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A Note on the Exponential Diophantine Equation $a^x + db^y = c^z$

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1. Introduction.

Let $\mathbf{N}$ be the set of all positive integers. Let a, b, c be fixed positive integers, and let d be a fixed prime with $d \equiv 3 \pmod 8$. In [2], using a lower bound for linear forms in two logarithms due to Laurent, Mignotte and Nesterenko [1], Terai and Takakuwa proved that if a, b, c, d satisfy

(1) $$a^2 + db^2 = c^2, \quad a \equiv 3 \pmod 8, \quad 4 \parallel b, \quad \left(\frac{b}{a}\right) = -1$$

(2) $$a \geq \lambda b, \quad d < 23865310019,$$

where (*/*) denote the Jacobi symbol and

$$\lambda = \sqrt{d}\left(\exp\left(2\left(\frac{\log d + 2}{\log 5} + 3231\right)^{-1}\right) - 1\right)^{-1/2},$$

then the equation

(3) $$a^x + db^y = c^z, \quad x, y, z \in \mathbf{N},$$

has only the solution $(x, y, z) = (2, 2, 2)$. In this paper, by an elementary approach, we prove the following result.

THEOREM *If a, b, c, d satisfy* (1) *and*

(4) $$a = db_2^2 - b_1^2 \quad b = 2b_1b_2 \quad c = db_2^2 + b_1^2,$$

where b_1, b_2 are positive integers satisfying $b_1 > 1$, $b_1 \equiv 1 \pmod 4$, $2 \parallel b_2$ and $\gcd(b_1, b_2) = 1$, *then* (3) *has only the solution* $(x, y, z) = (2, 2, 2)$.

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By the above result, we see that for any fixed prime d with $d \equiv 3 \pmod 8$, there exist infinitely many pairs (a, b, c) which (3) has only the solution $(x, y, z) = (2, 2, 2)$. This is an improvement on the results of [2].

2. Proofs.

LEMMA 1. *If a, b, c, d satisfy* (1), *then* (4) *holds for some positive integers b_1, b_2 satisfying $2 \nmid b_1$, $2 \| b_2$ and $\gcd(b_1, b_2) = 1$.*

PROOF. Since $4 \| b$, we get from (1) that

$$c + a = \begin{cases} 2b_1^2, \\ 2db_2^2, \end{cases} \quad c - a = \begin{cases} 2db_2^2, \\ 2b_1^2, \end{cases} \quad b = 2b_1 b_2, \tag{5}$$

where b_1, b_2 are positive integers satisfying $2 \| b_1 b_2$ and $\gcd(b_1, b_2) = 1$. By (5), we get

$$a = \begin{cases} b_1^2 - db_2^2, \\ db_2^2 - b_1^2, \end{cases} \quad c = db_2^2 + b_1^2. \tag{6}$$

Since $d \equiv 3 \pmod 8$, if $a = b_1^2 - db_2^2$, then $a \equiv 1 \not\equiv 3 \pmod 4$, a contradiction. So we have $a = db_2^2 - b_1^2$ by (6). Further, if $2 \mid b_1$, then $2 \| b_1$ and $a \equiv 7 \not\equiv 3 \pmod 8$, a contradiction. Thus, we get $2 \nmid b_1$, $2 \| b_2$ and (5). The lemma is proved.

LEMMA 2. *Let a, b, c, d be as in* (1). *If* (3) *has a solution (x, y, z) with $(x, y, z) \neq (2, 2, 2)$, then we have $2 \mid x$, $y = 1$ and $2 \nmid z$.*

PROOF. Let (x, y, z) be a solution of (3) with $(x, y, z) \neq (2, 2, 2)$. By [2, Lemma 4], then either $2 \mid x$, $y = 1$ and $2 \nmid z$ or $2 \mid x$, $y = 2$ and $2 \mid z$. Since $(x, y, z) \neq (2, 2, 2)$, if x, y, z satisfy $2 \mid x$, $y = 2$ and $2 \mid z$, then $x \geq 4$ and $z \geq 4$. Hence, by (1) and (4), we get $db^2 = c^z - a^x = (c^{z/2} + q^{x/2})(c^{z/2} - a^{x/2}) \geq c^{z/2} + a^{x/2} \geq c^2 + a^2 > c^2 - a^2 = db^2$, a contradiction. The lemma is proved.

PROOF OF THEOREM. We suppose that (3) has a solution (x, y, z) with $(x, y, z) \neq (2, 2, 2)$. By Lemma 2, we get

$$a^x + db = c^z, \quad 2|x, \quad 2 \nmid z. \tag{7}$$

Further, by Lemma 1, we see from (4) and (7) that

$$1 = \left(\frac{c}{b_1}\right) = \left(\frac{db_2^2 + b_1^2}{b_1}\right) = \left(\frac{d}{b_1}\right). \tag{8}$$

On the other hand, since $(b/a) = -1$ by (1), we obtain from (4) that

$$
\begin{aligned}
-1 &= \left(\frac{b}{a}\right) = \left(\frac{2b_1b_2}{a}\right) = \left(\frac{b_1}{a}\right)\left(\frac{b_2/2}{a}\right) \\
&= (-1)^{(b_1-1)/2+(b_2/2-1)/2}\left(\frac{a}{b_1}\right)\left(\frac{a}{b_2/2}\right) \qquad (9)\\
&= (-1)^{(b_1-1)/2+(b_2/2-1)/2}\left(\frac{db_2^2-b_1^2}{b_1}\right)\left(\frac{db_2^2-b_1^2}{b_2/2}\right) = (-1)^{(b_1-1)/2}\left(\frac{d}{b_1}\right).
\end{aligned}
$$

The combination of (8) and (9) yields $b_1 \equiv 3 \pmod 4$. Thus, if $b_1 \not\equiv 3 \pmod 4$, then (3) has only the solution $(x, y, z) = (2, 2, 2)$. The theorem is proved.

References

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