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Gap Invariance of a Symmetric Invariant Lamination

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Abstract. We shall show that any symmetric, forward and backward invariant lamination under $z \mapsto z^d$ is gap invariant in the sense of W. P. Thurston. As a corollary, we have the gap invariance of a quadratic invariant lamination.

1. Introduction.

Let $\bar{\mathbf{D}}$ (resp. $\mathbf{D}$) be the closed (resp. open) unit disk in the complex plane $\mathbf{C}$. For each subset A of $\bar{\mathbf{D}}$, denote the convex hull of A by $co\,A$, and let $-A = \{z \in \bar{\mathbf{D}} \mid -z \in A\}$. We call a subset S of $\bar{\mathbf{D}}$ a chord if $S = co\{\zeta, \eta\}$ for some ζ and η in $\partial\mathbf{D}$ (we often write $S = \overline{\zeta\eta}$ if $\zeta \neq \eta$), and S a degenerate chord if $\zeta = \eta$. When $S = \overline{\zeta\eta}$, let $ex S = \{\zeta, \eta\}$. Following Thurston ([4] and [1]), a lamination $\mathcal{L}$ is a family of chords (an element of $\mathcal{L}$ is called a leaf) such that $\cup\mathcal{L}$ is closed in $\mathbf{C}$, and $\mathcal{L}$ is non-crossing i.e. no two distinct leaves intersect in $\mathbf{D}$. A *gap* of $\mathcal{L}$ is the closure of a component of $\bar{\mathbf{D}} \setminus (\cup\mathcal{L})$.

For each positive integer d with $d \geq 2$, define a mapping $p_d : \partial\mathbf{D} \to \partial\mathbf{D}$ by $p_d(\zeta) = \zeta^d$. For a non-degenerate chord S, define $P_d S = co\, p_d(ex S)$. A lamination $\mathcal{L}$ is said to be forward invariant under p_d if for any $S \in \mathcal{L}$, $P_d S \in \mathcal{L}$ or $P_d S$ is degenerate. Furthermore $\mathcal{L}$ is said to be backward invariant if for any $S = \overline{pq} \in \mathcal{L}$, there is a collection of d disjoint chords in $\mathcal{L}$, each joining an inverse image of p to an inverse image of q. Finally $\mathcal{L}$ is said to be gap invariant if $co\, p_d(G \cap \partial\mathbf{D})$ is a gap of $\mathcal{L}$, a leaf, or degenerate for any gap G of $\mathcal{L}$.

Consider the case $d = 2$. Write $h = p_2$. In this case, for a forward invariant lamination $\mathcal{L}$, its backward invariance is equivalent to the following property: for each nondegenerate leaf S, $-S \in \mathcal{L}$ and there is $R \in \mathcal{L}$ such that $hR = S$. Indeed suppose $\mathcal{L}$ is backward invariant. It suffices to show that $-S \in \mathcal{L}$ for each non-degenerate $S \in \mathcal{L}$. When S is a diameter, it is clear. Suppose S is not a diameter. Since hS is non-degenerate, $hS \in \mathcal{L}$ by the forward invariance. By the backward invariance, $-S \in \mathcal{L}$. The converse is clear.

C. Bandt and K. Keller constructed a forward and backward invariant lamination under h and states, in Theorem 5.2 ([1]), that it is gap invariant. In this paper, we shall give the proof of the gap invariance theorem for *symmetric* and (forward and backward) invariant lamination

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under p_d where $d \geq 2$ (see Section 3), and indicate some systematic approach to study gaps of a lamination. To end the introduction, we mention that a result of K. M. Pilgrim ([2, p. 1324]) states that there is a lamination which fails to be gap invariant under h.

2. Boundary chords of a gap and 1-sided families.

For each $x \in \mathbf{C}$ and $\varepsilon > 0$, denote $B_\varepsilon(x) = \{y \in \mathbf{C} \mid |y-x| < \varepsilon\}$. For each subset A of $\mathbf{C}$, denote by $cl\ A$ (resp. $int\ A$) the closure (resp. interior) of A in $\mathbf{C}$ and for each $B \subset A$, denote by $int_A B$ (resp. $\partial_A B$) the relative interior (resp. relative boundary) of B in A (notice that the relative closure of B in A is $A \cap cl\ B$). Since $\bar{\mathbf{D}}$ is closed in $\mathbf{C}$, we have $cl\ B = int_{\bar{\mathbf{D}}} B \cup \partial_{\bar{\mathbf{D}}} B = B \cup \partial_{\bar{\mathbf{D}}} B$ for each $B \subset \bar{\mathbf{D}}$. For each $B \subset \partial\mathbf{D}$, we write $\widetilde{int}\, B = int_{\partial\mathbf{D}} B$ and $\tilde{\partial} B = \partial_{\partial\mathbf{D}} B$. Similarly since $\partial\mathbf{D}$ is closed in $\mathbf{C}$, we have $cl\ B = \widetilde{int}\, B \cup \tilde{\partial} B = B \cup \tilde{\partial} B$ for each $B \subset \partial\mathbf{D}$. Denote $\mathcal{S} = \{S \subset \bar{\mathbf{D}} \mid S \text{ is a chord}\}$ and $\mathcal{S}_+ = \{S \in \mathcal{S} \mid S \text{ is non-degenerate}\}$. For $S \in \mathcal{S}_+$, denote $S^\circ = S \setminus ex\, S$, and for each connected subset V of $\bar{\mathbf{D}} \setminus S$, the connected component of $\bar{\mathbf{D}} \setminus S$ containing V by $D_V\langle S\rangle$, in particular when $V = \{x\}$, write $D_x\langle S\rangle = D_V\langle S\rangle$. We often use the following fact. Suppose that $l \subset \bar{\mathbf{D}}$ is a non-degenerate line segment (for example, $l = co\{z, w\}$ for some $z, w \in \bar{\mathbf{D}}$ with $z \neq w$). Let $A \subset \bar{\mathbf{D}}$. If $l \cap A \neq \emptyset$ and $l \setminus A \neq \emptyset$, then $l \cap \partial_{\bar{\mathbf{D}}} A \neq \emptyset$ (indeed notice that l is connected).

We also use the following general lemmas (see Appendix). Let X be a topological space.

LEMMA 1. *Let D be a connected and open subset of X and E is a non-empty subset of D. If $\partial E \subset \partial D$, then $E = D$.*

LEMMA 2. *Let $E \subset X$. Then $\partial E \supset \partial\, cl\, E$. Furthermore $\partial E \subset \partial\, cl\, E$ if and only if $E \supset int\, cl\, E$.*

For each $E \subset X$, denote by $Comp(E)$ the family of connected components of E. X is said to be locally connected if for any open subset U of X and any $C \in Comp(U)$, C is open in X. Hence $\bar{\mathbf{D}}$ and $\partial\mathbf{D}$ is locally connected with respect to their relative topology.

LEMMA 3. *Suppose that X is locally connected. Let F be a closed subset of X. Then for any $C \in Comp(X \setminus F)$, $\partial C \subset F$. Furthermore $\partial F = cl(\bigcup_{C \in Comp(X\setminus F)} \partial C)$.*

For a subfamily $\mathcal{M}$ of $\mathcal{S}$, define $\mathcal{M}_+ = \mathcal{M} \cap \mathcal{S}_+$, $\mathcal{M}^\circ = \{S^\circ \mid S \in \mathcal{M}_+\}$ and $ex\mathcal{M} = \partial\mathbf{D} \cap (\cup\mathcal{M}_+)$. Then $(\cup\mathcal{M}) \cap \mathbf{D} = \cup\mathcal{M}^\circ$ and $ex\mathcal{M} = \{x \in \partial\mathbf{D} \mid x \in S \text{ for some } S \in \mathcal{M}_+\}$.

DEFINITION 1 (Non-crossing family of chords and Lamination). Suppose a subfamily $\mathcal{L}$ of $\mathcal{S}$ satisfies that $\mathcal{L}_+ \neq \emptyset$ and $\bar{\mathbf{D}} \setminus cl(\cup\mathcal{L}) \neq \emptyset$ (hence $\mathbf{D} \setminus \cup\mathcal{L} \neq \emptyset$). We say $\mathcal{L}$ is non-crossing if for any $R, S \in \mathcal{L}_+$, $R = S$ or $S^\circ \cap R = \emptyset$, or equivalently if for any (or some) $x \in \mathbf{D} \setminus \cup\mathcal{L}$ and any $R \neq S \in \mathcal{L}_+$, $S^\circ \subset D_x\langle R\rangle$ or $S^\circ \cap cl\ D_x\langle R\rangle = \emptyset$. Let $\mathcal{L}$ be non-crossing. For each $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$, define $\mathcal{L}_V = \{S \in \mathcal{S}_+ \mid S \subset \partial_{\bar{\mathbf{D}}} V\}$. We say $\mathcal{L}$ is a lamination if $\mathcal{L}$ is non-crossing and $\cup\mathcal{L}$ is closed in $\mathbf{C}$.

REMARK 1. *Let $\mathcal{L}$ be a lamination and $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$. Then V is relatively open in $\bar{\mathbf{D}}$, $\partial_{\bar{\mathbf{D}}} V \subset \cup\mathcal{L}$, $\mathbf{D} \cap \partial_{\bar{\mathbf{D}}} V = \cup\mathcal{L}_V^\circ$, and furthermore $\sharp\mathcal{L}_V \geq 1$.*

PROOF. By Lemma 3, V is relatively open in $\bar{\mathbf{D}}$ and $\partial_{\bar{\mathbf{D}}} V \subset \cup\mathcal{L}$ because $\bar{\mathbf{D}}$ is locally connected and $\cup\mathcal{L}$ is relatively closed in $\bar{\mathbf{D}}$. Hence $\mathbf{D} \cap \partial_{\bar{\mathbf{D}}} V = \cup\mathcal{L}_V^\circ$. (Indeed $\cup\mathcal{L}_V^\circ = \mathbf{D} \cap (\cup\mathcal{L}_V) \subset \mathbf{D} \cap \partial_{\bar{\mathbf{D}}} V$. Conversely let $x \in \mathbf{D} \cap \partial_{\bar{\mathbf{D}}} V$. Since $x \in \mathbf{D} \cap (\cup\mathcal{L})$, there is $S \in \mathcal{L}$ such that $x \in S^\circ$. Assume $S \notin \mathcal{L}_V$. Let $y \in S \setminus \partial_{\bar{\mathbf{D}}} V$. So $y \notin V \cup \partial_{\bar{\mathbf{D}}} V = cl\, V$. Pick a point $z \in V$ (maybe $z \in \partial\mathbf{D}$). Then $l = (co\{y, z\}) \setminus \{y, z\} \subset \mathbf{D} \cap D_z\langle S\rangle$. Since $y \notin cl\, V$, $l \setminus V \neq \emptyset$. Since V is relatively open in $\bar{\mathbf{D}}$, $l \cap V \neq \emptyset$. Hence there is $w \in l \cap \partial_{\bar{\mathbf{D}}} V$. Since $w \in \mathbf{D} \cap (\cup\mathcal{L})$, there is $R \in \mathcal{L}$ such that $w \in R^\circ$. Since $w \in D_z\langle S\rangle$ and $\mathcal{L}$ is non-crossing, we have $S^\circ \cap cl\, V \subset S^\circ \cap cl\, D_z\langle R\rangle = \emptyset$, contradicting the fact that $x \in S^\circ \cap cl\, V$.)

Notice that $V \cap \mathbf{D}$ is relatively open, that is, $int_{\bar{\mathbf{D}}}(V \cap \mathbf{D}) = V \cap \mathbf{D}$. So $\partial_{\bar{\mathbf{D}}}(V \cap \mathbf{D}) = cl(V \cap \mathbf{D}) \setminus (V \cap \mathbf{D}) \subset (cl\, V \setminus V) \cup (cl\, V \setminus \mathbf{D}) \subset \partial_{\bar{\mathbf{D}}} V \cup \partial\mathbf{D}$. Assume $\mathcal{L}_V = \emptyset$. Then $\partial_{\bar{\mathbf{D}}} V \subset \partial\mathbf{D}$, hence $\partial_{\bar{\mathbf{D}}}(V \cap \mathbf{D}) \subset \partial\mathbf{D} = \partial_{\bar{\mathbf{D}}}\mathbf{D}$. By Lemma 1, we have $V \cap \mathbf{D} = \mathbf{D}$, contradicting the assumption $\mathcal{L}_+ \neq \emptyset$. □

Let $\mathcal{L}$ be a lamination. A subset G of $\bar{\mathbf{D}}$ is called a *gap* of $\mathcal{L}$ if $G = cl\, V$ for some $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$. We shall consider a representation for $G = cl\, V$ via a family $\mathcal{L}_V$. At first notice that $\mathcal{L}_V \subset \mathcal{L}_+$ and since V is relatively open in $\bar{\mathbf{D}}$,

$$V \subset int_{\bar{\mathbf{D}}}\left(\bigcap_{S \in \mathcal{L}_V} D_V\langle S\rangle \right).$$

Furthermore if $\sharp\mathcal{L}_V \geq 2$, then $\mathcal{L}_V$ is 1-sided in the following sense.

DEFINITION 2 (1-sided family of chords). Let $\mathcal{N}$ be a non-crossing subfamily of $\mathcal{S}_+$ with $\sharp\mathcal{N} \geq 2$. We say that $\mathcal{N}$ is 1-sided if $\cup\mathcal{N} \subset cl\, D_{R^\circ}\langle S\rangle$ for any two distinct chords R and S in $\mathcal{N}$.

In order to get representation for gaps of a lamination, we shall study the property of a 1-sided family. Let $\mathcal{N}$ be 1-sided. Then for each $S \in \mathcal{N}$, the set $D_{R^\circ}\langle S\rangle$ is independent of the choice of $R \in \mathcal{N} \setminus \{S\}$ and so denote it by $D_{\mathcal{N}}(S)$. Furthermore the cardinality of $\mathcal{N}$ is at most countable. Indeed the first statement holds by the definition of 1-sidedness. For each $S, R \in \mathcal{N}$ with $S \neq R$, we have that $(\bar{\mathbf{D}} \setminus cl\, D_{\mathcal{N}}(S)) \cap (\bar{\mathbf{D}} \setminus cl\, D_{\mathcal{N}}(R)) = \emptyset$. For each $S \in \mathcal{N}$, $\bar{\mathbf{D}} \setminus cl\, D_{\mathcal{N}}(S)$ is relatively open in $\bar{\mathbf{D}}$. Since $\bar{\mathbf{D}}$ is separable, the second statement holds. We define the center of a 1-sided family $\mathcal{N}$.

DEFINITION 3 (Center of a 1-sided family). Let $\mathcal{N}$ be a 1-sided subfamily of $\mathcal{S}_+$. Define $C_{\mathcal{N}} = int_{\bar{\mathbf{D}}}(\bigcap_{S \in \mathcal{N}} D_{\mathcal{N}}(S))$.

If $\mathcal{L}$ is a lamination, then $V \subset C_{\mathcal{L}_V}$ for each $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$ with $\sharp\mathcal{L}_V \geq 2$, and $D_{\mathcal{L}_V}(S) = D_V\langle S\rangle$ for any $S \in \mathcal{L}_V$. At the end of this section, we shall show that $G = cl\, V = cl\, C_{\mathcal{L}_V}$ (Lemma 5) and that if $\partial\mathbf{D} = cl\, ex\mathcal{L}$, then $\sharp\mathcal{L}_V \geq 2$ for each $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$ (Corollary 1).

REMARK 2. *Let $\mathcal{N}$ be 1-sided. Then $C_{\mathcal{N}} \neq \emptyset$ and $cl(\cup\mathcal{N}) \subset cl\, C_{\mathcal{N}} \subset \bigcap_{S \in \mathcal{N}} cl\, D_{\mathcal{N}}(S)$.*

PROOF. Let S_1, S_2 be distinct chords in $\mathcal{N}$. Since $\mathcal{N}$ is 1-sided, $(int\, co(S_1 \cup S_2)) \cap (\cup\mathcal{N}) = \emptyset$. Furthermore $int\, co(S_1 \cup S_2) \subset \bigcap_{S\in\mathcal{N}} D_{\mathcal{N}}(S)$. Hence $\emptyset \neq int\, co(S_1 \cup S_2) \subset C_{\mathcal{N}}$ and $S_1 \cup S_2 \subset cl\, C_{\mathcal{N}}$. □

Denote $\mathcal{F} = \{F \mid F$ is a closed disconnected subset of $\partial\mathbf{D}$ with $\sharp F \geq 3\}$ and $\Gamma = \{\mathcal{N} \mid \mathcal{N}$ is a 1-sided family$\}$. We can construct a 1 to 1 correspondence between Γ and $\mathcal{F}$. Indeed if $F \in \mathcal{F}$, then the family

$$\mathcal{N}_F = \{co\, \tilde{\partial}\gamma \mid \gamma \in Comp(\partial\mathbf{D} \setminus F)\},$$

is 1-sided (notice that γ is an open arc for each $\gamma \in Comp(\partial\mathbf{D} \setminus F)$). Conversely suppose $\mathcal{N}$ is 1-sided. Define

$$F_{\mathcal{N}} = \bigcap_{S\in\mathcal{N}} (\partial\mathbf{D} \cap cl\, D_{\mathcal{N}}(S)).$$

Then $Comp(\partial\mathbf{D} \setminus F_{\mathcal{N}}) = \{\partial\mathbf{D} \setminus cl\, D_{\mathcal{N}}(S)\}_{S\in\mathcal{N}}$ because $(\bar{\mathbf{D}} \setminus cl\, D_{\mathcal{N}}(S)) \cap (\bar{\mathbf{D}} \setminus cl\, D_{\mathcal{N}}(R)) = \emptyset$ for each $S, R \in \mathcal{N}$ with $S \neq R$. So $F_{\mathcal{N}}$ is closed disconnected with $\sharp F_{\mathcal{N}} \geq 3$. Since $\tilde{\partial}(\partial\mathbf{D} \cap cl\, D_{\mathcal{N}}(S)) = ex S$, we have $ex\mathcal{N} \subset F_{\mathcal{N}}$, and

$$\text{if } F = F_{\mathcal{N}}, \text{ then } \mathcal{N} = \mathcal{N}_F.$$

Notice that $\tilde{\partial} F_{\mathcal{N}} = cl\, ex\mathcal{N}$ by the local connectivity of $\partial\mathbf{D}$ and Lemma 3. Conversely let $F \in \mathcal{F}$. We have that

$$\text{if } \mathcal{N} = \mathcal{N}_F, \text{ then } F = F_{\mathcal{N}}$$

because $\partial\mathbf{D} \setminus F = \bigcup_{S\in\mathcal{N}} \partial\mathbf{D} \setminus cl\, D_{\mathcal{N}}(S) = \cup Comp(\partial\mathbf{D} \setminus F_{\mathcal{N}})$. Thus we have a 1 to 1 correspondence $\mathcal{N} \leftrightarrow F_{\mathcal{N}}$. We shall show that if $\partial\mathbf{D} = cl\, ex\mathcal{L}$, then $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D}\cap cl\, V}$ for each $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$ (Corollary 1). So we can reconstruct the family $\mathcal{L}_V$ from $\partial\mathbf{D} \cap cl\, V$ by the above method. This equality $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D}\cap cl\, V}$ is the key to prove gap invariance theorem (Section 3).

We use the following simple fact. Let $\varepsilon > 0$, $x \in \partial\mathbf{D}$ and $S \in \mathcal{S}_+$. If $S \cap co(\partial\mathbf{D} \cap B_\varepsilon(x)) \neq \emptyset$, then $ex S \cap (\partial\mathbf{D} \cap B_\varepsilon(x)) \neq \emptyset$ (indeed if $ex S \subset \partial\mathbf{D} \setminus B_\varepsilon(x)$, then $S = co\, ex S \subset co(\partial\mathbf{D} \setminus B_\varepsilon(x)) \subset \bar{\mathbf{D}} \setminus co(\partial\mathbf{D} \cap B_\varepsilon(x))$). We also use Caratheodory's theorem (Theorem 17.1 in [3]): For any $A \subset \mathbf{C}$ and any $x \in co\, A$, $x \in co\{x_1, x_2, x_3\}$ for some $x_1, x_2.x_3 \in A$. We show that $F_{\mathcal{N}} = \partial\mathbf{D} \cap cl\, C_{\mathcal{N}}$, $\widetilde{int}\, F_{\mathcal{N}} = \partial\mathbf{D} \cap C_{\mathcal{N}}$, $\tilde{\partial} F_{\mathcal{N}} = \partial\mathbf{D} \cap \partial_{\bar{\mathbf{D}}} C_{\mathcal{N}}$ and $cl\, C_{\mathcal{N}} = co\, F_{\mathcal{N}}$. More precisely,

LEMMA 4. *Let $\mathcal{N}$ be 1-sided. Then*

a) $F_{\mathcal{N}} = \partial\mathbf{D} \cap cl\, C_{\mathcal{N}}$.
b) $\widetilde{int}\, F_{\mathcal{N}} = \partial\mathbf{D} \cap C_{\mathcal{N}}$.
c) $cl\, C_{\mathcal{N}} = co\, F_{\mathcal{N}}$.
d) $\partial_{\bar{\mathbf{D}}} C_{\mathcal{N}} = \tilde{\partial} F_{\mathcal{N}} \cup (\cup\mathcal{N}^\circ)$.
e) $C_{\mathcal{N}} = int_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}}$ *(or equivalently,* $\partial_{\bar{\mathbf{D}}} C_{\mathcal{N}} = \partial_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}}$*).*
f) $C_{\mathcal{N}} \in Comp(\bar{\mathbf{D}} \setminus cl(\cup\mathcal{N}))$.

PROOF. Firstly we show the statements a) and b). By Remark 2, $F_{\mathcal{N}} \supset \partial\mathbf{D} \cap cl\, C_{\mathcal{N}}$. Since $\cup\mathcal{N} \subset cl\, C_{\mathcal{N}}$ (by Remark 2), we have $\tilde{\partial} F_{\mathcal{N}} = cl\, ex\mathcal{N} \subset \partial\mathbf{D} \cap cl\, C_{\mathcal{N}}$. We show

$\widetilde{int}\, F_{\mathcal{N}} = \partial\mathbf{D} \cap C_{\mathcal{N}}$ (then $F_{\mathcal{N}} = \partial\mathbf{D} \cap cl\, C_{\mathcal{N}}$). If $x \in \partial\mathbf{D} \cap C_{\mathcal{N}}$, then $B_{\varepsilon_0}(x) \cap \partial\mathbf{D} \subset (\bigcap_{S\in\mathcal{N}} D_{\mathcal{N}}(S)) \cap \partial\mathbf{D} \subset F_{\mathcal{N}}$ for some $\varepsilon_0 > 0$, hence $x \in \widetilde{int}\, F_{\mathcal{N}}$. Conversely let $x \in \widetilde{int}\, F_{\mathcal{N}}$. Then $\partial\mathbf{D} \cap B_\varepsilon(x) \subset \partial\mathbf{D} \setminus ex\mathcal{N}$ for some $\varepsilon > 0$. Assume there is $S \in \mathcal{N}$ with $S \cap co(\partial\mathbf{D} \cap B_\varepsilon(x)) \neq \emptyset$. Then $ex S \cap (\partial\mathbf{D} \cap B_\varepsilon(x)) \neq \emptyset$ contradicting the choice of ε. Hence $co(\partial\mathbf{D} \cap B_\varepsilon(x)) \subset \bar{\mathbf{D}} \setminus \cup\mathcal{N}$. Since $x \in F_{\mathcal{N}} \subset \bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S)$ and $co(\partial\mathbf{D} \cap B_\varepsilon(x))$ is connected and relatively open in $\bar{\mathbf{D}}$, we have $co(\partial\mathbf{D} \cap B_\varepsilon(x)) \subset C_{\mathcal{N}}$.

c) We show the equality

$$\bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S) = co\, F_{\mathcal{N}} = cl\, C_{\mathcal{N}}\,. \tag{1}$$

By Remark 2, it suffices to show $\bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S) \subset co\, F_{\mathcal{N}} \subset cl\, C_{\mathcal{N}}$. Let $x \in \bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S)$. If $x \in \partial\mathbf{D}$, then $x \in F_{\mathcal{N}}$. If $x \in \cup\mathcal{N}$, then $x \in S = co\, ex S$ for some $S \in \mathcal{N}$, hence $x \in co\, F_{\mathcal{N}}$ since $ex\mathcal{N} \subset F_{\mathcal{N}}$. Suppose $x \in \mathbf{D} \setminus \cup\mathcal{N}$. Let $\overline{ab} \in \mathcal{N}$ (note $\{a, b\} \subset F_{\mathcal{N}}$), $\overline{aa'}$ be the chord passing through x, $\overline{bb'}$ the chord passing through x and $[a', b']$ the closed arc between a' and b' containing neither a nor b. If $[a', b'] \cap F_{\mathcal{N}} \neq \emptyset$, then $x \in \Delta aby \subset co\, F_{\mathcal{N}}$ where $y \in [a', b'] \cap F_{\mathcal{N}}$. If $[a', b'] \subset \partial\mathbf{D} \setminus F_{\mathcal{N}}$, then $[a', b'] \subset \gamma$ for some $\gamma \in Comp(\partial\mathbf{D} \setminus F_{\mathcal{N}})$. Note that γ is an open arc and $\tilde{\partial}\gamma = \{a_0, b_0\} \subset F_{\mathcal{N}}$. Since $x \in \bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S)$ and $\gamma \subset \partial\mathbf{D} \setminus F_{\mathcal{N}}$, we have that the points a, a_0, b_0, b are distinct and $x \in co\{a, a_0, b_0, b\} \subset co\, F_{\mathcal{N}}$. Next we show $co\, F_{\mathcal{N}} \subset cl\, C_{\mathcal{N}}$. It suffices, by the theorem of Caratheodory, to show for any (closed) triangle Δ whose vertices belong to $F_{\mathcal{N}}$, $int\, \Delta \subset C_{\mathcal{N}}$. Let Δ be such a triangle and $S \in \mathcal{N}$. If $int\, \Delta \setminus D_{\mathcal{N}}(S) \neq \emptyset$, then some vertex of Δ does not belong to $\partial\mathbf{D} \cap cl\, D_{\mathcal{N}}(S) (\supset F_{\mathcal{N}})$ contradicting the choice of Δ. Thus $int\, \Delta \subset \bigcap_{S\in\mathcal{N}} D_{\mathcal{N}}(S)$.

d) At first we show the equality

$$\tilde{\partial} F_{\mathcal{N}} = \partial\mathbf{D} \cap cl(\cup\mathcal{N})\,. \tag{2}$$

Since $\tilde{\partial} F_{\mathcal{N}} = cl\, ex\mathcal{N}$, $\tilde{\partial} F_{\mathcal{N}} \subset \partial\mathbf{D} \cap cl(\cup\mathcal{N})$. Conversely let $x \in \partial\mathbf{D} \cap cl(\cup\mathcal{N})$. Since $co(\partial\mathbf{D} \cap B_\varepsilon(x))$ is relatively open in $\bar{\mathbf{D}}$, we see that for any $\varepsilon > 0$, $S \cap co(\partial\mathbf{D} \cap B_\varepsilon(x)) \neq \emptyset$ for some $S \in \mathcal{N}$. So $B_\varepsilon(x) \cap ex S \neq \emptyset$, that is, $x \in cl\, ex\mathcal{N} = \tilde{\partial} F_{\mathcal{N}}$. We show the equality

$$\tilde{\partial} F_{\mathcal{N}} \cup (\cup\mathcal{N}^\circ) = \partial_{\bar{\mathbf{D}}} C_{\mathcal{N}} = cl(\cup\mathcal{N})\,. \tag{3}$$

By a) and Remark 2, $F_{\mathcal{N}} \cup cl(\cup\mathcal{N}) \subset cl\, C_{\mathcal{N}}$. It suffices to show $cl\, C_{\mathcal{N}} \setminus (\tilde{\partial} F_{\mathcal{N}} \cup (\cup\mathcal{N}^\circ)) = C_{\mathcal{N}} = cl\, C_{\mathcal{N}} \setminus cl(\cup\mathcal{N})$. We show that $x \in C_{\mathcal{N}}$ for each $x \in cl\, C_{\mathcal{N}} \setminus (\tilde{\partial} F_{\mathcal{N}} \cup (\cup\mathcal{N}^\circ))$. If $x \in \partial\mathbf{D}$, then $x \in \widetilde{int}\, F_{\mathcal{N}} \subset C_{\mathcal{N}}$ by a). Suppose $x \in \mathbf{D}$. By (1), $x \in \bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S)$. Assume that $B_\varepsilon(x) \cap (\cup\mathcal{N}) \neq \emptyset$ for any $\varepsilon > 0$ with $B_\varepsilon(x) \subset \mathbf{D}$. Then since $x \notin \cup\mathcal{N}^\circ$, $\sharp\{S \in \mathcal{N} \mid B_\varepsilon(x) \cap S \neq \emptyset\} = \infty$. In particular $B_\varepsilon(x) \cap S_i \neq \emptyset$ for some $S_1, S_2, S_3 \in \mathcal{N}$ with $S_i^\circ \cap S_j^\circ = \emptyset$ $(i \neq j)$. We can suppose $S_3 \subset (cl\, D_{S_2^\circ}\langle S_1\rangle) \cap (cl\, D_{S_1^\circ}\langle S_2\rangle)$. Hence $D_{S_1^\circ}\langle S_3\rangle \cap D_{S_2^\circ}\langle S_3\rangle = \emptyset$, contradicting the 1-sidedness of $\mathcal{N}$. So $B_{\varepsilon_0}(x) \subset \mathbf{D} \setminus \cup\mathcal{N}$ for some $\varepsilon_0 > 0$. Since $B_{\varepsilon_0}(x)$ is connected and $x \in \bigcap_{S\in\mathcal{N}} cl\, D_{\mathcal{N}}(S)$, we have $B_{\varepsilon_0}(x) \subset \bigcap_{S\in\mathcal{N}} D_{\mathcal{N}}(S)$, in particular $x \in C_{\mathcal{N}}$.

Next we show $C_{\mathcal{N}} \subset cl\, C_{\mathcal{N}} \setminus cl(\cup\mathcal{N})$, that is, $C_{\mathcal{N}} \cap cl(\cup\mathcal{N}) = \emptyset$. If $x \in C_{\mathcal{N}} \cap \partial\mathbf{D}$, then $x \in \widetilde{int}\, F_N$ by b). Since $x \in \partial\mathbf{D}$, $x \notin cl(\cup\mathcal{N})$ by (2). When $x \in C_{\mathcal{N}} \cap \mathbf{D}$ we have $B_{\varepsilon_0}(x) \subset \bigcap_{S\in\mathcal{N}} D_{\mathcal{N}}(S)$ for some $\varepsilon_0 > 0$, so $B_{\varepsilon_0}(x) \cap (\cup\mathcal{N}) = \emptyset$ i.e. $x \notin cl(\cup\mathcal{N})$.

Finally by (2), we have $cl\, C_{\mathcal{N}} \setminus cl(\cup\mathcal{N}) \subset cl\, C_{\mathcal{N}} \setminus (\tilde{\partial} F_{\mathcal{N}} \cup (\cup\mathcal{N}))$.

e) It is clear that $C_{\mathcal{N}} \subset int_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}}$. By Lemma 2, it suffices to show $\partial_{\bar{\mathbf{D}}} C_{\mathcal{N}} \subset \partial_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}}$. Let $x \in \partial_{\bar{\mathbf{D}}} C_{\mathcal{N}}$. Then for any $\varepsilon > 0$, there is $R \in \mathcal{N}$ such that $(B_\varepsilon(x) \cap \bar{\mathbf{D}}) \setminus cl\, D_{\mathcal{N}}(R) \neq \emptyset$. (Indeed since $\partial_{\bar{\mathbf{D}}} C_{\mathcal{N}} = (\cup\mathcal{N}^\circ) \cup cl\, ex\mathcal{N}$ by d), we see that $x \in R^\circ$ for some $R \in \mathcal{N}$ or $x \in cl\, ex\mathcal{N}$. When $x \in cl\, ex\mathcal{N}$, there is $R \in \mathcal{N}$ such that $B_\varepsilon(x) \cap ex R \neq \emptyset$.) So by (1), $(B_\varepsilon(x) \cap \bar{\mathbf{D}}) \setminus cl\, C_{\mathcal{N}} = \bigcup_{S \in \mathcal{N}} (B_\varepsilon(x) \cap \bar{\mathbf{D}}) \setminus cl\, D_{\mathcal{N}}(S) \neq \emptyset$. Thus $x \in cl\, C_{\mathcal{N}} \setminus int_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}} = \partial_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}}$.

f) By (1), $cl\, C_{\mathcal{N}}$ is closed and convex in $\mathbf{C}$. Since $F_{\mathcal{N}} \in \mathcal{F}$, we have that $D = int\, cl\, C_{\mathcal{N}}$ is convex and $cl\, D = cl\, C_{\mathcal{N}}$ by Theorems 6.2 and 6.3 in [3] (see also p. 44 in [3]). Therefore we see $D \subset int_{\bar{\mathbf{D}}}\, cl\, C_{\mathcal{N}} = C_{\mathcal{N}} \subset cl\, D$ by e). So $C_{\mathcal{N}}$ is connected. By (3), $C_{\mathcal{N}} \in Comp(\bar{\mathbf{D}} \setminus cl(\cup\mathcal{N}))$. □

LEMMA 5. *Let $\mathcal{L}$ be a lamination and $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$ with $\sharp\mathcal{L}_V \geq 2$. Then*

a) $F_{\mathcal{L}_V} = \partial\mathbf{D} \cap cl\, V$. *(Hence $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D} \cap cl\, V}$.)*

b) *For each $R \in \mathcal{L}_+$, there is $Q \in \mathcal{L}_V$ such that $D_V\langle Q\rangle \subset D_V\langle R\rangle$.*

c) $C_{\mathcal{L}_V} = int_{\bar{\mathbf{D}}}(\bigcap_{S \in \mathcal{L}_+} D_V\langle S\rangle)$.

d) $cl\, V = cl\, C_{\mathcal{L}_V}$.

PROOF. We prepare the following claim. Suppose $S \in \mathcal{S}_+$ satisfies the condition that $\mathcal{L} \cup \{S\}$ is non-crossing and $S^\circ \cap V = \emptyset$. (Notice that $S \cap V = \emptyset$ because V is relatively open in $\bar{\mathbf{D}}$ by Remark 1.) Then the following conditions are equivalent.

1) $S^\circ \cap cl\, V \neq \emptyset$.

2) $S^\circ \subset D_V\langle R\rangle$ for any $R \in \mathcal{L}_+ \setminus \{S\}$.

3) $S \in \mathcal{L}_V$.

4) $ex S \subset cl\, V$.

$\neg 2) \Rightarrow \neg 1)$ Since $\mathcal{L} \cup \{S\}$ is non-crossing, $S^\circ \cap cl\, D_V\langle R\rangle = \emptyset$ for some $R \in \mathcal{L}_+ \setminus \{S\}$. Then $S^\circ \cap cl\, V \subset S^\circ \cap cl\, D_V\langle R\rangle = \emptyset$. 2) $\Rightarrow$ 3) Pick $x \in S^\circ$ and $y \in V$ (note $y \notin S$ and $x \neq y$). Let A be the half-open segment $(co\{x, y\}) \setminus \{x\}$. So $D_V\langle S\rangle = D_y\langle S\rangle = D_A\langle S\rangle$. Let $R \in \mathcal{L}_+ \setminus \{S\}$ (note $y \notin R$). Then $R \cap A = \emptyset$. (Indeed when $R^\circ \cap cl\, D_V\langle S\rangle = \emptyset$, it is clear. If $R^\circ \subset D_V\langle S\rangle$, then, by 2), $R \cap A = \emptyset$.) Therefore since $\mathcal{L} \cup \{S\}$ is non-crossing, $R \cap int\, \Delta = \emptyset$ where $\Delta = co(S \cup \{y\})$. Hence $int\, \Delta \subset \mathbf{D} \setminus \cup\mathcal{L}$. Furthermore since $y \in V \cap \Delta$ and $int\, \Delta$ is connected, we have $int\, \Delta \subset V$. Hence $S = \Delta \cap S \subset cl\, V \setminus V = \partial_{\bar{\mathbf{D}}} V$. 3) $\Rightarrow$ 1) is trivial. Thus 1), 2), 3) are equivalent. 3) $\Rightarrow$ 4) is trivial. $\neg 2) \Rightarrow \neg 4)$ Since $S^\circ \cap cl\, D_V\langle R\rangle = \emptyset$ for some $R \in \mathcal{L}_+ \setminus \{S\}$, we see $ex S \setminus cl\, V \supset ex S \setminus cl\, D_V\langle R\rangle \neq \emptyset$.

Now we prove the lemma. Recall that $V \subset C_{\mathcal{L}_V}$ and $D_{\mathcal{L}_V}(R) = D_V\langle R\rangle$ for any $R \in \mathcal{L}_V$.

a) We show $Comp(\partial\mathbf{D} \setminus F_{\mathcal{L}_V}) = Comp(\partial\mathbf{D} \setminus cl\, V)$ (hence $F_{\mathcal{L}_V} = \partial\mathbf{D} \cap cl\, V$).

Let $\gamma \in Comp(\partial\mathbf{D} \setminus F_{\mathcal{L}_V})$. By the definition of $F_{\mathcal{L}_V}$, $\gamma = \partial\mathbf{D} \setminus cl\, D_{\mathcal{L}_V}(R) = \partial\mathbf{D} \setminus cl\, D_V\langle R\rangle$ for some $R \in \mathcal{L}_V$. Since $\tilde{\partial}\gamma = ex R \subset cl\, V$, we have $\gamma \in Comp(\partial\mathbf{D} \setminus cl\, V)$. Conversely let $\gamma \in Comp(\partial\mathbf{D} \setminus cl\, V)$ and $\tilde{\partial}\gamma = \{a_0, a_1\}$. We show $\gamma = \partial\mathbf{D} \setminus cl\, D_{\mathcal{L}_V}(R)$ for some $R \in \mathcal{L}_V$ i.e. $S = \overline{a_0 a_1} \in \mathcal{L}_V$. Since $ex S \subset cl\, V$, it suffices to show S satisfies the assumption in the above claim, that is, $\mathcal{L} \cup \{S\}$ is non-crossing and $S^\circ \cap V = \emptyset$. If there is $R \in \mathcal{L}_+$ with $R^\circ \cap S \neq \emptyset$, then $a_i \notin cl\, V$ for some $i \in \{0, 1\}$, contradicting the fact that

$a_i \in cl\ V$. So the family $\mathcal{L} \cup \{S\}$ is non-crossing. Assume there is $x \in S^\circ \cap V$. Pick $y \in \gamma$. Since $y \in \partial\mathbf{D} \setminus cl\ V$, there is $z \in \mathbf{D} \cap co\{x, y\}$ with $z \in \partial_{\bar{\mathbf{D}}} V$. By Remark 1, $z \in R^\circ$ for some $R \in \mathcal{L}_V$. Since $\mathcal{L} \cup \{S\}$ is non-crossing, $R^\circ \subset co\,\gamma$. Hence $\gamma \cap ex R \neq \emptyset$, contradicting the fact that $\gamma \cap ex R \subset \gamma \cap cl\ V = \emptyset$. So $S^\circ \cap V = \emptyset$.

b) Let $R \in \mathcal{L}_+$. Since $\partial\mathbf{D} \setminus cl\ D_V\langle R\rangle \subset \partial\mathbf{D} \setminus cl\ V = \partial\mathbf{D} \setminus F_{\mathcal{L}_V}$, there is $\gamma \in Comp(\partial\mathbf{D} \setminus F_{\mathcal{L}_V})$ such that $\partial\mathbf{D} \setminus cl\ D_V\langle R\rangle \subset \gamma$. Letting $Q = co\,\tilde{\partial}\gamma$, we have $Q \in \mathcal{N}_{F_{\mathcal{L}_V}} = \mathcal{L}_V$ and $D_V\langle Q\rangle \subset D_V\langle R\rangle$.

c) By b), $\bigcap_{S \in \mathcal{L}_V} D_{\mathcal{L}_V}(S) = \bigcap_{S \in \mathcal{L}_V} D_V\langle S\rangle = \bigcap_{S \in \mathcal{L}_+} D_V\langle S\rangle$. Hence $C_{\mathcal{L}_V} = int_{\bar{\mathbf{D}}}(\bigcap_{S \in \mathcal{L}_+} D_V\langle S\rangle)$.

d) By Lemma 4-a), $cl\ C_{\mathcal{L}_V} = (\mathbf{D} \cap cl\ C_{\mathcal{L}_V}) \cup F_{\mathcal{L}_V}$. Since $\mathbf{D}$ is open in $\mathbf{C}$, we see $\mathbf{D} \cap cl\ C_{\mathcal{L}_V} \subset cl(\mathbf{D} \cap C_{\mathcal{L}_V})$. Hence $cl\ C_{\mathcal{L}_V} = cl(\mathbf{D} \cap C_{\mathcal{L}_V}) \cup F_{\mathcal{L}_V}$. By a), we can show that $cl\ V = cl(\mathbf{D} \cap V) \cup F_{\mathcal{L}_V}$ in the same way as above. Therefore since $V \subset C_{\mathcal{L}_V}$, it suffices to show that $\mathbf{D} \cap C_{\mathcal{L}_V} \subset \mathbf{D} \cap V$. Let $x \in \mathbf{D} \cap C_{\mathcal{L}_V}$. By c), $B_{\varepsilon_0}(x) \subset \mathbf{D} \cap (\bigcap_{S \in \mathcal{L}_+} D_V\langle S\rangle)$ for some $\varepsilon_0 > 0$. In particular $B_{\varepsilon_0}(x) \subset \mathbf{D} \setminus \cup\mathcal{L}$. Since $B_{\varepsilon_0}(x)$ is connected, $B_{\varepsilon_0}(x) \subset V$. □

COROLLARY 1. *Suppose that a lamination $\mathcal{L}$ satisfies the condition $\partial\mathbf{D} = cl\ ex\mathcal{L}$. Then for any $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$,*

1) *V is open in $\mathbf{C}$ and $\sharp\mathcal{L}_V \geq 2$. (Hence $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D} \cap cl\ V}$.)*
2) *$\widetilde{int}(\partial\mathbf{D} \cap cl\ V) = \emptyset$.*
3) *$V = C_{\mathcal{L}_V}$.*

PROOF. 1) Since $\cup\mathcal{L}$ is closed in $\mathbf{C}$ and $\partial\mathbf{D} = cl\ ex\mathcal{L}$, we have $\partial\mathbf{D} \subset \cup\mathcal{L}$. Hence $V \subset \mathbf{D}$ and furthermore V is open in $\mathbf{C}$. (Indeed since V is relatively open in $\bar{\mathbf{D}}$ and $V \subset \mathbf{D}$, we see that for each $x \in V$, there is $\varepsilon > 0$ with $B_\varepsilon(x) \subset V$.) We show $\sharp\mathcal{L}_V \geq 2$. Assume that $\mathcal{L}_V = \{S\}$ (recall Remark 1). Then $\mathbf{D} \cap D_V\langle S\rangle \subset V$. (Indeed assume there is $y \in \mathbf{D} \cap D_V\langle S\rangle \setminus V$. Let $x \in V$. Then there is $z \in co\{x, y\}$ such that $z \in \partial_{\bar{\mathbf{D}}} V$. Notice that $co\{x, y\} \subset \mathbf{D} \cap D_V\langle S\rangle$. Since $z \in \mathbf{D} \cap \partial_{\bar{\mathbf{D}}} V$, there is $R \in \mathcal{L}_V$ such that $z \in R^\circ$ by Remark 1. Since $z \in D_V\langle S\rangle$, $R \neq S$, contradicting the assumption that $\mathcal{L}_V = \{S\}$.) There is $T \in \mathcal{L}_+$ with $exT \cap \partial\mathbf{D} \cap D_V\langle S\rangle \neq \emptyset$ because $\partial\mathbf{D} = cl\ ex\mathcal{L}$ and $\partial\mathbf{D} \cap D_V\langle S\rangle$ is relatively open in $\partial\mathbf{D}$. Since $\mathcal{L}$ is non-crossing, $T^\circ \subset \mathbf{D} \cap D_V\langle S\rangle \subset V$, contradicting the fact that $T \cap V = \emptyset$. By Lemma 5-a), $F_{\mathcal{L}_V} = \partial\mathbf{D} \cap cl\ V$ and $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D} \cap cl\ V}$.

2) By 1), Lemma 4-b) and Lemma 5-c), we see that $\widetilde{int}(\partial\mathbf{D} \cap cl\ V) = \widetilde{int}\, F_{\mathcal{L}_V} \subset C_{\mathcal{L}_V} \subset \bigcap_{S \in \mathcal{L}_+} D_V\langle S\rangle$. Assume $\widetilde{int}(\partial\mathbf{D} \cap cl\ V) \neq \emptyset$. Since $\partial\mathbf{D} = cl\ ex\mathcal{L}$, there is $S \in \mathcal{L}_+$ with $ex S \cap \widetilde{int}(\partial\mathbf{D} \cap cl\ V) \neq \emptyset$, contradicting the fact that $\widetilde{int}(\partial\mathbf{D} \cap cl\ V) \subset D_V\langle S\rangle \subset \bar{\mathbf{D}} \setminus S$.

3) It suffices to show $C_{\mathcal{L}_V} \subset V$. By Lemma 5-c), $C_{\mathcal{L}_V} \cap (\cup\mathcal{L}_+) = \emptyset$. By 2) and Lemma 4-b), $C_{\mathcal{L}_V} \subset \mathbf{D}$. Hence $C_{\mathcal{L}_V} \subset \bar{\mathbf{D}} \setminus \cup\mathcal{L}$. Since $C_{\mathcal{L}_V}$ is connected (by Lemma 4-f)), $C_{\mathcal{L}_V} \subset V$. □

3. Gap invariance theorem.

Let $d \in \mathbf{N}$ be $d \geq 2$. Define the mapping $p_d : \partial\mathbf{D} \to \partial\mathbf{D}$ by $p_d(z) = z^d$, and its iterated mappings, $p_d^n = p_d^{n-1} \circ p_d$ for each $n \in \mathbf{N}$ where p_d^0 is the identity mapping on

$\partial\mathbf{D}$. For $z \in \partial\mathbf{D}$ and $n \geq 0$, define $p_d^{-n}(z) = \{w \in \partial\mathbf{D} \mid p_d^n(w) = z\}$. Note that for any $z \in \partial\mathbf{D}$, $\bigcup_{n\geq 0} p_d^{-n}(z)$ is dense in $\partial\mathbf{D}$. For each $S \in \mathcal{S}_+$, denote $P_d S = co\, p_d(exS)$. Let $\Omega_d = \{\exp(2\pi i \frac{k}{d}) \mid k = 0, 1, \cdots, d-1\}$. For $S \in \mathcal{S}_+$ and $\omega \in \Omega_d$, let $\omega S = \{\omega z \mid z \in S\} \in \mathcal{S}_+$.

DEFINITION 4. Let $\mathcal{L}$ be a lamination. We say that $\mathcal{L}$ is a symmetric and invariant lamination under p_d (we say d-SIL briefly) if the following three conditions hold:

(Symmetry) For any $S \in \mathcal{L}_+$ and $\omega \in \Omega_d$, $\omega S \in \mathcal{L}$.

(Forward invariance) For any $S \in \mathcal{L}_+$, $P_d S \in \mathcal{L}_+$ or $P_d S$ is degenerate.

(Backward invariance) For any $S \in \mathcal{L}_+$, there is $R \in \mathcal{L}$ such that $P_d R = S$.

Let $S \in \mathcal{S}$. Then $S = co\{\exp(2\pi i\alpha), \exp(2\pi i\beta)\}$ for some $\alpha, \beta \in [0, 1)$. Define $l(S) = \min\{|\alpha - \beta|, 1 - |\alpha - \beta|\}$ and we call $l(S)$ the length of S. Then $0 \leq l(S) \leq 1/2$. Clearly $l(S) = 0$ if and only if S is degenerate, and $l(S) = 1/2$ if and only if S is a diameter.

REMARK 3. *Let $\mathcal{L}$ be a d-SIL. Then $\partial\mathbf{D} = cl\, ex\mathcal{L}$ and for each $S \in \mathcal{L}_+$, $l(S) \leq 1/d$.*

PROOF. Firstly we claim that for any $n \geq 0$, $p_d^{-n} ex\mathcal{L} \subset ex\mathcal{L}$. It suffices to show $p_d^{-1} ex\mathcal{L} \subset ex\mathcal{L}$. Let $z \in exS$ where $S \in \mathcal{L}_+$. By the backward invariance of $\mathcal{L}$, $P_d R = S$ for some $R \in \mathcal{L}$, in particular $y \in exR$ for some $y \in p_d^{-1}(z)$. By the symmetry of $\mathcal{L}$, $p_d^{-1}(z) = \{\omega y \mid \omega \in \Omega_d\} \subset \bigcup_{\omega\in\Omega_d} ex(\omega R) \subset ex\mathcal{L}$. We show $\partial\mathbf{D} = cl\, ex\mathcal{L}$. Let $z \in ex\mathcal{L}$. Since $\bigcup_{n\geq 0} p_d^{-n}(z)$ is dense in $\partial\mathbf{D}$, we have for any $x \in \partial\mathbf{D}$ and $\varepsilon > 0$, there is $y \in \bigcup_{n\geq 0} p_d^{-n}(z) \cap B_\varepsilon(x)$. By the above claim, $y \in ex\mathcal{L}$. Next let $S \in \mathcal{L}_+$. By the non-crossing property and symmetry of $\mathcal{L}$, we see that the family $\{\omega S \mid \omega \in \Omega_d\}$ is non-crossing. Then $\sum_{\omega\in\Omega_d} l(\omega S) \leq 1$. Since $l(\omega S) = l(S)$, $l(S) \leq 1/d$. □

Let $\mathcal{L}$ be a d-SIL. Then for each $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$, $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D}\cap cl\, V}$ by Remark 3 and Corollary 1-1). This is the key relation to prove gap invariance theorem. Define $P_d(cl\, V) = co\, p_d(\partial\mathbf{D} \cap cl\, V)$. Observe that

1) If $\sharp p_d(\partial\mathbf{D} \cap cl\, V) = 1$, then $P_d(cl\, V)$ is a degenerate chord, that is, $P_d(cl\, V) \in \mathcal{S} \setminus \mathcal{S}_+$.

2) If $\sharp p_d(\partial\mathbf{D} \cap cl\, V) = 2$, then $P_d(cl\, V) \in \mathcal{L}_+$.

Indeed let $p_d(\partial\mathbf{D} \cap cl\, V) = \{w_1, w_2\}$ where $w_1 \neq w_2$. Since $\sharp(\partial\mathbf{D} \cap cl\, V) < \infty$ and $\mathcal{L}_V = \mathcal{N}_{\partial\mathbf{D}\cap cl\, V}$, we see that each chord in $\mathcal{L}_V$ has to join two consecutive points in $\partial\mathbf{D} \cap cl\, V$ (recall the definition of $\mathcal{N}_{\partial\mathbf{D}\cap cl\, V}$). Hence there is $S \in \mathcal{L}_V$ such that $exS \cap p_d^{-1}(w_k) \neq \emptyset$ for each $k = 1, 2$. So $P_d(cl\, V) = co\{w_1, w_2\} = co\, p_d(exS) = P_d S \in \mathcal{L}$ by the forward invariance of $\mathcal{L}$.

In order to show gap invariance theorem, we use the following technical lemmas and will show them later. Let $\mathbf{0}$ be the origin of $\mathbf{C}$. When $S \in \mathcal{S}_+$ is not a diameter (so $\mathbf{0} \notin S$), let γ_S the open arc subtended by S, that is, $\gamma_S = \partial\mathbf{D} \setminus cl\, D_{\mathbf{0}}\langle S\rangle$.

LEMMA 6. *Let $\mathcal{L}$ be a d-SIL. Then $\widetilde{int}\, p_d(\partial\mathbf{D} \cap cl\, V) = \emptyset$ for each $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$.*

LEMMA 7. *Let $\mathcal{L}$ be a d-SIL and $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$.*

Suppose that $R \in \mathcal{S}_+$ *satisfies the conditions*: $l(R) < 1/d$, $p_d(ex R) \subset p_d(\partial \mathbf{D} \cap cl\, V)$ *and* $\bigcup_{\omega \in \Omega_d} \gamma_{\omega R} \subset \partial \mathbf{D} \setminus cl\, V$. *Then* $P_d R \in P_d(\mathcal{L}_V)_+$.

GAP INVARIANCE THEOREM. *Let* $\mathcal{L}$ *be d-SIL and* $V \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$. *Then* $P_d(cl\, V) \in \mathcal{S} \setminus \mathcal{S}_+$ *or* $P_d(cl\, V) \in \mathcal{L}$, *otherwise* $P_d(cl\, V)$ *is a gap of* $\mathcal{L}$: *more precisely,* $int\, P_d(cl\, V) \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$ *and* $P_d(cl\, V) = cl\, int\, P_d(cl\, V)$.

PROOF. It suffices to show that if $\sharp p_d(\partial \mathbf{D} \cap cl\, V) \geq 3$, then $int\, P_d(cl\, V) \in Comp(\bar{\mathbf{D}} \setminus \cup\mathcal{L})$ and $P_d(cl\, V) = cl\, int\, P_d(cl\, V)$. Suppose that $\sharp p_d(\partial \mathbf{D} \cap cl\, V) \geq 3$. By Lemma 6, $p_d(\partial \mathbf{D} \cap cl\, V)$ is disconnected. Hence we can define the 1-sided family $\mathcal{N} = \mathcal{N}_{p_d(\partial \mathbf{D} \cap cl\, V)}$. Notice that $F_{\mathcal{N}} = p_d(\partial \mathbf{D} \cap cl\, V)$. Since $P_d(cl\, V) = co\, F_{\mathcal{N}}$, we have by Lemma 4-c) and e),

$$\begin{aligned} P_d(cl\, V) &= cl\, C_{\mathcal{N}}\,, \\ int_{\bar{\mathbf{D}}}\, P_d(cl\, V) &= C_{\mathcal{N}}\,, \\ \partial_{\bar{\mathbf{D}}} P_d(cl\, V) &= \partial_{\bar{\mathbf{D}}} C_{\mathcal{N}}\,. \end{aligned}$$

At first we show that

$$\mathcal{N} = P_d(\mathcal{L}_V)_+\,. \tag{4}$$

Let $S \in \mathcal{N}$. By the definition of $\mathcal{N}$, $S = co\, \tilde{\partial}\gamma$ for some $\gamma \in Comp(\partial \mathbf{D} \setminus F_{\mathcal{N}})$. Denote $R = co\, \tilde{\partial}\delta$ where $\delta \in Comp(p_d^{-1}(\gamma))$. Then $p_d(ex R) = \tilde{\partial}\gamma \subset F_{\mathcal{N}}$ (so $l(R) < 1/d$). By the property of the mapping p_d, $\bigcup_{\omega \in \Omega_d} \gamma_{\omega R} = p_d^{-1}(\gamma) \subset p_d^{-1}(\partial \mathbf{D} \setminus F_{\mathcal{N}}) \subset \partial \mathbf{D} \setminus cl\, V$. By Lemma 7, $S = co\, \tilde{\partial}\gamma = P_d(R) \in P_d(\mathcal{L}_V)_+$. Conversely let $R \in \mathcal{L}_V$ satisfy that $P_d R \in \mathcal{S}_+$. We show $P_d(R) \in \mathcal{N}$. By Remark 3, $l(R) < 1/d$. Hence we see that $\omega_1 R \cap \omega_2 R = \emptyset$ if $\omega_1, \omega_2 \in \Omega_d$ with $\omega_1 \neq \omega_2$. By the symmetry of $\mathcal{L}$, $\omega R \in \mathcal{L}_+$ for any $\omega \in \Omega_d$. Hence $V \subset C_R := (\bigcap_{\omega \in \Omega_d} D_0\langle \omega R \rangle)$ or $V \subset \bar{\mathbf{D}} \setminus cl\, D_0\langle \omega_0 R \rangle$ for some $\omega_0 \in \Omega_d$. We have that $\partial \mathbf{D} \cap cl\, V \subset \partial \mathbf{D} \cap cl\, C_R$ or $\partial \mathbf{D} \cap cl\, V \subset cl\, \gamma_{\omega_0 R}$ for some $\omega_0 \in \Omega_d$. Here notice that $p_d(cl\, \gamma_{\omega R}) = p_d(cl\, \gamma_R) = cl\, p_d(\gamma_R)$ for each $\omega \in \Omega_d$ and $p_d(\partial \mathbf{D} \cap cl\, C_R) \cap p_d(\bigcup_{\omega \in \Omega_d} \gamma_{\omega R}) = \emptyset$ by the property of the mapping p_d. So $F_{\mathcal{N}} \cap p_d(\gamma_R) = \emptyset$ or $F_{\mathcal{N}} \subset cl\, p_d(\gamma_R)$. Since $\tilde{\partial} p_d(\gamma_R) = p_d(\tilde{\partial}\gamma_R) = p_d(ex R) \subset F_{\mathcal{N}}$, we have that $p_d(\gamma_R) \in Comp(\partial \mathbf{D} \setminus F_{\mathcal{N}})$ or $\partial \mathbf{D} \setminus cl\, p_d(\gamma_R) \in Comp(\partial \mathbf{D} \setminus F_{\mathcal{N}})$. By the definition of $\mathcal{N}$, $P_d R = co\, \tilde{\partial} p_d(\gamma_R) \in \mathcal{N}$.

By the forward invariance of $\mathcal{L}$ and equality (4), we see that $\mathcal{N} \subset \mathcal{L}_+$.

Next we show the following equalities

$$\partial \mathbf{D} \cap cl\, C_{\mathcal{N}} \cap (\cup\mathcal{L}) = \tilde{\partial} F_{\mathcal{N}}\,, \tag{5}$$

$$\mathbf{D} \cap cl\, C_{\mathcal{N}} \cap (\cup\mathcal{L}) = \cup\mathcal{N}^{\circ}\,. \tag{6}$$

By Remark 3, Lemma 4-a) and Lemma 6,

$$\partial \mathbf{D} \cap cl\, C_{\mathcal{N}} \cap (\cup\mathcal{L}) = \partial \mathbf{D} \cap cl\, C_{\mathcal{N}} = F_{\mathcal{N}} = \tilde{\partial} F_{\mathcal{N}}\,.$$

To prove (6), we give some preparation.

Define $\mathcal{B} = \{S \in \mathcal{L}_+ \mid S^{\circ} \cap cl\, C_{\mathcal{N}} \neq \emptyset\}$. Then

$$\cup\mathcal{B} \subset cl\, C_{\mathcal{N}} \quad \text{and} \quad \mathbf{D} \cap cl\, C_{\mathcal{N}} \cap (\cup\mathcal{L}) = \cup\mathcal{B}^{\circ}\,.$$

(Indeed let $S \in \mathcal{B}$. Assume that $S^\circ \setminus cl\, C_{\mathcal{N}} \neq \emptyset$. Then $S^\circ \cap \partial_{\bar{\mathbf{D}}} cl\, C_{\mathcal{N}} \neq \emptyset$. By Lemma 4-d) and e), $S^\circ \cap (\cup \mathcal{N}^\circ) \neq \emptyset$, contradicting the fact that $\mathcal{N} \subset \mathcal{L}_+$ and $\mathcal{L}$ is non-crossing. Hence we have $S \subset cl\, C_{\mathcal{N}}$. Since $\mathbf{D} \cap (\cup \mathcal{L}) = \cup \mathcal{L}^\circ$, the second statement holds.) Thus to prove (6), it suffices to show $\mathcal{N} = \mathcal{B}$. By Lemma 4-d), $\cup \mathcal{N}^\circ \subset cl\, C_{\mathcal{N}}$. Since $\mathcal{N} \subset \mathcal{L}_+$, we have that $\mathcal{N} \subset \mathcal{B}$. Conversely let $S \in \mathcal{B}$. By the backward invariance of $\mathcal{L}$, $S = P_d R$ for some $R = \overline{xy} \in \mathcal{L}_+$. Then $l(R) < 1/d$ and $p_d(ex R) = ex S \subset ex \mathcal{B} \subset \partial \mathbf{D} \cap cl\, C_{\mathcal{N}} = F_{\mathcal{N}} = p_d(\partial \mathbf{D} \cap cl\, V)$. Since $p_d^{-1}(p_d(x)) = \{\omega x \mid \omega \in \Omega_d\}$, we have that $\omega_0 x \in cl\, V$ for some $\omega_0 \in \Omega_d$. By the symmetry of $\mathcal{L}$, we can suppose $x \in cl\, V$. Furthermore by the symmetry of $\mathcal{L}$, we see that $V \subset C_R = (\bigcap_{\omega \in \Omega_d} D_{\mathbf{0}}\langle \omega R \rangle)$ or $V \subset \bar{\mathbf{D}} \setminus cl\, D_{\mathbf{0}}\langle R \rangle$. Consider the case $cl\, V \subset cl\, C_R$. Then $\bigcup_{\omega \in \Omega_d} \gamma_{\omega R} \subset \partial \mathbf{D} \setminus cl\, V$. So by Lemma 7 and the equality (4), we have that $S = P_d R \in P_d(\mathcal{L}_V)_+ = \mathcal{N}$. Suppose that $cl\, V \subset \bar{\mathbf{D}} \setminus D_{\mathbf{0}}\langle R \rangle$. Then $\partial \mathbf{D} \cap cl\, V \subset cl\, \gamma_R$. Furthermore since $p_d(y) \in p_d(\partial \mathbf{D} \cap cl\, V)$ and $l(R) < 1/d$, we see that y has to belong to $cl\, V$. Then $\partial \mathbf{D} \setminus cl\, \gamma_R \in Comp(\partial \mathbf{D} \setminus cl\, V)$. Hence $R \in \mathcal{N}_{\partial \mathbf{D} \cap cl\, V} = \mathcal{L}_V$. By the equality (4), $S = P_d R \in P_d(\mathcal{L}_V)_+ = \mathcal{N}$.

By equalities (5), (6) and Lemma 4-d), we have

$$cl\, C_{\mathcal{N}} \cap (\cup \mathcal{L}) = \tilde{\partial} F_{\mathcal{N}} \cup (\cup \mathcal{N}^\circ) = \partial_{\bar{\mathbf{D}}} C_{\mathcal{N}}.$$

Thus $C_{\mathcal{N}} \subset \bar{\mathbf{D}} \setminus \cup \mathcal{L}$. Note $cl(\cup \mathcal{N}) \subset \cup \mathcal{L}$ since $\mathcal{N} \subset \mathcal{L}_+$ and $\cup \mathcal{L}$ is closed. Hence $C_{\mathcal{N}} \in Comp(\bar{\mathbf{D}} \setminus \cup \mathcal{L})$ by Lemma 4-f). By Corollary 1-1), $C_{\mathcal{N}}$ is open in $\mathbf{C}$.

Therefore $C_{\mathcal{N}} = int\, C_{\mathcal{N}} = int(int_{\bar{\mathbf{D}}}\, P_d(cl\, V)) = int\, P_d(cl\, V)$, hence $int\, P_d(cl\, V) \in Comp(\bar{\mathbf{D}} \setminus \cup \mathcal{L})$ and furthermore we have $P_d(cl\, V) = cl\, C_{\mathcal{N}} = cl\, int\, P_d(cl\, V)$. □

(PROOF OF LEMMA 6) Case 1. $l(S) = 1/2$ for some $S \in \mathcal{L}_V$.

By Remark 3, $d = 2$. Notice that $p_2(\partial \mathbf{D} \cap cl\, V) = p_2(\partial \mathbf{D} \cap cl\, V \setminus ex S) \cup p_2(ex S)$, $\partial \mathbf{D} \cap cl\, V \setminus ex S \subset \partial \mathbf{D} \cap D_V\langle S \rangle$ and p_2 is injective on $\partial \mathbf{D} \cap D_V\langle S \rangle$. By Corollary 1-2), $\widetilde{int}\, p_2(\partial \mathbf{D} \cap cl\, V \setminus ex S) = \emptyset$. Hence $\widetilde{int}\, p_2(\partial \mathbf{D} \cap cl\, V) = \emptyset$.

Case 2. $l(S) < 1/2$ for some $S \in \mathcal{L}_V$.

Then $\mathbf{0} \notin \partial_{\bar{\mathbf{D}}} V$ by Remark 1. Thus $\mathbf{0} \notin cl\, V$ or $\mathbf{0} \in V$.

Suppose $\mathbf{0} \notin cl\, V$. Let $x \in V$. Then there is $y \in co\{\mathbf{0}, x\} \cap \partial_{\bar{\mathbf{D}}} V \setminus \{\mathbf{0}, x\}$. By Remark 1, $y \in R$ for some $R \in \mathcal{L}_V$. Since $\mathbf{0} \notin cl\, V$, we see that $\partial \mathbf{D} \cap cl\, V \setminus ex R \subset \gamma_R$. Since $l(R) \leq 1/d$ (by Remark 3), p_d is injective on γ_R. By Corollary 1-2), $\widetilde{int}\, p_d(\partial \mathbf{D} \cap cl\, V \setminus ex R) = \emptyset$. Hence $\widetilde{int}\, p_d(\partial \mathbf{D} \cap cl\, V) = \emptyset$.

Suppose $\mathbf{0} \in V$. Then $\omega R \in \mathcal{L}_V$ for any $R \in \mathcal{L}_V$ and $\omega \in \Omega_d$. (Indeed let $R \in \mathcal{L}_V$. Assume that $\omega R \notin \mathcal{L}_V$ for some $\omega \in \Omega_d$. By the symmetry of $\mathcal{L}$, $\omega R \in \mathcal{L}_+$. By Corollary 1-1) and Lemma 5-b), $(\omega R)^\circ \cap D_{\mathbf{0}}\langle Q \rangle = \emptyset$ for some $Q \in \mathcal{L}_V$. By the symmetry of $\mathcal{L}$, $\omega^{-1} Q \in \mathcal{L}_+$. Furthermore $R^\circ \cap D_{\mathbf{0}}\langle \omega^{-1} Q \rangle = \emptyset$, contradicting the fact that $R \in \mathcal{L}_V$.) So since $\mathcal{N}_{\partial \mathbf{D} \cap cl\, V} = \mathcal{L}_V$, we see that $\{\omega z \mid z \in \gamma\} \in Comp(\partial \mathbf{D} \setminus cl\, V)$ for each $\gamma \in Comp(\partial \mathbf{D} \setminus cl\, V)$ and $\omega \in \Omega_d$. Thus $\partial \mathbf{D} \cap cl\, V$ is invariant under the rotation of angle $2\pi/d$. Let $z \in cl\, V$ and $z_1 = z \exp(2\pi i/d)$. So $p_d(\partial \mathbf{D} \cap cl\, V) = p_d(\partial \mathbf{D} \cap cl\, V \cap \gamma_{\overline{zz_1}}) \cup \{p_d(z)\}$. Since p_d is injective on $\gamma_{\overline{zz_1}}$, $\widetilde{int}\, p_d(\partial \mathbf{D} \cap cl\, V) = \emptyset$ by Corollary 1-2). □

(PROOF OF LEMMA 7) Suppose that $R = \overline{yz} \in \mathcal{S}_+$ satisfies the conditions: $l(R) < 1/d$, $p_d(ex R) \subset p_d(\partial\mathbf{D} \cap cl\ V)$ and $\bigcup_{\omega\in\Omega_d} \gamma_{\omega R} \subset \partial\mathbf{D} \setminus cl\ V$. So $P_d R \in \mathcal{S}_+$. We shall show that $P_d R \in P_d(\mathcal{L}_V)$. Note that $p_d^{-1}(p_d(z)) = \{\omega z \mid \omega \in \Omega_d\}$. Hence $\omega_0 z \in cl\ V$ for some $\omega_0 \in \Omega_d$ because $p_d(z) \in p_d(\partial\mathbf{D} \cap cl\ V)$. Since $P_d R = P_d(\omega_0 R)$, we can suppose $z \in cl\ V$ without loss of generality. Since $z \in cl\ V$ and $\gamma_R \subset \partial\mathbf{D} \setminus cl\ V$, we see that

$$\text{there is } \gamma \in Comp(\partial\mathbf{D} \setminus cl\ V) \text{ such that } \gamma_R \subset \gamma \text{ and } z \in \tilde{\partial}\gamma\,.$$

Let $\tilde{\partial}\gamma = \{x, z\}$. Hence $S = \overline{xz} \in \mathcal{N}_{\partial\mathbf{D}\cap cl\ V} = \mathcal{L}_V$ and $x \in cl\ V$. Notice that

$$\text{if } l(S) < \frac{1}{2}, \quad \text{then } \gamma = \gamma_S \text{ or } \gamma = \partial\mathbf{D} \setminus cl\ \gamma_S\,.$$

Let $y_k = y\exp(2\pi ik/d)$ and $z_k = z\exp(2\pi ik/d)$ where $k \in \{1, d-1\}$.

Case 1. $x = z_k$ where $k \in \{1, d-1\}$ (or equivalently, $l(S) = 1/d$).

Suppose $d = 2$. Then $x = -z$. Since $l(R) < 1/2$, $y \in \gamma$. In particular $y \notin cl\ V$. Since $p_2(y) \in p_2(\partial\mathbf{D}\cap cl\ V)$, $-y$ has to belong to $cl\ V$. Since $\gamma_{-R}\cap cl\ V = \emptyset$ and $ex(-R) \subset cl\ V$, we have $-R \in \mathcal{N}_{\partial\mathbf{D}\cap cl\ V} = \mathcal{L}_V$. Hence $P_2 R = P_2(-R) \in P_2(\mathcal{L}_V)$.

Suppose $d \geq 3$. Then $l(s) = 1/d < 1/2$. Firstly we show $\gamma = \partial\mathbf{D} \setminus cl\ \gamma_s$.

Assume that $\gamma = \gamma_S$. Then $V \subset D_0\langle S\rangle$. Furthermore

$$cl\ V = P \quad \text{where } P = co\ p_d^{-1}(p_d(z))\,.$$

(Indeed by the symmetry of $\mathcal{L}$, we see that $\omega S \in \mathcal{L}$ for any $\omega \in \Omega_d$. Notice that ωS is a side of the polygon P. If there is $x \in (\cup\mathcal{L}) \cap int\ P$, then there is $R \in \mathcal{L}_+$ such that $x \in R^\circ$, but $R^\circ \cap (\bigcup_{\omega\in\Omega_d} \omega S) \neq \emptyset$, contradicting the non-crossing property of $\mathcal{L}$. So $int\ P \subset \bar{\mathbf{D}} \setminus \cup\mathcal{L}$. Since $int\ P$ is connected and contained in $D_0\langle S\rangle$, we have that $int\ P = V$.) So since $p_d(y) \in p_d(\partial\mathbf{D}\cap cl\ V)$, we see that $p_d^{-1}(p_d(y))\cap p_d^{-1}(p_d(z)) = p_d^{-1}(p_d(y))\cap\partial\mathbf{D}\cap cl\ V \neq \emptyset$. Hence $p_d(y) = p_d(z)$, contradicting the fact that $P_d R \in \mathcal{S}_+$.

Therefore $\partial\mathbf{D} \setminus cl\ \gamma_S = \gamma \subset \partial\mathbf{D} \setminus cl\ V$ i.e. $\partial\mathbf{D} \cap cl\ V \subset cl\ \gamma_S$. Since $\gamma_R \subset \partial\mathbf{D} \setminus cl\ \gamma_S$ and $l(R) < l(S)$, we see that $y \notin cl\ \gamma_S$. Since $S = \overline{zz_k}$, there is unique $\omega \in \Omega_d$ such that $\omega y \in cl\ \gamma_S$. Hence $\omega y = y_k$ i.e. $\omega = \exp(2\pi ik/d)$. Furthermore y_k has to belong to $cl\ V$ because $p_d(y) \in p_d(\partial\mathbf{D} \cap cl\ V) \subset p_d(cl\ \gamma_S)$. Hence $ex(\omega R) = \{z_k, y_k\} = \{x, y_k\} \subset cl\ V$. Since $\gamma_{\omega R} \cap cl\ V = \emptyset$, we have $\omega R \in \mathcal{N}_{\partial\mathbf{D}\cap cl\ V} = \mathcal{L}_V$. Hence $P_d R = P_d(\omega R) \in P_d(\mathcal{L}_V)$.

Case 2. $x \notin \{z_1, z_{d-1}\}$.

Then $l(S) < 1/d$ by Remark 3. There is unique $k \in \{1, d-1\}$ with $x \in \gamma_{\overline{zz_k}}$, and there is unique $l \in \{1, d-1\}$ with $y \in \gamma_{\overline{zz_l}}$.

Suppose $k \neq l$. Then $x \in \gamma_{\overline{y_kz}} \cup\{y_k\}$ because $\bigcup_{\omega\in\Omega_d} \gamma_{\omega R} \subset \partial\mathbf{D}\setminus cl\ V$ and $x \in cl\ V\setminus\{z\}$. So $\gamma_R\cap cl\ \gamma_S = \emptyset$, and if $x \neq y_k$, then $p_d^{-1}(p_d(y))\cap cl\ \gamma_S = \emptyset$. Since $\gamma_R \subset \gamma$, $\gamma = \partial\mathbf{D}\setminus cl\ \gamma_S$, that is, $\partial\mathbf{D} \cap cl\ V \subset cl\ \gamma_S$. Since $p_d(y) \in p_d(cl\ \gamma_S)$, $x = y_k$. So $P_d R = P_d S \in P_d(\mathcal{L}_V)$.

Suppose $k = l$. Assume that $R \neq S$. Then $y \in \gamma_S = \gamma$. (Indeed if $y \notin \gamma_S$, then since $x \neq y$, $x \in \gamma_R \cap cl\ V$, contradicting the assumption that $\gamma_R \cap cl\ V = \emptyset$. So $\gamma = \gamma_S$ since $y \in \gamma_S$ and $\gamma_R \subset \gamma$.) Therefore $\partial\mathbf{D} \cap cl\ V \subset \partial\mathbf{D} \setminus \gamma_S$ because $\gamma_S = \gamma$. Since $y \in \gamma_S$ and $p_d(y) \in p_d(\partial\mathbf{D} \cap cl\ V)$, we have that there is $\omega \in \Omega_d$ such that $\omega y \in cl\ V$ and $\omega \neq 1$. Since

$\omega S \in \mathcal{L}_+$ (by the symmetry of $\mathcal{L}$) and $\omega y \in \gamma_{\omega S} \cap cl\ V$, we have $\partial \mathbf{D} \cap cl\ V \subset cl\ \gamma_{\omega S}$. Since $\omega \neq 1$ and $l(\omega S) < 1/d$, we see $z \notin cl\ \gamma_{\omega S}$, contradicting the assumption that $z \in cl\ V$. □

4. Appendix.

(PROOF OF LEMMA 1) Since $\partial E \subset \partial D \subset X \setminus D$, $D \cap cl\ E = D \cap int\ E \subset int\ E \subset E \subset D \cap cl\ E$ and hence $int\ E = E$ and furthermore $E = D \cap cl\ E$ i.e. E is a non-empty and relative clopen subset of D. Then $E = D$ by the connectivity of D. □

(PROOF OF LEMMA 2) Notice that $\partial E = (cl\ E) \setminus (int\ E)$ and $\partial\, cl\ E = (cl\ E) \setminus (int\, cl\ E)$. □

(PROOF OF LEMMA 3) Since C is relatively closed in $X \setminus F$, $C = (cl\ C) \cap (X \setminus F)$. Since C is open in X, $\partial C = (cl\ C) \setminus C = (cl\ C) \cap F \subset F$.

For the second statement, it suffices to show that $\partial F \subset cl(\bigcup_{C \in Comp(X \setminus F)} \partial C)$. Let $x \in \partial F$ and U be an open subset of X with $x \in U$. Denote by D the component of U containing x. Then $D \setminus F \neq \emptyset$ since D is open in X and $x \in \partial F$. Hence there is $C_0 \in Comp(X \setminus F)$ such that $D \cap C_0 \neq \emptyset$. Here $x \in D \cap F \subset D \setminus C_0$. Thus $D \cap C_0 \neq \emptyset \neq D \setminus C_0$. Since D is connected, $U \cap (\bigcup_{C \in Comp(X \setminus F)} \partial C) \supset D \cap \partial C_0 \neq \emptyset$. Hence $x \in cl(\bigcup_{C \in Comp(X \setminus F)} \partial C)$. □

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