 f)^2 + 2(\partial_4 f)^2\,, \tag{3.22}$$

$$\partial_1\partial_4 f = 0\,, \tag{3.23}$$

$$\partial_1\partial_2 f = \frac{4}{f}\partial_1 f\partial_2 f\,, \tag{3.24}$$

$$\partial_1\partial_3 f = 0\,, \tag{3.25}$$

$$\partial_2\partial_4 f = 0\,, \tag{3.26}$$

$$f\partial_3\partial_4 f = -2\partial_3 f\partial_4 f\,, \tag{3.27}$$

$$\partial_2\partial_3 f = 0\,. \tag{3.28}$$

By (3.11), we have the following

LEMMA 3.4. *The almost Kähler manifold (M, J, g) in Lemma 3.2 is weakly $*$-Einstein if and only if*

$$\begin{aligned}\frac{1}{2f^3}(\partial_1\partial_1 f - \partial_2\partial_2 f) &+ \frac{f}{2}(\partial_3\partial_3 f - \partial_4\partial_4 f)\\ &= \frac{2}{f^4}\{(\partial_1 f)^2 - (\partial_2 f)^2\} - \{(\partial_3 f)^2 - (\partial_4 f)^2\}\,,\end{aligned} \tag{3.29}$$

$$\frac{\partial_1\partial_2 f}{f^3} - f\partial_3\partial_4 f = 4\frac{\partial_1 f\partial_2 f}{f^4} + 2\partial_3 f\partial_4 f\,, \tag{3.30}$$

$$\partial_1\partial_3 f = \partial_2\partial_4 f\,, \tag{3.31}$$

$$\partial_1\partial_3 f - \partial_2\partial_4 f = \frac{2}{f}(\partial_1 f\partial_3 f - \partial_2 f\partial_4 f)\,. \tag{3.32}$$

By Lemma 3.3, we get the following

PROPOSITION 3.5. *If the almost Kähler manifold* (M, J, g) *in Lemma* 3.2 *is Einstein, then* (M, J, g) *is Kähler.*

PROOF. First, we show that if f is a function of two variables x_1, x_2 or x_3, x_4, then f is constant. Suppose that f is a function of x_1, x_2; $f = f(x_1, x_2)$. Then the conditions in Lemma 3.3 reduce to

$$f\partial_2\partial_2 f = -(\partial_1 f)^2 + 3(\partial_2 f)^2, \tag{3.33}$$

$$f\partial_1\partial_1 f = 3(\partial_1 f)^2 - (\partial_2 f)^2, \tag{3.34}$$

$$f\partial_1\partial_2 f = 4\partial_1 f\partial_2 f. \tag{3.35}$$

Differentiating (3.33) with respect to x_1 yields

$$(\partial_1 f)\partial_2\partial_2 f + f\partial_1\partial_2^2 f = -2(\partial_1 f)\partial_1\partial_1 f + 6(\partial_2 f)\partial_1\partial_2 f.$$

By using (3.33), (3.34) and (3.35), we have

$$f^2\partial_1\partial_2^2 f = -5(\partial_1 f)^3 + 23(\partial_1 f)(\partial_2 f)^2. \tag{3.36}$$

On one hand, differentiating (3.35) with respect to x_2, and making use of (3.33) and (3.35), we have

$$f^2\partial_1\partial_2^2 f = -4(\partial_1 f)^3 + 24(\partial_1 f)(\partial_2 f)^2. \tag{3.37}$$

From (3.36) and (3.37), we obtain

$$(\partial_1 f)\|\mathrm{grad}_1 f\|^2 = 0. \tag{3.38}$$

Similarly, differentiating (3.34) with respect to x_2, we have

$$f^2\partial_1^2\partial_2 f = 23(\partial_1 f)^2\partial_2 f - 5(\partial_2 f)^3. \tag{3.39}$$

Differentiating (3.35) with respect to x_1, we have

$$f^2\partial_1^2\partial_2 f = 24(\partial_1 f)^2\partial_2 f - 4(\partial_2 f)^3. \tag{3.40}$$

From (3.39) and (3.40), we get

$$(\partial_2 f)\|\mathrm{grad}_1 f\|^2 = 0. \tag{3.41}$$

By virtue of (3.38) and (3.41), we obtain

$$\|\mathrm{grad}_1 f\|^6 = 0.$$

This shows that f must be constant.

Next, let f be a function of x_3, x_4; $f = f(x_3, x_4)$. Then the conditions in Lemma 3.3 reduce to

$$f\partial_3\partial_3 f = -(\partial_3 f)^2 + (\partial_4 f)^2, \tag{3.42}$$

$$f\partial_4\partial_4 f = (\partial_3 f)^2 - (\partial_4 f)^2, \tag{3.43}$$

$$f\partial_3\partial_4 f = -2\partial_3 f\partial_4 f. \tag{3.44}$$

By the same argument as above, we obtain

$$\|\mathrm{grad}_2 f\|^6 = 0$$

and it follows that f is constant.

We now proceed to complete the proof. By (3.23) and (3.26), we can see that $\partial_4 f$ is a function of x_3 and x_4. Similarly, by (3.25), (3.28), $\partial_3 f$ is also a function of x_3, x_4. Assume that there exists a point $p \in M$ such that $\partial_3\partial_3 f - \partial_4\partial_4 f \neq 0$ or $\partial_3\partial_4 f \neq 0$ at p. Then, by (3.22), (3.27), f is a function of only two variables x_3, x_4 on some neighborhood of p. It follows that f is constant around p, but this contradicts with the assumption.

Thus $\partial_3\partial_3 f - \partial_4\partial_4 f = 0$ and $\partial_3\partial_4 f = 0$. Then, by (3.22) and (3.27), we have

$$(\partial_3 f)^2 = (\partial_4 f)^2 \quad \text{and} \quad \partial_3 f \partial_4 f = 0 .$$

This implies that $\partial_3 f = \partial_4 f = 0$ and f is a function of two variables x_1, x_2. Therefore we can conclude that f is a constant function, and $\nabla J = 0$ by (3.5). □

Now, we provide an almost Kähler structure (J, g) which has pointwise constant holomorphic sectional curvature. First, we assume that f is a function of two variables x_1, x_2; $f = f(x_1, x_2)$. Then (3.13)–(3.19) reduce to

$$\partial_1\partial_1 f - \partial_2\partial_2 f = \frac{4}{f}\{(\partial_1 f)^2 - (\partial_2 f)^2\}, \tag{3.45}$$

$$c = -\frac{1}{8}(\tau^* - \tau) = -\frac{\|\mathrm{grad}_1 f\|^2}{4f^4}, \tag{3.46}$$

$$f\Delta_1 f = \|\mathrm{grad}_1 f\|^2, \tag{3.47}$$

$$f\partial_1\partial_2 f = 4\partial_1 f \partial_2 f . \tag{3.48}$$

If we choose $f = K(x_1^2 + x_2^2)^\alpha$, where K and α are constants, then it is easy to see that f satisfies (3.47). Substituting f in (3.45), we obtain $\alpha = -1/3$. In this case, f also satisfies (3.48), and $c = -(1/9K^2)(x_1^2 + x_2^2)^{-1/3}$. Consequently, we obtain the following

THEOREM 3.6. *Let*

$$M = \{(x_1, x_2, x_3, x_4) \in \boldsymbol{R}^4 \mid (x_1, x_2) \neq 0\} .$$

We define a Riemannian metric g and an almost complex structure J on M by

$$g = 2\begin{pmatrix} K^2(x_1^2 + x_2^2)^{-2/3} & 0 & 0 & 0 \\ 0 & K^2(x_1^2 + x_2^2)^{-2/3} & 0 & 0 \\ 0 & 0 & \frac{1}{K^2}(x_1^2 + x_2^2)^{2/3} & 0 \\ 0 & 0 & 0 & \frac{1}{K^2}(x_1^2 + x_2^2)^{2/3} \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & 0 & -K^2(x_1^2+x_2^2)^{-2/3} \\ 0 & 0 & -K^2(x_1^2+x_2^2)^{-2/3} & 0 \\ 0 & \frac{1}{K^2}(x_1^2+x_2^2)^{2/3} & 0 & 0 \\ \frac{1}{K^2}(x_1^2+x_2^2)^{2/3} & 0 & 0 & 0 \end{pmatrix},$$

where K is a non-zero constant. Then the almost Kähler manifold (M, J, g) *is of pointwise constant holomorphic sectional curvature with* $c = -(1/9K^2)(x_1^2+x_2^2)^{-1/3}$. *It is also weakly* $*$*-Einstein, and* $\tau = -(4/3K^2)(x_1^2+x_2^2)^{-1/3}$, $\tau^* = -(4/9K^2)(x_1^2+x_2^2)^{-1/3}$.

It should be remarked that the pointwise constant c is negative contrary to the Nurowski-Przanowski example.

Next, we assume that f is a function of x_3, x_4; $f = f(x_3, x_4)$. Then (3.13)–(3.19) reduce to

$$f(\partial_3\partial_3 f - \partial_4\partial_4 f) = -2\{(\partial_3 f)^2 - (\partial_4 f)^2\}, \tag{3.49}$$

$$c = -\frac{1}{8}(\tau^* - \tau) = -\frac{1}{4}\|\mathrm{grad}_2 f\|^2, \tag{3.50}$$

$$f\Delta_2 f = \|\mathrm{grad}_2 f\|^2, \tag{3.51}$$

$$f\partial_3\partial_4 f = -2\partial_3 f\partial_4 f. \tag{3.52}$$

By the same way, we can easily see that $f = L(x_3^2+x_4^2)^{1/3}$ satisfies (3.49), (3.51) and (3.52). Therefore we have the following

THEOREM 3.7. *Let*

$$M' = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid (x_3, x_4) \neq 0\}.$$

We define a Riemannian metric g' *and an almost complex structure* J' *on* M' *by*

$$g' = 2\begin{pmatrix} L^2(x_3^2+x_4^2)^{2/3} & 0 & 0 & 0 \\ 0 & L^2(x_3^2+x_4^2)^{2/3} & 0 & 0 \\ 0 & 0 & \frac{1}{L^2}(x_3^2+x_4^2)^{-2/3} & 0 \\ 0 & 0 & 0 & \frac{1}{L^2}(x_3^2+x_4^2)^{-2/3} \end{pmatrix},$$

$$J' = \begin{pmatrix} 0 & 0 & 0 & -L^2(x_3^2+x_4^2)^{2/3} \\ 0 & 0 & -L^2(x_3^2+x_4^2)^{2/3} & 0 \\ 0 & \frac{1}{L^2}(x_3^2+x_4^2)^{-2/3} & 0 & 0 \\ \frac{1}{L^2}(x_3^2+x_4^2)^{-2/3} & 0 & 0 & 0 \end{pmatrix},$$

where L is a non-zero constant. Then the almost Kähler manifold (M', J', g') *is of pointwise constant holomorphic sectional curvature with* $c = -(L^2/9)(x_3^2+x_4^2)^{-1/3}$. *It is also weakly* $*$*-Einstein, and* $\tau = -(4L^2/3)(x_3^2+x_4^2)^{-1/3}$, $\tau^* = -(4L^2/9)(x_3^2+x_4^2)^{-1/3}$.

We define a mapping $\varphi : M \to M'$ by

$$\varphi(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2)\,.$$

Then it is easy to show that the almost Kähler manifold $(M', -J', g')$ is holomorphically isometric to (M, J, g) in Theorem 3.6, when $L^2 = 1/K^2$.

Finally, we remark that there exist other examples of weakly $*$-Einstein almost Kähler manifolds. For example, we can choose

$$f = K(x_1 + x_2)^{-1/3}\,, \quad f = K(x_3 + x_4)^{1/3} \quad \text{and} \quad f = K\left(\frac{x_3 + x_4}{x_1 + x_2}\right)^{1/3}\,,$$

which all satisfy the conditions in Lemma 3.4, and we have

$$\rho^* = \frac{1}{9K^2(x_1 + x_2)^{4/3}}g\,, \quad \rho^* = \frac{K^2}{9(x_3 + x_4)^{4/3}}g\,,$$

$$\rho^* = \frac{K^4(x_1 + x_2)^{2/3} + (x_3 + x_4)^{2/3}}{9K^2\{(x_1 + x_2)(x_3 + x_4)\}^{4/3}}g\,,$$

respectively. Moreover, from (3.9), we see that the Ricci tensors of these examples are J-anti-invariant, i.e., $\rho(JX, JY) = -\rho(X, Y)$.

References

[1] J. T. CHO and K. SEKIGAWA, Four-dimensional almost Kähler manifolds of pointwise constant holomorphic sectional curvatue, Arab. J. Math. Sci. **2** (1996), 39–53.

[2] S. I. GOLDBERG, Integrability of almost Kähler manifolds, Proc. Amer. Math. Soc. **21** (1969), 96–100.

[3] A. GRAY and L. VANHECKE, Almost Hermitian manifolds with constant holomorphic sectional curvature, Časopis Pěst. Mat. **104** (1979), 170–179.

[4] P. NUROWSKI and M. PRZANOWSKI, A four-dimensional example of Ricci-flat metric admitting an almost-Kähler non-Kähler structure, preprint (1998).

[5] T. SATO, Almost Kähler manifolds of constant holomorphic sectional curvature, Tsukuba J. Math. **20** (1996), 517–524.

[6] T. SATO, Some remarks on almost Kähler 4-manifolds of pointwise constant holomorphic sectional curvature, Kodai Math. J. **22** (1999), 286–303.

Present Address:
FACULTY OF ENGINEERING, KANAZAWA UNIVERSITY,
KANAZAWA, 920–8667 JAPAN