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Edge-Magic Labelings of Wheel Graphs

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Abstract. A graph G is called edge-magic if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ such that $f(x) + f(y) + f(xy) = C$ is a constant for any $xy \in E(G)$. In this paper, we show that a wheel graph W_n is edge-magic if $n \not\equiv 0 \pmod 4$.

1. Statement of the main result.

Let G be a simple undirected graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A bijection f from $V(G) \cup E(G)$ to $\{1, 2 \cdots, |V(G)| + |E(G)|\}$ is called an *edge-magic* labeling of G if there exists a constant C (called the *magic number* of f) such that $f(x) + f(y) + f(xy) = C$ for any edge $xy \in E(G)$. An edge-magic labeling f of G is called a *super edge-magic* labeling if $f(V(G)) = \{1, 2, \cdots, |V(G)|\}$ and $f(E(G)) = \{|V(G)|+1, |V(G)|+2, \cdots, |V(G)|+|E(G)|\}$. A graph G is called edge-magic (resp. super edge-magic) if there exists an edge-magic (resp. super edge-magic) labeling of G.

For $n \geq 4$, the wheel graph W_n of order n is defined as the join $C_{n-1} + K_1$ of the cycle C_{n-1} of order $n - 1$ and the complete graph K_1 of order 1. In this paper, we prove the following theorem:

THEOREM. *Let n be an integer with $n \geq 5$ and $n \not\equiv 0 \pmod 4$. Then W_n is edge-magic.*

This theorem was conjectured by Enomoto, Llado, Nakamigawa and Ringel [1, Conjecture 3.3]. They verified that the theorem is true when $n \leq 30$. They also gave an important result about edge-magic labeling of W_n, which we state as a remark:

REMARK 1.1 ([1, Theorem 2.4]). Let $n \geq 4$ be an integer. Then W_n is not super edge-magic. Moreover, W_n is not edge-magic if $n \equiv 0 \pmod 4$.

Our proof of the Theorem is constructive. However, the edge-magic labeling given in this paper is rather complicated. Thus for completeness' sake, we include in Section 2 a proof of the fact that W_n does not have certain simpler forms of edge-magic labelings.

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2. Nonexistence and characterization of certain types of edge-magic labelings.

In this section, we determine certain types of edge-magic labelings of W_n. Write $V(W_n) = \{v_0, v_1, \cdots, v_{n-1}\}$ so that $E(W_n) = \{v_i v_{i+1} \mid 1 \le i \le n-2\} \cup \{v_{n-1}v_1\} \cup \{v_0 v_i \mid 1 \le i \le n-1\}$. For subsets $X, Y \subset \{1, 2, \cdots, |V(W_n)| + |E(W_n)|\}$, we write $X < Y$ to mean $\max X < \min Y$.

REMARK 2.1. If $n = 2k+1$, then W_n has no edge-magic labeling f of the form

$$f(v_{2m-1}) = a + m \quad (1 \le m \le k)\,, \qquad f(v_{2m}) = b + m \quad (1 \le m \le k)\,.$$

PROOF. Suppose that such a labeling f exists. Then $f(v_1) + f(v_{2k}) = (a+1) + (b+k) = a+b+k+1$. If $k \equiv 0 \pmod 2$, $f(v_k) + f(v_{k+1}) = (b+k/2) + (a+k/2+1) = a+b+k+1$; if $k \equiv 1 \pmod 2$, $f(v_k) + f(v_{k+1}) = (a+(k+1)/2) + (b+(k+1)/2) = a+b+k+1$. Thus in both cases, letting C be the magic number of f, we have $f(v_1 v_{2k}) = C - (f(v_1) + f(v_{2k})) = C - (f(v_k) + f(v_{k+1})) = f(v_k v_{k+1})$, which is a contradiction. □

REMARK 2.2. If $n = 2k+2$, then W_n has no edge-magic labeling f of the form

$$f(v_{2m-1}) = a + m \quad (1 \le m \le k+1)\,, \qquad f(v_{2m}) = b + m \quad (1 \le m \le k)\,.$$

PROOF. By Remark 1.1, we may assume $k \ge 2$ and $k \equiv 0 \pmod 2$. Suppose that such a labeling f exists. Then $\bar{f}(x) = |V(G)| + |E(G)| + 1 - f(x)$ $(x \in V(G) \cup E(G))$ is also the same type of labeling. Thus we may assume that $a+k+1 < b+1$, and hence $f(v_0 v_2) = C - (b+1+f(v_0)) < C - (a+k+1+f(v_0)) = f(v_0 v_{2k+1})$, where C is the magic number of f. Write $A = \{a+1, \cdots, a+k+1\}$, $\bar{A} = \{C-(a+k+1+f(v_0)), \cdots, C-(a+1+f(v_0))\}$, $B = \{b+1, \cdots, b+k\}$, $\bar{B} = \{C-(b+k+f(v_0)), \cdots, C-(b+1+f(v_0))\}$, $U = \{C-(a+b+2k+1), \cdots, C-(a+b+2)\}$. Thus $|A| = |\bar{A}| = k+1$, $|B| = |\bar{B}| = k$, $|U| = 2k$ and

$$A \cup \bar{A} \cup B \cup \bar{B} \cup U \cup \{f(v_0), f(v_1 v_{2k+1})\} = \{1, \cdots, 6k+4\}\,. \tag{1}$$

Set $d_1 = b-a-k-1$, $d_2 = C-(a+b+2k+2+f(v_0))$ and $d_3 = a+b+1+f(v_0)-C$. Thus $d_1 = \min B - \max A - 1 = \min \bar{A} - \max \bar{B} - 1 = f(v_1 v_{2k+1}) - \max U - 1$, $d_2 = \min \bar{B} - \max A - 1 = \min \bar{A} - \max B - 1$ and $d_3 = \min A - \max \bar{B} - 1 = \min B - \max \bar{A} - 1$. Having in mind the equality

$$\min A + \max \bar{A} = \min B + \max \bar{B}\,, \tag{2}$$

we divide our proof into four cases:

Case 1 $A < B < \bar{B} < \bar{A}$

Since $d_1 = \min B - \max A - 1 = \min \bar{A} - \max \bar{B} - 1$, it follows from (1) that $d_1 = 0$ or 1. Suppose $d_1 = 0$. Then since $d_1 = f(v_1 v_{2k+1}) - \max U - 1$, $f(v_1 v_{2k+1}) = \max U + 1$. First assume that $U < A < B < \bar{B} < \bar{A}$. Then $f(v_1 v_{2k+1}) \le |U| + 2 = 2k+2$. On the other hand, $\min \bar{A} - \min A \ge |A| + |B| + |\bar{B}| = 3k+1$ and, since f is edge-magic, we also have $f(v_1 v_{2k+1}) + \min A = f(v_1 v_{2k+1}) + f(v_1) = f(v_0 v_{2k+1}) + f(v_0) = \min \bar{A} + f(v_0)$. Consequently $f(v_1 v_{2k+1}) = \min \bar{A} - \min A + f(v_0) > 3k+1$, which is a contradiction. Next assume that $A < B < U < \bar{B} < \bar{A}$. Then $f(v_1 v_{2k+1}) \le 4k+3$. On the other hand,

$\min \bar{A} - \min A \geq 5k+2$, and hence $f(v_1v_{2k+1}) = \min \bar{A} - \min A + f(v_0) > 5k+2$, which is a contradiction. Finally assume that $A < B < \bar{B} < \bar{A} < U$. Then $f(v_1v_{2k+1}) = 6k+3$ or $6k+4$, and $f(v_0) = 1, 2k+2, 4k+3$ or $6k+4$. On the other hand, $\min \bar{A} - \min A = 3k+1$ or $3k+2$, and hence $f(v_0) = f(v_1v_{2k+1}) - (\min \bar{A} - \min A) = 3k+1, 3k+2$ or $3k+3$, which is a contradiction.

Suppose now that $d_1 = 1$. Then since $\max A + 1 = \min B - 1$ and since each of $\bar{A}$, $\bar{B}$ and U has cardinality at least 2, $\max A + 1 \notin A \cup \bar{A} \cup B \cup \bar{B} \cup U$. Similarly $\max \bar{B} + 1$, $\max U + 1 \notin A \cup \bar{A} \cup B \cup \bar{B} \cup U$. Since $\max A + 1$, $\max \bar{B} + 1$, $\max U + 1$ are distinct and belong to $\{1, 2, \cdots, 6k+4\}$, this contradicts (1).

Case 2 $\quad A < \bar{B} < B < \bar{A}$

By (1) and (2), $d_2 = 0$ or 1, and we also have $d_1 = \min B - \max A - 1 \geq |\bar{B}| = k$. Suppose $d_2 = 0$. Then since $f(v_1v_{2k+1}) - \max U - 1 = d_1 \geq k \geq 2$, at least one of $A \cup \bar{B}$ and $B \cup \bar{A}$ is between U and $f(v_1v_{2k+1})$, and hence $d_1 \geq |A| + |\bar{B}| = 2k+1 \geq k+3$. Since $d_1 = \min B - \max A - 1$, this implies U is between $\bar{B}$ and B. Consequently $d_1 = \min B - \max A - 1 \geq |\bar{B}| + |U| \geq 3k$, and $d_1 = f(v_1v_{2k+1}) - \max U - 1 \leq |B \cup \bar{A}| + 1 \leq 2k+2$. This forces $k = 2$ and $A < \bar{B} < U < B < \bar{A} < \{f(v_0)\} < \{f(v_1v_{2k+1})\}$. In particular, $\max B < \min \bar{A}$ and $\min U < f(v_0)$. Therefore $\max B + \min U < \min \bar{A} + f(v_0)$, which contradicts the fact that $\max B + \min U = f(v_{2k}) + f(v_{2k}v_{2k+1}) = f(v_0v_{2k+1}) + f(v_0) = \min \bar{A} + f(v_0)$.

Suppose now that $d_2 = 1$. Since $|U| \geq 4$, this implies $\max A + 1$, $\max B + 1 \notin A \cup \bar{A} \cup B \cup \bar{B} \cup U$, and hence by (1), $\{\max A + 1, \max B + 1\} = \{f(v_0), f(v_1v_{2k+1})\}$. Since $\max A + 1 < \max B + 1$, this implies that $\min A < f(v_0)$ and $f(v_1v_{2k+1}) < \max B + 2 = \min \bar{A}$. Consequently $\min A + f(v_1v_{2k+1}) < f(v_0) + \min \bar{A}$, which contradicts the fact that $\min A + f(v_1v_{2k+1}) = f(v_0) + \min \bar{A}$.

Case 3 $\quad \bar{B} < A < \bar{A} < B$

By (1) and (2), $d_3 = 0$ or 1, and we also have $d_1 \geq k+1$. Suppose $d_3 = 0$. Then at least one of $\bar{B} \cup A$ and $\bar{A} \cup B$ is between U and $f(v_1v_{2k+1})$. On the other hand, since $d_1 \leq |\bar{A}| + |U| + 2 \leq 3k+3$, it is not possible that both $\bar{B} \cup A$ and $\bar{A} \cup B$ are between U and $f(v_1v_{2k+1})$. Therefore $2k+1 = |\bar{B} \cup A| \leq d_1 \leq |\bar{B} \cup A| + 1 \leq 2k+2$, which implies that U cannot be between A and $\bar{A}$, and hence $d_1 \leq |\bar{A}| + 2 \leq k+3$. Consequently $k = 2$ and $U < \bar{B} < A < \{f(v_1v_{2k+1})\} < \{f(v_0)\} < \bar{A} < B$. This implies that $f(v_1v_{2k+1}) < \min \bar{A}$ and $\min A < f(v_0)$, which contradicts the fact that $\min A + f(v_1v_{2k+1}) = \min \bar{A} + f(v_0)$.

Suppose now that $d_3 = 1$. Since $|U| \geq 4$, this implies $\min A - 1$, $\min B - 1 \notin A \cup \bar{A} \cup B \cup \bar{B} \cup U$, and hence by (1), $\{\min A - 1, \min B - 1\} = \{f(v_0), f(v_1v_{2k+1})\}$. Since $\min A < \min \bar{A} < \max \bar{A} = \min B - 2$, this implies that $f(v_0) < \min A < \min \bar{A} < f(v_1v_{2k+1})$ or $f(v_1v_{2k+1}) < \min A < \min \bar{A} < f(v_0)$, which contradicts the fact that $\min A + f(v_1v_{2k+1}) = \min \bar{A} + f(v_0)$.

Case 4 $\quad \bar{B} < \bar{A} < A < B$

By (1) and (2), $d_1 = 0$ or 1. Suppose $d_1 = 0$. Then $f(v_1v_{2k+1}) = \max U + 1$, If $U < \bar{B} < \bar{A} < A < B$, then $f(v_1v_{2k+1}) = 2k+1$ or $2k+2$, and $\min A - \min \bar{A} = k+1$ or

$k+2$; if $\bar{B} < \bar{A} < U < A < B$, then $f(v_1 v_{2k+1}) = 4k+2$ or $4k+3$, and $\min A - \min \bar{A} = 3k+2$ or $3k+3$; if $\bar{B} < \bar{A} < A < B < U$, then $f(v_1 v_{2k+1}) = 6k+3$ or $6k+4$, and $\min A - \min \bar{A} = k+1$ or $k+2$. In all cases, $f(v_0) = 1, 2k+2, 4k+3$ or $6k+4$. From these observations, we see that $\min A - \min \bar{A} \neq f(v_0) - f(v_1 v_{2k+1})$, which contradicts the fact that $\min A + f(v_1 v_{2k+1}) = f(v_0) + \min \bar{A}$.

Suppose now that $d_1 = 1$. Then as in Case 1, $\max A + 1$, $\max \bar{B} + 1$, $\max U + 1$ are distinct and $\max A + 1, \max \bar{B} + 1, \max U + 1 \notin A \cup \bar{A} \cup B \cup \bar{B} \cup U$, which contradicts (1). □

REMARK 2.3. Let $n = 2k+1$ with $k \geq 3$, and suppose that W_n has an edge-magic labeling f of the form

$$f(v_{2m-1}) = \begin{cases} a+m & (1 \leq m \leq l) \\ a'+m & (l+1 \leq m \leq k) \end{cases}$$

$$f(v_{2m}) = b+m \quad (1 \leq m \leq k)\,.$$

Then $k \equiv 0 \pmod 2$ (i.e., $n \equiv 1 \pmod 4$), $l = k/2$, and one of the following holds:

(i) $a = 7l+1, a' = 3l+1$ and $b = 9l+1$;
(ii) $a = 11l+1, a' = 7l+1$ and $b = 5l+1$;
(iii) $a = 3l, a' = -l$ and $b = 5l$;
(iv) $a = 7l, a' = 3l$ and $b = l$;
(v) $a = 11l+1, a' = 71+1$ and $b = l$; or
(vi) $a = 3l, a' = -l$ and $b = 9l+1$.

PROOF. If $1 \leq k \leq 2l-1$, then as in Remark 2.1, we have $f(v_1) + f(v_{2k}) = a + b + k + 1 = f(v_k) + f(v_{k+1})$. Thus $k \geq 2l$. Set $A = \{a+1, \cdots, a+l\}$, $\bar{A} = \{C - (a+l+f(v_0)), \cdots, C - (a+1+f(v_0))\}$, $A' = \{a'+l+1, \cdots, a'+k\}$, $\bar{A}' = \{C - (a'+k+f(v_0)), \cdots, C - (a'+l+1+f(v_0))\}$, $B = \{b+1, \cdots, b+k\}$, $\bar{B} = \{C - (b+k+f(v_0)), \cdots, C - (b+1+f(v_0))\}$, $U = \{C - (a+b+2l), \cdots, C - (a+b+2)\}$, $U' = \{C - (a'+b+2k), \cdots, C - (a'+b+2l+1)\}$, where C is the magic number of f. Thus $|A| = |\bar{A}| = l$, $|A'| = |\bar{A}'| = k-l$, $|B| = |\bar{B}| = k$, $|U| = 2l-1$, $|U'| = 2k-2l$ and

$$A \cup \bar{A} \cup A' \cup \bar{A}' \cup B \cup \bar{B} \cup U \cup U' \cup \{f(v_0), f(v_1 v_{2k})\} = \{1, \cdots, 6k+1\}\,. \tag{3}$$

Note that

$$\min A + \max \bar{A} = \min A' + \max \bar{A}' = \min B + \max \bar{B}\,, \tag{4}$$

and that if $k \geq 2l+1$, then

$$|U| < |U'| - 2\,. \tag{5}$$

We prove four claims.

CLAIM 1. Suppose that $k \geq 2l+1$ and $\{f(v_1 v_{2k})\} < A < U < B < \bar{A}$, and that $\bar{B}$ is outside A and U, i.e., either $\bar{B} < A$ or $U < \bar{B}$. Then $l = 1$, and either $\{f(v_1 v_{2k})\} < A < U < \bar{B} < B < \{f(v_0)\} < \bar{A}$ or $\{f(v_1 v_{2k})\} < A < U < B < \bar{B} < \{f(v_0)\} < \bar{A}$.

PROOF. By (4) and the assumption that $A < B < \bar{A}$, we have $A < \bar{B} < \bar{A}$, and hence $U < \bar{B} < \bar{A}$ by the assumption that $\bar{B}$ is outside A and U. Assume first that $\bar{B} < B$. By (4), either both A' and $\bar{A}'$ are between B and $\bar{B}$ or both A' and $\bar{A}'$ are outside B and $\bar{B}$, and either both A' and $\bar{A}'$ are between A and $\bar{A}$ or both A' and $\bar{A}'$ are outside A and $\bar{A}$. Consequently, if $B < U' < \bar{A}$, then $(\max \bar{A} - \min B) - (\max \bar{B} - \min A) = |U'| - (|U| + 1)$ or $|U'| - |U|$ according as $f(v_0)$ is between A and $\bar{B}$ or not, and hence $\max \bar{A} - \min B > \max \bar{B} - \min A$ by (5), which contradicts (4). Thus U' is outside B and $\bar{A}$, and hence $(\max \bar{B} - \min A) - (\max \bar{A} - \min B) \geq |U| - 1$. Since $\max \bar{B} - \min A = \max \bar{A} - \min B$ by (4), this implies that $l = 1$ and $f(v_0)$ is between B and $\bar{A}$ (and U' is outside A and $\bar{B}$), as desired. If $B < \bar{B}$, we similarly obtain $l = 1$ and $\bar{B} < \{f(v_0)\} < \bar{A}$. □

CLAIM 2. Suppose that $k \geq 2l+1$, $\{f(v_1v_{2k})\} < A < U < \bar{A}$, B is outside $f(v_1v_{2k})$ and $\bar{A}$, and $\bar{B}$ is outside $f(v_1v_{2k})$ and A. Then either $\bar{B} < \{f(v_1v_{2k})\} < A < U < \bar{A} < \{f(v_0)\} < B$ or $B < \{f(v_1v_{2k})\} < A < U < \bar{A} < \{f(v_0)\} < \bar{B}$, and U' is either between A and $\bar{A}$ or outside B and $\bar{B}$.

PROOF. Assume first that $\bar{A} < B$. Then by (4) and by assumption, $\bar{B} < \{f(v_1v_{2k})\}$. If $\bar{A} < U' < B$, then arguing as in Claim 1, we obtain $(\min B - \max \bar{A}) - (\min A - \max \bar{B}) = |U'| - 2$ or $|U'| - 1$ according as $f(v_0)$ is between $\bar{B}$ and A or not, and hence $\min B - \max \bar{A} > \min A - \max \bar{B}$ by (5), which contradicts (4). Thus U' is outside $\bar{A}$ and B. Similarly U' is outside $\bar{B}$ and A. Consequently U' is either between A and $\bar{A}$ or outside B and $\bar{B}$. Since $\min A - \max \bar{B} = \min B - \max \bar{A}$, this implies $\bar{A} < \{f(v_0)\} < B$, as desired. If $B < \{f(v_1v_{2k})\}$, we similarly obtain $\bar{A} < \{f(v_0)\} < \bar{B}$. □

CLAIM 3. Suppose that $k \geq 2l + 1$, $\bar{A} < B < \{f(v_1v_{2k})\} < A < U$, and $\bar{B}$ is outside $f(v_1v_{2k})$ and A. Then $\bar{A} < \{f(v_0)\} < \bar{B} < B < \{f(v_1v_{2k})\} < A < U$ or $\bar{A} < \{f(v_0)\} < B < \bar{B} < \{f(v_1v_{2k})\} < A < U$.

PROOF. By (4) and by assumption, $\bar{A} < \bar{B} < \{f(v_1v_{2k})\}$. Assume first that $\bar{B} < B$. If $\bar{A} < U' < \bar{B}$, then $(\max \bar{B} - \max \bar{A}) - (\min A - \min B) = |U'| - 2$ or $|U'| - 1$, and hence $\max \bar{B} - \max \bar{A} > \min A - \min B$, which contradicts (4). Thus U' is outside $\bar{A}$ and $\bar{B}$. Since $\min A - \min B = \max \bar{B} - \max \bar{A}$ by (4), this implies that $f(v_0)$ is between $\bar{A}$ and $\bar{B}$ (and U' is outside B and A), as desired. If $B < \bar{B}$, we similarly obtain $\bar{A} < \{f(v_0)\} < B$. □

CLAIM 4. Suppose that $k \geq 2l + 1$, $\bar{A} < A < U$, B is outside $\bar{A}$ and U, and $\bar{B}$ is outside A and U. Then $l = 1$, U' is either between $\bar{A}$ and A or outside $\bar{B}$ and B, and one of the following holds:

(i) $\bar{B} < \bar{A} < A < U < B$, one of $f(v_1v_{2k})$ and $f(v_0)$ is between $\bar{B}$ and $\bar{A}$, and the other one is either between $\bar{A}$ and A or outside $\bar{B}$ and B; or

(ii) $B < \bar{A} < A < U < \bar{B}$, one of $f(v_1v_{2k})$ and $f(v_0)$ is between B and $\bar{A}$, and the other one is either between $\bar{A}$ and A or outside B and $\bar{B}$.

PROOF. Assume first that $U < B$. Then by (4), $\bar{B} < \bar{A}$. If $\bar{B} < U' < \bar{A}$, then $(\max \bar{A} - \max \bar{B}) - (\min B - \min A) \geq |U'| - (|U| + 2)$, and hence $\max \bar{A} - \max \bar{B} > \min B - \min A$ by (5), which contradicts (4). Thus U' is outside $\bar{B}$ and $\bar{A}$. Similarly, if $A < U' < B$, then $(\min B - \min A) - (\max \bar{A} - \max \bar{B}) \geq (|U| + |U'|) - 2 > 0$, which contradicts (4). Thus U'

is outside A and B. Consequently, U' is either between $\bar{A}$ and A or outside $\bar{B}$ and B. Now if $l \geq 2$, then $|U| = 2l - 1 \geq 3$, and hence $(\min B - \min A) - (\max \bar{A} - \max \bar{B}) \geq |U| - 2 > 0$, which contradicts (4). Thus $l = 1$, and hence $|U| = 1$. Since $\min B - \min A = \max \bar{A} - \max \bar{B}$ by (4), this implies that one of $f(v_1 v_{2k})$ and $f(v_0)$ is between $\bar{B}$ and $\bar{A}$, and the other one is neither between $\bar{B}$ and $\bar{A}$ nor between A and B. Consequently (i) holds. If $B < \bar{A}$, we similarly see that (ii) holds. □

Now set $T = \{C - (a + b + k), \cdots, C - (a + b + 2l + 1)\}$. If $T \neq \emptyset$, then

$$\min T - 1 = C - (a + 1) - (b + k) = f(v_1 v_{2k}) \tag{6}$$

and

$$\max T + 1 = C - (a + b + 2l) = \min U . \tag{7}$$

From (3), (6) and (7), it follows that T is the union of some of $A, \bar{A}, A', \bar{A}', B, \bar{B}, U'$ and $\{f(v_0)\}$. Since $|T| = k - 2l$ is less than each of $|A'|, |\bar{A}'|, |B|, |\bar{B}|$ and $|U'|$, this means that T is the union of some of A, $\bar{A}$ and $\{f(v_0)\}$. Note that if $T \neq \emptyset$, then $k \geq 2l + 1$. We now divide our proof into seven cases:

Case 1 $T = \{f(v_0)\}$

In this case, $\max U - f(v_0) = |U| = 2l - 1$. We also have $\max \bar{A} - \min B \geq |\bar{A}| + |B| - 1 = k + l - 1$ or $\max \bar{A} - \min B < 0$, according as $B < \bar{A}$ or $\bar{A} < B$. Consequently $(\max U + \min B) - (\max \bar{A} + f(v_0)) = (\max U - f(v_0)) - (\max \bar{A} - \min B) \neq 0$. On the other hand, since f is edge-magic, we have $\max U + \min B = f(v_1 v_2) + f(v_2) = f(v_0 v_1) + f(v_0) = \max \bar{A} + f(v_0)$, which is a contradiction.

Case 2 $T = A$

Since $k - 2l = l$, $k = 3l$. Suppose $A < \bar{A}$. We first show that B is outside A and $\bar{A}$. Suppose that $A < B < \bar{A}$. Then since $\max A + 1 = \max T + 1 = \min U$ by the assumption of Case 2 and (7), $A < U < B < \bar{A}$ and $\bar{B}$ cannot be between A and U. We also have $\{f(v_1 v_{2k})\} < T = A$ by (6). Hence by Claim 1, $f(v_0) > \max B$. On the other hand, since $A < \bar{A}$, $\max \bar{A} > \min A = \min T > f(v_1 v_{2k})$ by (6). Consequently $f(v_0) + \max \bar{A} > \max B + f(v_1 v_{2k})$, which contradicts the fact that $f(v_0) + \max \bar{A} = f(v_0) + f(v_0 v_1) = f(v_{2k}) + f(v_1 v_{2k}) = \max B + f(v_1 v_{2k})$. Thus B is outside A and $\bar{A}$. Hence in view of (6) and (7), we can apply Claim 2 to see that

$$\begin{aligned} &\bar{B} < \{f(v_1 v_{2k})\} < A < U < \bar{A} < \{f(v_0)\} < B \\ \text{or } &B < \{f(v_1 v_{2k})\} < A < U < \bar{A} < \{f(v_0)\} < \bar{B} \end{aligned} \tag{8}$$

and

$$U' \text{ is between } A \text{ and } \bar{A} \text{ or outside } B \text{ and } \bar{B} . \tag{9}$$

Now if A' is between A and $\bar{A}$, then arguing as in Claim 1 with B replaced by A', we obtain $\{f(v_0)\} < \bar{A}$, which contradicts (8). Thus A' is outside A and $\bar{A}$. Hence arguing as in Claim

2, we get

$$\begin{aligned} &\bar{A}' < \{f(v_1v_{2k})\} < A < U < \bar{A} < \{f(v_0)\} < A' \\ \text{or } &A' < \{f(v_1v_{2k})\} < A < U < \bar{A} < \{f(v_0)\} < \bar{A}'. \end{aligned} \tag{10}$$

By (3), (8), (9) and (10), $f(v_0) = \max \bar{A} + 1$, $\min B - f(v_0)$ is either equal to 1 or $|A'| + 1 = k - l + 1$ or less than 0, and $\max \bar{A} - \max U$ is equal to $|\bar{A}| = l$ or $|\bar{A}| + |U'| = 2k - l$. On the other hand, since $\min B + \max U = f(v_2) + f(v_1v_2) = f(v_0v_1) + f(v_0) = \max \bar{A} + f(v_0)$, $\min B - f(v_0) = \max \bar{A} - \max U$. Since $k = 3l$, these imply that $l = 1$, and $U' = \{1, 2, 3, 4\}$ or $\{16, 17, 18, 19\}$. Consequently $f(v_0) - \min U' = -6$ or 13, and $\max A' - \min \bar{B} = 6$, and hence $f(v_0) - \min U' \neq \max A' - \min \bar{B}$, which contradicts the fact that $\max A' + \min U' = f(v_0) + \min \bar{B}$.

Suppose now that $\bar{A} < A$. If $\bar{A} < B < A$, then by Claim 3, $\max \bar{B} < \max U$ and $f(v_0) < \min A$, and hence $\max \bar{B} + f(v_0) < \max U + \min A$, which contradicts the fact that $\max U + \min A = \max \bar{B} + f(v_0)$. Thus B is outside $\bar{A}$ and A, and hence it follows from Claim 4 that $l = 1$. Since $\bar{A} < \{f(v_1v_{2k})\} < A$ by (6), it also follows from Claim 4 that $B < \{f(v_0)\} < \bar{A} < \{f(v_1v_{2k})\} < A < U < \bar{B}$ or $\bar{B} < \{f(v_0)\} < \bar{A} < \{f(v_1v_{2k})\} < A < U < B$. Hence arguing as in the preceding paragraph, we see that A' is also outside $\bar{A}$ and A, $f(v_0) = \min \bar{A} - 1$, $\min A - f(v_0)$ is equal to $|\bar{A} \cup \{f(v_1v_{2k})\}| + 1 = 3$ or $|\bar{A} \cup \{f(v_1v_{2k})\}| + |U'| + 1 = 7$, and $\max \bar{B} - \max U$ is either equal to $|\bar{B}| = 3$ or $|\bar{B}| + |A'| = 5$ or less than 0. On the other hand, since $\min A + \max U = \max \bar{B} + f(v_0)$, $\min A - f(v_0) = \max \bar{B} - \max U$. Consequently $\min A - f(v_0) = \max \bar{B} - \max U = 3$, and $U' = \{1, 2, 3, 4\}$ or $\{16, 17, 18, 19\}$. This implies that $\min U' - \min A = 7$ or -12, and $f(v_1v_{2k}) - \max A' = 6$ or -7, and hence $\min U' - \min A \neq f(v_1v_{2k}) - \max A'$, which contradicts the fact that $f(v_1v_{2k}) + \min A = \min U' + \max A'$.

Case 3 $\quad T = \bar{A}$

Since $k - 2l = l$, $k = 3l$. Suppose $A < \bar{A}$. If $A < B < \bar{A}$, then arguing as in the second paragraph of Case 2, we obtain $f(v_0) < \min B$ and $\max \bar{A} < \max U$, which contradicts the fact that $f(v_0) + \max \bar{A} = \min B + \max U$. Thus B is outside A and $\bar{A}$. Hence arguing again as in the second paragaph of Case 2, we see that $l = 1$, $\min A - f(v_0) = 1$, and $\max \bar{B} - \max U$ is either equal to 3 or 5 or less than or equal to -5. Consequently $\min A - f(v_0) \neq \max \bar{B} - \max U$, which contradicts the fact that $\min A + \max U = f(v_0) + \max \bar{B}$.

Suppose now that $\bar{A} < A$. If $\bar{A} < B < A$, then arguing as in the first paragraph of Case 2, we obtain $f(v_0) > \max B$ and $\max \bar{A} > f(v_1v_{2k})$, which contradicts the fact that $f(v_0) + \max \bar{A} = \max B + f(v_1v_{2k})$. Thus B is outside $\bar{A}$ and A. Hence arguing as in the first paragraph of Case 2, we see that $f(v_0) = \max A + 1$, $f(v_0) - \min A = l$, and $\max U - \max \bar{B} \geq 3l$ or < 0. Consequently $f(v_0) - \min A \neq \max U - \max \bar{B}$, which contradicts the fact that $f(v_0) + \max \bar{B} = \min A + \max U$.

Case 4 $\quad T = A \cup \{f(v_0)\}$ or $T = \bar{A} \cup \{f(v_0)\}$

Suppose first that $T = A \cup \{f(v_0)\}$. Then by (6) and (7), the assumption of one of Claims 1 through 4 is satisfied. On the other hand, it follows from (6) that $A \cup \{f(v_0), f(v_1v_{2k})\}$

consists of consecutive integers, and hence the conclusion of none of Claims 1 through 4 can hold, which is a contradiction. If $T = \bar{A} \cup \{f(v_0)\}$, then arguing as in Claims 1 through 4 with the roles of A and $\bar{A}$ replaced by each other, we can similarly get a contradiction.

Case 5 $T = A \cup \bar{A}$

In this case, B is outside A and $\bar{A}$. Suppose $\bar{A} < A$. Then it follows from Claim 4 that $l = 1$, and $f(v_0)$ is outside B and $\bar{B}$. In view of (6), it also follows from Claim 4 that $\bar{B} < \{f(v_1v_{2k})\} < \bar{A} < A < U < B < \{f(v_0)\}$ or $B < \{f(v_1v_{2k})\} < \bar{A} < A < U < \bar{B} < \{f(v_0)\}$. Consequently, arguing as in Case 2, we see that $\max\bar{A} - f(v_1v_{2k}) = 1$, and $\max B - f(v_0) \leq 0$, which contradicts the fact that $\max\bar{A} + f(v_0) = f(v_1v_{2k}) + \max B$. If $A < \bar{A}$, then similarly $\max\bar{A} - f(v_1v_{2k}) = 2$ and $\max B - f(v_0) \leq 0$, which again contradicts the fact that $\max\bar{A} + f(v_0) = f(v_1v_{2k}) + \max B$.

Case 6 $T = A \cup \bar{A} \cup \{f(v_0)\}$

In this case, B is outside A and $\bar{A}$. Suppose $\bar{A} < A$. Then it follows from Claim 4 that $l = 1$. In view of (6), it also follows from Claim 4 that $\bar{B} < \{f(v_1v_{2k})\} < \bar{A} < \{f(v_0)\} < A < U < B$ or $B < \{f(v_1v_{2k})\} < \bar{A} < \{f(v_0)\} < A < U < \bar{B}$. Consequently, arguing as in Case 2, we see that $\max\bar{A} - f(v_1v_{2k}) = 1$, and $\max B - f(v_0) \geq 5$ or < 0, which contradicts the fact that $\max\bar{A} + f(v_0) = f(v_1v_{2k}) + \max B$. If $A < \bar{A}$, then similarly, $\max\bar{A} - f(v_1v_{2k}) = 3$, and $\max B - f(v_0) \geq 5$ or < 0, which again contradicts the fact that $\max\bar{A} + f(v_0) = f(v_1v_{2k}) + \max B$.

Case 7 $T = \emptyset$

In this case, since $|T| = k - 2l = 0$, we have $k = 2l$, and hence $|A'| = |\bar{A}'| = l$ and $|B'| = |\bar{B}'| = |U'| = 2l$. We also have $\min U - 1 = f(v_1v_{2k})$ and $|\{f(v_1v_{2k})\} \cup U| = 2l$. Hence by (4), either

$$f(v_0) \text{ is between } A \text{ and } \bar{A}, \text{ between } A' \text{ and } \bar{A}', \text{ and between } B \text{ and } \bar{B}, \tag{11}$$

or

$$f(v_0) \text{ is outside } A \text{ and } \bar{A}, \text{ outside } A' \text{ and } \bar{A}', \text{ and outside } B \text{ and } \bar{B}. \tag{12}$$

It is easy to verify the assertion of the remark for $k = 4$. Thus we henceforth assume $k \geq 6$ (so $l \geq 3$).

First we consider the case where (12) holds. If $f(v_0)$ is outside A and U', and outside A' and U, then $\min U - \min A' \equiv 1 \pmod{l}$ and $\max U' - \max A \equiv 0 \pmod{l}$, which contradicts the face that

$$\min U - \min A' = \max U' - \max A. \tag{13}$$

Thus $f(v_0)$ is between A and U', or between A' and U. Assume first that $f(v_0)$ is between A' and U. Suppose $A' < \{f(v_0)\} < U$. Then $\min U - \min A' \equiv 2 \pmod{l}$. On the other hand, $A < U'$ by (13), and hence $\max U' - \max A \equiv 0$ or $1 \pmod{l}$ according as $f(v_0)$ is outside A and U' or between A and U'. Since $l \geq 3$, this contradicts (13). Thus $U < \{f(v_0)\} < A'$. This implies $\min A' - \min U \equiv 0 \pmod{l}$. Also by (13), $U' < A$. If $U' < \{f(v_0)\} < A$, then $\max A - \max U' \equiv 1 \pmod{l}$, which contradicts (13). Thus $f(v_0)$ is outside U' and A, and hence $U < \{f(v_0)\} < A' < U' < A$ or $U < \{f(v_0)\} < U' < A' < A$ by (12) and (13). Suppose $U < \{f(v_0)\} < A' < U' < A$. Then by (12), $f(v_0) - \max U = 1$. Since

$$f(v_0) - \max U = \min A - \max \bar{B}\,, \tag{14}$$

this implies $\min A - \max \bar{B} = 1$. In particular, $U' < \bar{B} < A$, and hence $\max A - \max U' \geq 3l$. In view of (13), this implies $\bar{A}'$ or $\bar{A} \cup B$ is between U and A', i.e., $U < \{f(v_0)\} < \bar{A}' < A'$ or $U < \{f(v_0)\} < \bar{A} < B < A'$. But if $\bar{A}' < A'$, then by (4), $\bar{A} < B < \bar{A}' < A'$, and hence $U < \{f(v_0)\} < \bar{A} < B < A'$. Thus $U < \{f(v_0)\} < \bar{A} < B < A'$. Consequently we obtain $\min A' - \min U \geq 5l$ and $\max A - \max U' \leq |\bar{A}' \cup \bar{B} \cup A| = 4l$, which contradicts (13). Thus $U < \{f(v_0)\} < U' < A' < A$. Again by (12), $f(v_0) - \max U = 1$, and hence $\min A - \max \bar{B} = 1$ by (14). If $\bar{A}'$ or $\bar{A} \cup B$ is between $f(v_0)$ and U', then arguing as above, we obtain $\min A' - \min U \geq 7l$ and $\max A - \max U' \leq 5l$, a contradiction. Thus $\min U' - f(v_0) = 1$. Since

$$\min U' - f(v_0) = \min \bar{B} - \max A'\,, \tag{15}$$

this implies $\min \bar{B} - \max A' = 1$. Consequently $\{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U' < A' < \bar{B} < A < \bar{A} < B < \bar{A}'$ or $\{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U' < \bar{A} < B < \bar{A}' < A' < \bar{B} < A$ by (4) and (12), and hence (i) or (ii) holds.

Assume now that $f(v_0)$ is outside A' and U, and between A and U'. If $U' < \{f(v_0)\} < A$, then $U < A'$ by (13), and hence $\max A - \max U' \equiv 1 \pmod{l}$ and $\min A' - \min U \equiv -1 \pmod{l}$, which contradicts (13). Thus $A < \{f(v_0)\} < U'$. By (13), this implies $A' < U$, and hence $A' < \{f(v_1 v_{2k})\} \cup U < A < \{f(v_0)\} < U'$ or $A' < A < \{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U'$. If $A' < \{f(v_1 v_{2k})\} \cup U < A < \{f(v_0)\} < U'$, then arguing as in the preceding paragraph, we get a contradiction. Thus $A' < A < \{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U'$. By (12), $\min U' - f(v_0) = 1$, and hence $\min \bar{B} - \max A' = 1$ by (15). If $\bar{A}$ or $B \cup \bar{A}'$ is between U and $f(v_0)$, then again arguing as in the preceding paragraph, we get a contradiction. Thus $f(v_0) - \max U = 1$, and hence $f(v_0) - f(v_1 v_{2k}) = 2l$ because $\min U - 1 = f(v_1 v_{2k})$ and $|U| = 2l - 1$. Since

$$f(v_0) - f(v_1 v_{2k}) = \min A - \min \bar{B}\,, \tag{16}$$

this implies $\min A - \min \bar{B} = 2l$. Consequently $A' < \bar{B} < A < \bar{A} < B < \bar{A}' < \{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U'$ or $\bar{A} < B < \bar{A}' < A' < \bar{B} < A < \{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U'$, and hence (iii) or (iv) holds. This concludes the discussion for the case where (12) holds.

Next we consider the case where (11) holds. Arguing as in the case where (12) holds, we see that either

$$U < \{f(v_0)\} < A'\,, \quad U' < A, \text{ and } f(v_0) \text{ is outside } U' \text{ and } A\,, \tag{17}$$

or

$$A < \{f(v_0)\} < U'\,, \quad A' < U, \text{ and } f(v_0) \text{ is outside } A' \text{ and } U\,. \tag{18}$$

Assume first that (17) holds. Then by (13), $U' < \{f(v_1 v_{2k})\} \cup U < A < \{f(v_0)\} < A'$, $U' < A < \{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < A'$, $\{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < A' < U' < A$, or $\{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U' < A' < A$. Suppose $U' < \{f(v_1 v_{2k})\} \cup U < A <$

$\{f(v_0)\} < A'$. Then by (4) and (11), $U < \bar{A}' < \{fv_0)\} < \bar{A}$, one of B and $\bar{B}$ is between U and $f(v_0)$, and the other one is larger than $f(v_0)$. Hence we have $f(v_0) - f(v_1v_{2k}) = 6l$ and $\min A - \min \bar{B} \le 3l$, which contradicts (16). Next suppose $U' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < A'$. Then by (4) and (11), $U' < \bar{A}' < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < \bar{A}$, one of B and $\bar{B}$ is between U' and $\{f(v_1v_{2k})\} \cup U$, and the other one is larger than $f(v_0)$. Hence $f(v_0) - f(v_1v_{2k}) = 2l$ and $f(v_0) - \min U' = 8l$. From $f(v_0) - f(v_1v_{2k}) = 2l$, we get $\min A - \min \bar{B} = 2l$ by (16), and hence $\max \bar{B} + 1 = \min A$. Consequently $\max A' - \min \bar{B} = 9l$ or $7l$, according as $\bar{A}' < \bar{B} < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < \bar{A} < B < A'$ or $\bar{B} < A < \bar{A}' < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < A' < \bar{A} < B$. But in view of (15), this contradicts the earlier assertion that $f(v_0) - \min U' = 8l$. Now suppose $\{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < A' < U' < A$. Then by (4) and (11), $\bar{A} < \{f(v_1v_{2k})\} \cup U < \bar{A}' < \{f(v_0)\} < A' < U' < A$. By (13) and (4), this implies that one of B and $\bar{B}$ is between U' and A, and the other one is between A' and $\{f(v_1v_{2k})\} \cup U$. Hence $f(v_0) - f(v_1v_{2k}) = 3l$ and $\min A - \min \bar{B} = 2l$ or $10l$, which contradicts (16). Thus $\{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U' < A' < A$. By (4) and (11), this implies $\{f(v_1v_{2k})\} \cup U < \bar{A} < \bar{A}' < \{f(v_0)\} < U' < A' < A$ or $\bar{A} < \bar{A}' < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U' < A' < A$. If $\{f(v_1v_{2k})\} \cup U < \bar{A} < \bar{A}' < \{f(v_0)\} < U' < A' < A$, then $\min A' - \min U \ge 6l$ and $\max A - \max U' \le 4l$, which contradicts (13). Thus $\bar{A} < \bar{A}' < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U' < A' < A$. By (13) and (4), this implies that one of B and $\bar{B}$ is between A' and A, and the other one is between $\bar{A}$ and $\bar{A}'$. Consequently $f(v_0) - f(v_1v_{2k}) = 2l$, and hence $\min A - \min \bar{B} = 2l$ by (15). Therefore $\bar{A} < B < \bar{A}' < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U' < A' < \bar{B} < A$, and hence (v) holds.

Assume now that (18) holds. Then by (13), $A < \{f(v_0)\} < U' < A' < \{f(v_1v_{2k})\} \cup U$, $A < \{f(v_0)\} < A' < U' < \{f(v_1v_{2k})\} \cup U$, $A' < \{f(v_1v_{2k})\} \cup U < A < \{f(v_0)\} < U'$, or $A' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U'$. Suppose $A < \{f(v_0)\} < U' < A' < \{f(v_1v_{2k})\} \cup U$. Then by (4) and (11), $A' < \{f(v_0)\} < U' < \bar{A} < \{f(v_1v_{2k})\} \cup U$, one of B and $\bar{B}$ is between U' and $\{f(v_1v_{2k})\} \cup U$, and the other one is less than $f(v_0)$. Hence $f(v_1v_{2k}) - f(v_0) = 6l + 1$, which implies $\min \bar{B} - \min A = 6l + 1$ by (16). Consequently $A < \bar{A}' < B < \{f(v_0)\} < U' < \bar{B} < A' < \bar{A} < \{f(v_1v_{2k})\} \cup U$ or $B < A < \bar{A}' < \{f(v_0)\} < U' < A' < \bar{A} < \bar{B} < \{f(v_1v_{2k})\} \cup U$, which contradicts (13). Next suppose $A < \{f(v_0)\} < A' < U' < \{f(v_1v_{2k})\} \cup U$. Then by (4) and (11), $\bar{A}' < \{f(v_0)\} < \bar{A} < U'$, one of B and $\bar{B}$ is between $f(v_0)$ and U', and the other one is less than $f(v_0)$. Hence $f(v_1v_{2k}) - f(v_0) = 6l + 1$ and $\min \bar{B} - \min A \le 5l + 1$, which contradicts (16). Now suppose $A' < \{f(v_1v_{2k})\} \cup U < A < \{f(v_0)\} < U'$. Then $A' < \{f(v_1v_{2k})\} \cup U < A < \{f(v_0)\} < \bar{A} < U' < \bar{A}'$ by (4) and (11). By (4) and (13), this implies that one of B and $\bar{B}$ is between A' and $\{f(v_1v_{2k})\} \cup U$ and the other one is between U' and $\bar{A}'$, which contradicts (14). Thus $A' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U'$. By (4) and (11), this implies $A' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < \bar{A} < \bar{A}' < U'$ or $A' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U' < \bar{A} < \bar{A}'$. If $A' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < \bar{A} < \bar{A}' < U'$, then $\max U' - \max A \ge 6l + 1$ and $\min U - \min A' \le 4l + 1$, which contradicts (13). Thus $A' < A < \{f(v_1v_{2k})\} \cup U < \{f(v_0)\} < U' < \bar{A} < \bar{A}'$. By (13) and (4), this implies that one of B and $\bar{B}$ is between A' and A, and the other one is between $\bar{A}$ and $\bar{A}'$.

Consequently $f(v_0) - f(v_1 v_{2k}) = 2l$, and hence $\min A - \min \bar{B} = 2l$ by (15). Therefore $A' < \bar{B} < A < \{f(v_1 v_{2k})\} \cup U < \{f(v_0)\} < U' < \bar{A} < B < \bar{A}'$, and hence (vi) holds (note that the above argument shows that if one of (i) through (vi) holds, then the labeling under consideration is in fact edge-magic; in particular, (vi) yields the labeling described in Case 1 of Section 3).

3. Proof of theorem.

We give a constructive proof of the theorem. As in Section 2, write $V(W_n) = \{v_0, v_1, \cdots, v_{n-1}\}$ so that $E(W_n) = \{v_i v_{i+1} \mid 1 \le i \le n-2\} \cup \{v_{n-1} v_1\} \cup \{v_0 v_i \mid 1 \le i \le n-1\}$. The proof is divided into five cases as follows.

$$\begin{aligned}
&\text{Case } 1: && n \equiv 1 \pmod 4,\\
&\text{Case } 2: && n \equiv -2 \pmod 8,\\
&\text{Case } 3: && n \equiv 2 \pmod 8,\\
&\text{Case } 4: && n \equiv -1 \pmod 8,\\
&\text{Case } 5: && n \equiv 3 \pmod 8.
\end{aligned}$$

Case 1 $n \equiv 1 \pmod 4$
Write $n = 4k + 1$ $(k \ge 1)$. Thus $|V(W_n)| + |E(W_n)| = 12k + 1$. Define f by

$$\begin{aligned}
f(v_0) &= 6k + 1\\
f(v_{2m-1}) &= \begin{cases} 3k + m & (1 \le m \le k)\\ -k + m & (k + 1 \le m \le 2k) \end{cases}\\
f(v_{2m}) &= 9k + 1 + m \quad (1 \le m \le 2k)\\
f(v_0 v_{2m-1}) &= \begin{cases} 9k + 2 - m & (1 \le m \le k)\\ 13k + 2 - m & (k + 1 \le m \le 2k) \end{cases}\\
f(v_0 v_{2m}) &= 3k + 1 - m \quad (1 \le m \le 2k)\\
f(v_{2m-1} v_{2m}) &= \begin{cases} 6k + 2 - 2m & (1 \le m \le k)\\ 10k + 2 - 2m & (k + 1 \le m \le 2k) \end{cases}\\
f(v_{2m} v_{2m+1}) &= \begin{cases} 6k + 1 - 2m & (1 \le m \le k - 1)\\ 10k + 1 - 2m & (k \le m \le 2k - 1) \end{cases}\\
f(v_{4k} v_1) &= 4k + 1\,.
\end{aligned}$$

Then f is an edge-magic labeling of W_n with magic number $18k + 3$ (see the parenthetic remark made at the end of the proof of Remark 2.3).

Case 2 $n \equiv -2 \pmod 8$
Write $n = 8k - 2$ $(k \ge 1)$. Thus $|V(W_n)| + |E(W_n)| = 24k - 8$. Define

$$f(v_0) = 10k - 3$$

$$f(v_{2m-1}) = \begin{cases} 8k-2-m & (1 \le m \le k) \\ 18k-5-m & (k+1 \le m \le 3k-1) \\ 4k-m & (3k \le m \le 4k-1) \end{cases}$$

$$f(v_{2m}) = \begin{cases} 2k-m & (1 \le m \le k-1) \\ 4k-1-m & (k \le m \le 2k-1) \\ 8k-2-m & (2k \le m \le 3k-1) \\ 10k-3-m & (3k \le m \le 4k-2)\,, \end{cases}$$

and let $f(uv) = 30k - 9 - f(u) - f(v)$ for each $uv \in E(W_n)$. Then f is an edge-magic labeling of W_n with magic number $30k - 9$.

Case 3 $n \equiv 2 \pmod 8$
Write $n = 8k + 2$ $(k \ge 1)$. Thus $|V(W_n)| + |E(W_n)| = 24k + 4$. Define

$$f(v_0) = 10k + 2$$

$$f(v_{2m-1}) = \begin{cases} 8k+2-m & (1 \le m \le k) \\ 6k+2-m & (k+1 \le m \le 3k+1) \\ 4k+2-m & (3k+2 \le m \le 4k+1) \end{cases}$$

$$f(v_{2m}) = \begin{cases} 2k+1-m & (1 \le m \le k) \\ 16k+3-m & (k+1 \le m \le 2k) \\ 20k+4-m & (2k+1 \le m \le 3k) \\ 10k+2-m & (3k+1 \le m \le 4k)\,, \end{cases}$$

and let $f(uv) = 30k + 6 - f(u) - f(v)$ for each $uv \in E(W_n)$. Then f is an edge-magic labeling of W_n with magic number $30k + 6$.

Case 4 $n \equiv -1 \pmod 8$
Write $n = 8k - 1$ $(k \ge 1)$. Thus $|V(W_n)| + |E(W_n)| = 24k - 5$. Define

$$f(v_0) = 8k$$

$$f(v_{2m-1}) = \begin{cases} m & (1 \le m \le 3k-1) \\ m+1 & (3k \le m \le 4k-1) \end{cases}$$

$$f(v_{2m}) = \begin{cases} 16k-2+m & (1 \le m \le 2k-1) \\ 16k-1+m & (2k \le m \le 3k-2)\,, \quad k \ge 2 \\ 21k-4 & (m = 3k-1) \\ 16k-2+m & (3k \le m \le 4k-3)\,, \quad k \ge 3 \\ 18k-2 & (m = 4k-2)\,, \quad k \ge 2 \\ 16k-2 & (m = 4k-1)\,, \end{cases}$$

and let $f(uv) = 32k - 4 - f(u) - f(v)$ for each $uv \in E(W_n)$. Then f is an edge-magic labeling of W_n with magic number $32k - 4$.

Case 5 $n \equiv 3 \pmod 8$
Write $n = 8k + 3$ $(k \ge 1)$. Thus $|V(W_n)| + |E(W_n)| = 24k + 7$. Define

$$f(v_0) = 8k + 6\,.$$

In the case where $k = 1$, define

$$f(v_1) = 23\,,\quad f(v_2) = 2\,,\quad f(v_3) = 26\,,\quad f(v_4) = 3\,,\quad f(v_5) = 31\,,$$
$$f(v_6) = 4\,,\quad f(v_7) = 22\,,\quad f(v_8) = 5\,,\quad f(v_9) = 25\,,\quad f(v_{10}) = 8\,;$$

in the case where $k = 2$, define

$$f(v_1) = 38\,,\quad f(v_2) = 6\,,\quad f(v_3) = 39\,,\quad f(v_4) = 9\,,\quad f(v_5) = 40\,,$$
$$f(v_6) = 2\,,\quad f(v_7) = 41\,,\quad f(v_8) = 5\,,\quad f(v_9) = 42\,,\quad f(v_{10}) = 13\,,$$
$$f(v_{11}) = 44\,,\quad f(v_{12}) = 7\,,\quad f(v_{13}) = 45\,,\quad f(v_{14}) = 8\,,\quad f(v_{15}) = 46\,,$$
$$f(v_{16}) = 4\,,\quad f(v_{17}) = 55\,,\quad f(v_{18}) = 3\,.$$

Then f can be extended to an edge-magic labeling with magic number 46 and 78, respectively.

We henceforth assume $k \ge 3$. Define

$$f(v_{2m-1}) = \begin{cases} 16k+5+m & (1 \le m \le 3k-1) \\ 16k+6+m & (3k \le m \le 4k) \\ 24k+7 & (m = 4k+1)\,. \end{cases}$$

For labelings of v_{2m} $(1 \leq m \leq 4k+1)$, we consider three subcases.

Subcase 5.1 $k \equiv 0 \pmod 3$

Set $k = 3l$ $(l \geq 1)$. Thus $n = 24l + 3$, $|V(W_n)| + |E(W_n)| = 72l + 7$. Define

$$f(v_{2m}) = \begin{cases} m+5 & (1 \leq m \leq 6l \text{ and } m \not\equiv 0 \pmod 3) \\ m-1 & (1 \leq m \leq 6l \text{ and } m \equiv 0 \pmod 3) \\ 12l+1 & (m = 6l+1) \\ m+4 & (6l+2 \leq m \leq 9l-2 \text{ and } m \not\equiv 1 \pmod 3)\,, \quad l \geq 2 \\ m-2 & (6l+2 \leq m \leq 9l-2 \text{ and } m \equiv 1 \pmod 3)\,, \quad l \geq 2 \\ 15l+3 & (m = 9l-1) \\ 9l-1 & (m = 9l) \\ m+1 & (9l+1 \leq m \leq 12l-1) \\ 4 & (m = 12l) \\ 3 & (m = 12l+1)\,. \end{cases}$$

Subcase 5.2 $k \equiv 1 \pmod 3$

Set $k = 3l + 1$ $(l \geq 1)$. Thus $n = 24l + 11$, $|V(W_n)| + |E(W_n)| = 72l + 31$. Define

$$f(v_{2m}) = \begin{cases} m+5 & (1 \leq m \leq 6l \text{ and } m \not\equiv 0 \pmod 3) \\ m-1 & (1 \leq m \leq 6l+3 \text{ and } m \equiv 0 \pmod 3) \\ 12l+5 & (m = 6l+1) \\ 6l+5 & (m = 6l+2) \\ m+2 & (6l+4 \leq m \leq 9l+1) \\ 15l+8 & (m = 9l+2) \\ m+1 & (9l+3 \leq m \leq 12l+3) \\ 4 & (m = 12l+4) \\ 3 & (m = 12l+5)\,. \end{cases}$$

Subcase 5.3 $k \equiv 2 \pmod 3$

Set $k = 3l + 2$ $(l \geq 1)$. Thus $n = 24l + 19$, $|V(W_n)| + E(W_n)| = 72l + 55$. Define

$$f(v_{2m}) = \begin{cases} 6 & (m = 1) \\ m + 7 & (2 \leq m \leq 6l + 4 \text{ and } m \equiv 2 \pmod 3) \\ 2 & (m = 3) \\ m + 1 & (4 \leq m \leq 6l + 4 \text{ and } m \not\equiv 2 \pmod 3) \\ 12l + 9 & (m = 6l + 5) \\ m + 6 & (6l + 6 \leq m \leq 9l + 2 \text{ and } m \equiv 0 \pmod 3)\,, \quad l \geq 2 \\ m & (6l + 6 \leq m \leq 9l + 4 \text{ and } m \not\equiv 0 \pmod 3) \\ 9l + 7 & (m = 9l + 3) \\ 15l + 13 & (m = 9l + 5) \\ 9l + 5 & (m = 9l + 6) \\ m + 1 & (9l + 7 \leq m \leq 12l + 7) \\ 4 & (m = 12l + 8) \\ 3 & (m = 12l + 9)\,. \end{cases}$$

Then in all of the three subcases, f can be extended to an edge-magic labeling of W_n with magic number $32k + 14$. □

References

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