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Sheaf Cohomology of the Moduli Space of Spatial Polygons and Lattice Points

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Abstract. Let $\mathcal{M}_n$ be the moduli space of spatial polygons with n edges. We consider the case of odd n. Let $K_n^* = \Lambda^{n-3}T\mathcal{M}_n$ be the dual bundle of the canonical bundle on $\mathcal{M}_n$. In this paper we determine the sheaf cohomology $H^*(\mathcal{M}_n, K_n^*)$. We have $H^q(\mathcal{M}_n, K_n^*) = 0$ $(q \geq 1)$ and $\dim H^0(\mathcal{M}_n, K_n^*)$ is equal to the number of lattice points in the convex polytope Δ_n in $\mathbf{R}^{n-3}$.

1. Introduction.

Let $\mathcal{M}_n$ be the moduli space of spatial polygons $P = (a_1, a_2, \cdots, a_n)$ whose edges are vectors $a_i \in \mathbf{R}^3$ of length $|a_i| = 1$ $(1 \leq i \leq n)$. Two polygons are identified if they differ only by motions in $\mathbf{R}^3$. The sum of the vectors is assumed to be zero. Thus we first define the space X_n by

$$X_n = \{P = (a_1, \cdots, a_n) \in (S^2)^n : a_1 + \cdots + a_n = 0\}. \tag{1.1}$$

Then we set $\mathcal{M}_n = X_n/SO(3)$.

For odd n or $n = 4$, $\mathcal{M}_n$ is known to be a Fano manifold (i.e. the dual bundle of the canonical bundle is ample) of complex dimension $n - 3$ [4]. On the other hand, for even $n \geq 6$, $\mathcal{M}_n$ has cone-like singular points. For other properties of $\mathcal{M}_n$, see for example [3] and the references therein.

In this paper, we consider $\mathcal{M}_n$ for odd n. Since $\mathcal{M}_3 = \{\text{point}\}$, we assume that $n \geq 5$. Thus $\mathcal{M}_n$ has no singular points such that $\dim_{\mathbf{C}} \mathcal{M}_n \geq 2$. For a holomorphic vector bundle $E \to \mathcal{M}_n$, we abbreviate the sheaf cohomology group $H^*(\mathcal{M}_n, \mathcal{O}(E))$ to $H^*(\mathcal{M}_n, E)$. In [3] we proved that $H^q(\mathcal{M}_n, T\mathcal{M}_n) = 0$ $(q \geq 0)$, where $T\mathcal{M}_n$ denotes the tangent bundle of the complex manifold $\mathcal{M}_n$.

Let K_n be the canonical bundle on $\mathcal{M}_n$ and K_n^* be its dual bundle. Thus $K_n^* = \Lambda^{n-3}T\mathcal{M}_n$. The purpose of this paper is to determine $H^*(\mathcal{M}_n, K_n^*)$. In order to state the results, we recall some notations in [2]. First we define a map $\mu_n : \mathcal{M}_n \to \mathbf{R}^{n-3}$ as follows. Let $P =$

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$(a_1, \cdots, a_n) \in \mathcal{M}_n$. Then we set

$$\mu_n(P) = \left(|a_1 + a_2|, |a_1 + a_2 + a_3|, \cdots, \left| \sum_{i=1}^{n-2} a_i \right| \right). \tag{1.2}$$

Thus $\mu_n(P)$ is the lengths of the diagonals connecting the vertices to the origin. (Since $|a_1| = |\sum_{i=1}^{n-1} a_i| = 1$, only these $n-3$ lengths are new. And in fact, the restriction $\mu_n \mid \mathcal{M}'_n$ of μ_n to an open dense subspace $\mathcal{M}'_n$ of $\mathcal{M}_n$ is a moment map for the T^{n-3}-action of $\mathcal{M}'_n$.)

We set $\Delta_n = \mu_n(\mathcal{M}_n)$. Thus Δ_n is an $(n-3)$-dimensional convex polytope in $\mathbf{R}^{n-3}$. (Note that in [2], we wrote the image of μ_n by Δ_{n-3} in order to indicate the dimension of the polytope. But in this paper we write the image by Δ_n.) Now let $\sharp(\Delta_n \cap \mathbf{Z}^{n-3})$ be the number of lattice points in Δ_n. Then our first result is the following:

THEOREM A. *For odd $n \geq 5$, we have*
(i) $H^q(\mathcal{M}_n, K_n^*) = 1$ $(q \geq 1)$.
(ii) $\dim H^0(\mathcal{M}_n, K_n^*) = \sharp(\Delta_n \cap \mathbf{Z}^{n-3})$.

Next we give a formula for $\sharp(\Delta_n \cap \mathbf{Z}^{n-3})$ with $n \geq 4$, although Theorem A is valid only for odd n. For $n \geq 4$, we define α_n by

$$\alpha_n = -\frac{1}{2} \sum_{q=0}^{\left[\frac{n+1}{3}\right]} (-1)^q \binom{n}{q} \binom{2n-2-3q}{n-3}. \tag{1.3}$$

In fact, (1.3) is equivalent to

$$\alpha_n = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\sin 3\theta}{\sin \theta} \right)^n \sin^2 \theta d\theta. \tag{1.4}$$

Then we have the following:

THEOREM B. *For $n \geq 4$, we have*

$$\sharp(\Delta_n \cap \mathbf{Z}^{n-3}) = \alpha_n.$$

EXAMPLE 1.5. We have the following examples: $\alpha_4 = 3, \alpha_5 = 6, \alpha_6 = 15, \alpha_7 = 36$, $\alpha_8 = 91, \alpha_9 = 232$ and $\alpha_{10} = 603$.

REMARK 1.6. Let Δ be an integral convex polytope, $V(\Delta)$ the associated toric variety and $\mathcal{O}(D)$ the line bundle of a T-Cartier divisor D on $V(\Delta)$. Then a theorem similar to Theorem A holds for the cohomology $H^*(V(\Delta), \mathcal{O}(D))$ [5]. However the example that $\mathcal{M}_5$ is the del Pezzo surface of degree 5 (obtained from $\mathbf{C}P^2$ by blowing up four points in general position) [4], which is not a toric variety (see [5]), tells us that in general $\mathcal{M}_n$ is not a toric variety.

Finally note that in contrast to $H^*(\mathcal{M}_n, K_n^*)$, the cohomology $H^*(\mathcal{M}_n, K_n)$ is already known. In fact this is a special case of the fact $h^{i,j}(\mathcal{M}_n) = 0$ $(i \neq j)$ (see for example [4]).

This paper is organized as follows. In Section 2, we calculate $H^*(\mathcal{M}_n, K_n^*)$. About $H^0(\mathcal{M}_n, K_n^*)$, we prove that its dimension is equal to α_n (see Theorem 2.1). Here α_n is defined in (1.3) and (1.4). In Section 3, we prove Theorem B. In particular, the proof of Theorem A (ii) completes in this Section.

2. Cohomology of $\mathcal{M}_n$.

First we prove Theorem A (i). Note that $H^q(\mathcal{M}_n, K_n^*) = H^q(\mathcal{M}_n, \Omega^{n-3}((K_n^*)^{\otimes 2}))$. Since K_n^* is an ample line bundle [4], so is $(K_n^*)^{\otimes 2}$. Then from the Kodaira-Nakano vanishing theorem [1], we have $H^q(\mathcal{M}_n, K_n^*) = 0$ for $q + n - 3 > n - 3$. Thus Theorem A (i) holds.

Instead of Theorem A (ii), we prove the following Theorem in this section:

THEOREM 2.1. *For odd $n \geq 5$, the Euler characteristic $\chi(\mathcal{M}_n, K_n^*)$ is equal to α_n.*

PROOF. We follow the method of the calculations of $\chi(\mathcal{M}_n, T\mathcal{M}_n)$ in [3]. In that paper the previous results which are necessary to the calculations are summarized. We shall not repeat the results here, but we note that the following three facts are essential. For odd n, we set $n = 2m + 1$.

(i) The ring structure: $H^*(\mathcal{M}_n; \mathbf{R})$ is generated by $z_1, \cdots, z_n \in H^2(\mathcal{M}_n; \mathbf{R})$. Since $z_i^2 = z_j^2$, we define $D^2 \in H^4(\mathcal{M}_n; \mathbf{R})$ to be $z_i^2 = D^2$ $(1 \leq i \leq n)$.

(ii) The total Chern class: The description of $c(T\mathcal{M}_n)$ in terms of the ring generators $z_1, \cdots, z_n$ is known.

(iii) The intersection number: For a sequence $(d_1, \cdots, d_n)$ of nonnegative integers with $\sum_{i=1}^n d_i = n - 3$, the intersection number $\langle z_1^{d_1} \cdots z_n^{d_n}, [\mathcal{M}_n]\rangle$ is known. In particular, if we define $\langle \rho_{n,2k}\rangle$ $(0 \leq k \leq m-1)$ by $\langle \rho_{n,2k}\rangle = \langle z_1^{2k} z_2 \cdots z_{n-2k-2}, [\mathcal{M}_n]\rangle$, then $\langle \rho_{n,2k}\rangle$ is known.

Now we calculate $\chi(\mathcal{M}_n, K_n^*)$. From the Hirzebruch-Riemann-Roch formula [1], we have

$$\chi(\mathcal{M}_n, K_n^*) = \left\langle \frac{2}{D^2}(-1 + \cosh D) \prod_{i=1}^n \left(\frac{z_i e^{z_i}}{1 - e^{-z_i}} \right), [\mathcal{M}_n] \right\rangle. \tag{2.2}$$

We define even functions $f(x)$ and $g(x)$ to satisfy

$$\frac{xe^x}{1 - e^{-x}} = f(x) + xg(x).$$

Then (2.2) is equivalent to

$$\chi(\mathcal{M}_n, K_n^*) = \sum_{j=0}^n \binom{n}{j} \left\langle \frac{2}{D^2}(-1+\cosh D)(f(D))^j (g(D))^{n-j} z_1 z_2 \cdots z_{n-j}, [\mathcal{M}_n] \right\rangle. \tag{2.3}$$

Since $\dim_{\mathbf{C}} \mathcal{M}_n = n - 3$, which is even, we can assume that j in (2.3) is odd ≥ 3. Hence we set $j = 2l + 1$. We define a rational number $\zeta_{n,2l}$ by

$$\zeta_{n,2l} = \text{the coefficient of } x^{2l} \text{ in } [2(-1 + \cosh x)(f(x))^{2l+1}(g(x))^{2m-2l}].$$

Then (2.3) is equivalent to

$$\chi(\mathcal{M}_n, K_n^*) = \sum_{l=1}^{m} \binom{2m+1}{2l+1} \langle \rho_{n,2l-2} \rangle \zeta_{n,2l} \,. \tag{2.4}$$

Here as before $\langle \rho_{n,2l-2} \rangle$ is the intersection number and known to be equal to $(-1)^{l+1} \frac{\binom{m-1}{l-1}\binom{2m-1}{m}}{\binom{2m-1}{2l-1}}$.

Using standard arguments of binomial coefficients, we see that (2.4) is equivalent to (1.3) or (1.4). Thus we have Theorem 2.1.

3. Proof of Theorem B.

In order to calculate ${}^\sharp(\Delta_n \cap \mathbf{Z}^{n-3})$, it is convenient to construct a recursion relation. First we generalize the convex polytope Δ_n to $\Delta_{n,i}$. For $i \in \mathbf{N}$, we define $\mathcal{M}_{n,i}$ as follows. Let $S^2(i)$ be the sphere in $\mathbf{R}^3$ with center the origin and radius i. We define $X_{n,i}$ by

$$X_{n,i} = \{(a_1, a_2, \cdots, a_n) \in S^2(i) \times (S^2)^{n-1} : a_1 + a_2 + \cdots + a_n = 0\} \,.$$

(Compare (1.1).) Then we set $\mathcal{M}_{n,i} = X_{n,i}/SO(3)$. Thus $\mathcal{M}_{n,1} = \mathcal{M}_n$.

Note the map $\mu_n : \mathcal{M}_n \to \mathbf{R}^{n-3}$ in (1.2) is naturally generalized to a map $\mu_n : \mathcal{M}_{n,i} \to \mathbf{R}^{n-3}$. Then we set $\Delta_{n,i} = \mu_n(\mathcal{M}_{n,i})$. Thus $\Delta_{n,1} = \Delta_n$.

Finally we set $\beta_{n,i} = {}^\sharp(\Delta_{n,i} \cap \mathbf{Z}^{n-3})$. When $i = 0$, we define $\beta_{n,0}$, by

$$\beta_{n,0} = \beta_{n-1,1} \,. \tag{3.1}$$

Thus Theorem B is equivalent to $\beta_{n,1} = \alpha_n$, where α_n is defined in (1.3) or (1.4).

About $\beta_{n,i}$, we have the following:

PROPOSITION 3.2. (i) $\beta_{4,0} = 1, \beta_{4,1} = 3, \beta_{4,2} = 2$ *and* $\beta_{4,3} = 1$.
(ii) $\beta_{n,i} = \beta_{n-1,i-1} + \beta_{n-1,i} + \beta_{n-1,i+1}$ $(i \geq 1)$.
(iii) $\beta_{n,0} = \beta_{n-1,1}$.
(iv) $\beta_{n,i} = 0$ $(i \geq n)$.

PROOF. (i) Since $\Delta_{4,1} = [0, 2]$, $\Delta_{4,2} = [1, 2]$, $\Delta_{4,3} = \{2\}$ and $\beta_{n,i}$ is the number of lattice points in $\Delta_{n,i}$, the result follows.

(ii) Let $(x_1, x_2, \cdots, x_{n-3}) \in \Delta_{n,i} \cap \mathbf{Z}^{n-3}$. From the definition of $\Delta_{n,i}$, the triangle inequality shows that x_1 is either of $i-1$, i or $i+1$. If $x_1 = i-1$, then we have $(x_2, \cdots, x_{n-3}) \in \Delta_{n-1,i-1} \cap \mathbf{Z}^{n-4}$. The other two cases are considered similarly. Thus (ii) follows.

(iii) is the definition of $\beta_{n,0}$ in (3.1).

(iv) The triangle inequality shows that $\mathcal{M}_{n,i} = \emptyset$ $(i \geq n)$. Thus we have $\Delta_{n,i} = \emptyset$ $(i \geq n)$ and the result follows. □

Now we see that the solution of the recursion relation in Proposition 3.2 is given by

$$\beta_{n,i} = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\sin(1+2i)\theta}{\sin\theta} \right) \left(\frac{\sin 3\theta}{\sin\theta} \right)^{n-1} \sin^2\theta d\theta \,.$$

In particular we have $\beta_{n,1} = \alpha_n$ (see (1.4)). This completes the proof of Theorem B.

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References

[1] P. GRIFFITHS and J. HARRIS, *Principles of algebraic geometry*, John Wiley (1978).
[2] Y. KAMIYAMA and M. TEZUKA, Symplectic volume of the moduli space of spatial polygons, J. Math. Kyoto Univ. **39** (1999), 557–575.
[3] Y. KAMIYAMA and S. TSUKUDA, On deformations of the complex structure on the moduli space of spatial polygons, to appear in Canad. Math. Bull.
[4] A. KLYACHKO, Spatial polygons and stable configurations of points in the projective line, *Algebraic geometry and its applications* (Yaroslavl', 1992), Aspects of Math. E. **25** (1994) Vieweg, 67–84.
[5] T. ODA, *Convex bodies and algebraic geometry*, Springer (1988).

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