TOKYO J. MATH.
VOL. 24, NO. 2, 2001

The Higher Spin Dirac Operators on 3-Dimensional Manifolds

Yasushi HOMMA

Waseda University

(Communicated by S. Kaneyuki and R. Miyaoka)

Abstract. We study the higher spin Dirac operators on 3-dimensional manifolds and show that there exist two Laplace type operators for each associated bundle. Furthermore, we give lower bound estimations for the first eigenvalues of these Laplace type operators.

1. Introduction.

In this paper, we study the higher spin Dirac operator, which is a generalization of the Dirac operator defined as follows (see [3], [4], and [6]). Let M be an n-dimensional spin manifold and $\mathbf{Spin}(M)$ be the principal spin bundle on M. The irreducible unitary representation (ρ, V_ρ) of the structure group $Spin(n)$ induces the associated (irreducible) bundle $\mathbf{S}_\rho(M)$,

$$\mathbf{S}_\rho(M) := \mathbf{Spin}(M) \times_\rho V_\rho . \tag{1.1}$$

For example, the adjoint representation (Ad, $\mathbf{R}^n$) of $Spin(n)$ induces the cotangent bundle $T^*(M)$. Here, the representation (Ad, $\mathbf{R}^n$) is given by $\mathrm{Ad}(g)x = g \cdot x \cdot g^{-1}$ for g in $Spin(n)$ and x in $\mathbf{R}^n$ (see [9]). For each bundle, we have the covariant derivative ∇ associated to the Levi-Civita connection or the spin connection,

$$\nabla : \Gamma(\mathbf{S}_\rho(M)) \to \Gamma(\mathbf{S}_\rho(M) \otimes T^*(M)) . \tag{1.2}$$

We decompose the tensor bundle $\mathbf{S}_\rho(M) \otimes T^*(M)$ into irreducible bundles with respect to $Spin(n)$. Let $\pi_{\rho,\nu}$ be the orthogonal projection onto the irreducible bundle $\mathbf{S}_\nu(M)$ from $\mathbf{S}_\rho(M) \otimes T^*(M) \simeq \oplus_\nu \mathbf{S}_\nu(M)$. Then we define the higher spin Dirac operator $D_{\rho,\nu}$ to be the composed mapping $\pi_{\rho,\nu} \circ \nabla$,

$$D_{\rho,\nu} : \Gamma(\mathbf{S}_\rho(M)) \xrightarrow{\nabla} \Gamma(\mathbf{S}_\rho(M) \otimes T^*(M)) \xrightarrow{\pi_{\rho,\nu}} \Gamma(\mathbf{S}_\nu(M)) . \tag{1.3}$$

In fact, the Dirac operator is given in this way. To construct the Dirac operator, we take the spinor representation (Δ, V_Δ) and the associated bundle $\mathbf{S}_\Delta(M)$. Then the tensor bundle $\mathbf{S}_\Delta(M) \otimes T^*(M)$ decomposes into the direct sum of only two irreducible bundles, $\mathbf{S}_\Delta(M)$ and $\mathbf{S}_T(M)$. Then the differential operator $D := D_{\Delta,\Delta}$ is the Dirac operator and $D_{\Delta,T}$ is the

Received November 2, 2000
Revised September 13, 2001

twistor operator (see [1] and [2]). On the other hand, we know another definition of the Dirac operator by using the Clifford algebra, that is,

$$D = \sum_i e_i \cdot \nabla_{e_i} . \tag{1.4}$$

From the relations

$$e_i e_j + e_j e_i = -2\delta_{ij} \tag{1.5}$$

we show that the Dirac operator satisfies the Bochner type identity

$$D^2 = \nabla^* \nabla + \frac{1}{4}\kappa , \tag{1.6}$$

where κ is the scalar curvature of M.

The aim of this paper is to give the Bochner type identities for the higher spin Dirac operators on 3-dimensional spin manifolds. As mentioned above, the relations (1.5) is necessary to give the Bochner type identity for the Dirac operator. But the Clifford action does not exist on the representation spaces of $Spin(n)$ in general. So we consider linear mappings among the representation spaces given as follows: for X in $\mathbf{R}^n$ and v in V_ρ, we decompose $v \otimes X$ as $\sum_\nu (v \otimes X)^\nu$ with respect to the irreducible decomposition $V_\rho \otimes \mathbf{R}^n = \oplus_\nu V_\nu$. Then we have a homomorphism from V_ρ to V_ν,

$$p_\nu^\rho(X) : V_\rho \ni v \to (v \otimes X)^\nu \in V_\nu . \tag{1.7}$$

We call this homomorphism $p_\nu^\rho(X)$ *the Clifford homomorphism.* This is a generalization of the Clifford multiplication on the spinor space V_Δ. In general, it is difficult to give the explicit decomposition of $v \otimes X$. But, in the 3-dimensional case, we use the Clebsch-Gordan formula to give the Clifford homomorphisms explicitly because the structure group of $\mathbf{Spin}(M)$ is $Spin(3) = SU(2)$. This is the reason why we consider the 3-dimensional case. Then we obtain local formulas of the higher spin Dirac operators such as (1.4) and the Bochner type identities for them. Furthermore, the identities lead us to give lower bound estimations for the first eigenvalues of these operators.

In section 2, we explain the Clebsch-Gordan formula for the Lie group $SU(2)$. In section 3, we define the Clifford homomorphisms on the representation spaces and obtain some relations among these homomorphisms including the usual Clifford relations (1.5). In section 4, we have formulas of the higher spin Dirac operators by using the Clifford homomorphisms and investigate the properties of these operators (ellipticity, the Bochner type identities, and so on). The interesting fact is that we obtain two Laplace type operators for each associated bundle. In section 5, we have lower bound estimations for the first eigenvalues of the Laplace type operators. This estimation is a generalization of the one for the Dirac operator given in [1] or the Laplace-Beltrami operator in [7] and [10]. In section 6, we consider the case of the 3-dimensional manifold of constant curvature and show that some operators commute each other. In the last section, as an example, we calculate all the eigenvalues of the higher spin Dirac operators on the symmetric space S^3.

2. The Clebsch-Gordan formula.

In this section we shall explain the representations of $SU(2)$ and the Clebsch-Gordan formula. Let V_m be the $(m+1)$-dimensional complex vector space of polynomials of degree $\leq m$ in z_m. The inner product on V_m is set by

$$\langle v_m^k, v_m^l \rangle = \delta_{kl}\,, \tag{2.1}$$

where

$$v_m^k := \frac{z_m^k}{\sqrt{k!(m-k)!}}\,. \tag{2.2}$$

We define a representation ρ_m on V_m by $\rho_m(h)z_m^k = (bz_m + d)^{m-k}(az_m + c)^k$ for $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SU(2)$. Then (ρ_m, V_m) is a finite dimensional irreducible unitary representation of $SU(2)$ called the spin-$\frac{m}{2}$ representation and all such representations are given in this way.

We denote the infinitesimal representation of (ρ_m, V_m) by the same symbol (ρ_m, V_m). The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ has the following basis, that is, the Pauli matrices:

$$\sigma_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.3}$$

Then we show that

$$\begin{aligned} \rho_m\left(\frac{\sigma_1}{2}\right) z_m^k &= i\left(k - \frac{m}{2}\right) z_m^k, \\ \rho_m\left(\frac{\sigma_2}{2} + i\frac{\sigma_3}{2}\right) z_m^k &= -kz_m^{k-1}, \\ \rho_m\left(\frac{\sigma_2}{2} - i\frac{\sigma_3}{2}\right) z_m^k &= (m-k)z_m^{k+1}\,. \end{aligned} \tag{2.4}$$

EXAMPLE 2.1. The spin-$\frac{1}{2}$ representation (ρ_1, V_1) is the spinor representation on $\mathbf{C}^2$, where we identify $Spin(3)$ with $SU(2)$.

EXAMPLE 2.2. The spin-1 representation (ρ_2, V_2) is the adjoint representation on $\mathfrak{su}(2) \otimes \mathbf{C}$ of $SU(2)$, or the adjoint representation on $\mathbf{R}^3 \otimes \mathbf{C}$ of $Spin(3)$. Here, the correspondence of the bases is given as follows:

$$z_2^0 \leftrightarrow \frac{\sigma_2 + i\sigma_3}{2} \leftrightarrow e_2 + ie_3\,, \quad z_2^1 \leftrightarrow \frac{i\sigma_1}{2} \leftrightarrow ie_1\,, \quad z_2^2 \leftrightarrow \frac{\sigma_2 - i\sigma_3}{2} \leftrightarrow e_2 - ie_3\,, \tag{2.5}$$

where z_2^i is in V_2, σ_i in $\mathfrak{su}(2)$, and e_i in $\mathbf{R}^3$.

Now, we consider the unitary representation $(\rho_m \otimes \rho_n, V_m \otimes V_n)$. Then we can decompose $\rho_m \otimes \rho_n$ into its irreducible components,

$$\rho_m \otimes \rho_n \simeq \rho_{m+n} \oplus \rho_{m+n-2} \oplus \cdots \oplus \rho_{|m-n|}\,. \tag{2.6}$$

This formula is called Clebsch-Gordan formula. We need the orthogonal projection to each irreducible component from $V_m \otimes V_n$ in the next section.

3. The Clifford homomorphisms.

In this section we shall define the Clifford homomorphism, which is a generalization of the Clifford action. Let Cl_3 be the complex Clifford algebra associated to $\mathbf{R}^3$ and $\{e_i\}_{1\le i\le 3}$ be the standard basis of $\mathbf{R}^3$. We realize Cl_3 as matrix algebra $\mathbf{C}(2)\oplus\mathbf{C}(2)$ by the mapping

$$Cl_3 \ni e_i \mapsto (\sigma_i, -\sigma_i) \in \mathbf{C}(2)\oplus\mathbf{C}(2)\,. \tag{3.1}$$

Then the Clifford action of e_i on the spinor space $V_1 \simeq \mathbf{C}^2$ is given by $e_i \cdot v = \sigma_i v$. Since we would like to generalize this Clifford action on other representation spaces, we use another definition of the Clifford action as follows: we recall the irreducible decomposition

$$(\rho_1, V_1)\otimes(\rho_2, V_2) \simeq (\rho_3, V_3)\oplus(\rho_1, V_1) \tag{3.2}$$

and the isomorphism

$$(\rho_2, V_2) \simeq (ad, \mathfrak{su}(2)\otimes\mathbf{C}) \simeq (ad, \mathbf{R}^3\otimes\mathbf{C})\,. \tag{3.3}$$

For v in V_1 and e_i in $\mathbf{R}^3$, we project $v\otimes e_i$ onto V_1 along V_3 orthogonally. By calculating the Clebsch-Gordan coefficients, we show that $\mathrm{pr}(v\otimes e_i) = \sigma_i v = e_i\cdot v$.

Now, we consider the representation space V_m. In this case, we use the irreducible decomposition

$$(\rho_m, V_m)\otimes(\rho_2, V_2) \simeq (\rho_{m+2}, V_{m+2})\oplus(\rho_m, V_m)\oplus(\rho_{m-2}, V_{m-2})\,. \tag{3.4}$$

For v in V_m and X in $\mathbf{R}^3$, we decompose $v\otimes X$ as

$$v\otimes X = (v\otimes X)^+ + (v\otimes X)^0 + (v\otimes X)^-\,. \tag{3.5}$$

Here, $(v\otimes X)^0$ is in V_m and $(v\otimes X)^\pm$ in $V_{m\pm 2}$. Thus, we have linear mappings from V_m to V_m or $V_{m\pm 2}$ for any X in $\mathbf{R}^3$:

$$\begin{aligned}
\rho_m^0(X)v &:= -\frac{\sqrt{m(m+2)}}{2}(v\otimes X)^0 \in V_m\,,\\
\rho_m^+(X)v &:= -\frac{\sqrt{(m+1)(m+2)}}{\sqrt{2}}(v\otimes X)^+ \in V_{m+2}\,,\\
\rho_m^-(X)v &:= \frac{\sqrt{m(m+1)}}{\sqrt{2}}(v\otimes X)^- \in V_{m-2}\,,
\end{aligned} \tag{3.6}$$

where we multiply each mapping by a constant to let the calculations easier. We call these linear mappings *the Clifford homomorphisms*.

Calculating the Clebsch-Gordan coefficients in the decomposition (3.4), we deduce explicit formulas of the Clifford homomorphisms.

PROPOSITION 3.1. *The Clifford homomorphisms associated to* $\mathbf{R}^3$ *are given as follows: for the basis* $\{z_m^k\}_{0\le k\le m}$ *of* V_m *and* $\{e_i\}_{1\le i\le 3}$ *in* $\mathbf{R}^3$,

1. $\rho_m^0(\cdot) : V_m \to V_m$,

$$\begin{aligned} \rho_m^0\left(\frac{e_1}{2}\right) z_m^k &= i\left(k - \frac{m}{2}\right) z_m^k , \\ \rho_m^0\left(\frac{e_2}{2} + i\frac{e_3}{2}\right) z_m^k &= -k z_m^{k-1} , \\ \rho_m^0\left(\frac{e_2}{2} - i\frac{e_3}{2}\right) z_m^k &= (m-k) z_m^{k+1} . \end{aligned} \tag{3.7}$$

2. $\rho_m^+(\cdot) : V_m \to V_{m+2}$,

$$\begin{aligned} \rho_m^+\left(\frac{e_1}{2}\right) z_m^k &= i z_{m+2}^{k+1} , \\ \rho_m^+\left(\frac{e_2}{2} + i\frac{e_3}{2}\right) z_m^k &= -z_{m+2}^{k} , \\ \rho_m^+\left(\frac{e_2}{2} - i\frac{e_3}{2}\right) z_m^k &= -z_{m+2}^{k+2} . \end{aligned} \tag{3.8}$$

3. $\rho_m^-(\cdot) : V_m \to V_{m-2}$,

$$\begin{aligned} \rho_m^-\left(\frac{e_1}{2}\right) z_m^k &= ik(m-k) z_{m-2}^{k-1} , \\ \rho_m^-\left(\frac{e_2}{2} + i\frac{e_3}{2}\right) z_m^k &= k(k-1) z_{m-2}^{k-2} , \\ \rho_m^-\left(\frac{e_2}{2} - i\frac{e_3}{2}\right) z_m^k &= (m-k)(m-k-1) z_{m-2}^{k} . \end{aligned} \tag{3.9}$$

We remark that ρ_m^0 is the representation (ρ_m, V_m) of $\mathfrak{su}(2)$ under the isomorphism $\mathfrak{su}(2) \simeq \mathbf{R}^3$ and ρ_1^0 is the usual Clifford action on the spinor space V_1.

Now, we shall investigate some properties of the Clifford homomorphisms.

LEMMA 3.2. *For X in $\mathbf{R}^3 \simeq \mathfrak{su}(2)$, we have*

$$(\rho_m^0(X))^* = -\rho_m^0(X) , \tag{3.10}$$

$$(\rho_m^\pm(X))^* = -\rho_{m\pm 2}^\mp(X) , \tag{3.11}$$

where we denote by A^* the adjoint operator of A such that $\langle Av, w\rangle = \langle v, A^* w\rangle$.

PROOF. Because ρ_m^0 is the representation of $\mathfrak{su}(2)$, the relation (3.10) is trivial. So we shall prove that $(\rho_m^+(X))^* = -\rho_{m+2}^-(X)$. We take the complexification of (3.11) and may prove $(\rho_m^+(X+iY))^* = -\rho_{m+2}^-(X) + i\rho_{m+2}^-(Y)$. For example, we have

$$\begin{aligned} \left\langle \rho_m^+\left(\frac{\sigma_2}{2} - i\frac{\sigma_3}{2}\right) z_m^k, z_{m+2}^l \right\rangle &= -\langle z_{m+2}^{k+2}, z_{m+2}^l\rangle \\ &= -(k+2)!(m-k)!\delta_{k+2,l} , \quad \text{for any } k, l. \end{aligned}$$

On the other hands,

$$\begin{aligned} \left\langle z_m^k, -\rho_m^-\left(\frac{\sigma_2}{2} + i\frac{\sigma_3}{2}\right) z_{m+2}^l \right\rangle &= -l(l-1)\langle z_m^k, z_m^{l-2}\rangle \\ &= -l(l-1)k!(m-k)!\delta_{k,l-2} \\ &= -(k+2)!(m-k)!\delta_{k+2,l} , \quad \text{for any } k, l. \end{aligned}$$

So we have $(\rho_m^+(\sigma_2 - i\sigma_3))^* = -\rho_m^-(\sigma_2) + i\rho_m^-(\sigma_3)$. Similarly we can prove the other cases. □

LEMMA 3.3. *For X in $\mathbf{R}^3 \simeq \mathfrak{su}(2)$ and g in $SU(2)$, we have*

$$\rho_m^0(gXg^{-1}) = \rho_m(g)\rho_m^0(X)\rho_m(g^{-1}), \tag{3.12}$$

$$\rho_m^\pm(gXg^{-1}) = \rho_{m\pm2}(g)\rho_m^\pm(X)\rho_m(g^{-1}). \tag{3.13}$$

PROOF. The equation (3.12) is trivial. So we shall prove (3.13). For an orthonormal basis $\{v_{m+2}^k\}_k$ of V_{m+2}, we denote the corresponding one of the irreducible component V_{m+2} in $V_m \otimes V_2$ by $\{\omega_{m+2}^k\}_k$. Since ρ_m^+ is the orthogonal projection from $V_m \otimes V_2$ to V_{m+2}, the homomorphism ρ_m^+ is represented as

$$\rho_m^+(X)v = \sum_k \langle v \otimes X, \omega_{m+2}^k\rangle v_{m+2}^k, \tag{3.14}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $V_m \otimes V_2$. If we use another orthonormal basis $\{\rho_{m+2}(g)v_{m+2}^k\}_k$, then we have

$$\rho_m^+(X)v = \sum_k \langle v \otimes X, (\rho_m \otimes \rho_2)(g)\omega_{m+2}^k\rangle \rho_{m+2}(g)v_{m+2}^k.$$

It follows that

$$\begin{aligned}\rho_m^+(gXg^{-1})v &= \sum \langle v \otimes gXg^{-1}, \omega_{m+2}^k\rangle v_{m+2}^k \\ &= \sum \langle v \otimes \rho_2(g)X, \omega_{m+2}^k\rangle v_{m+2}^k \\ &= \sum \langle (\rho_m \otimes \rho_2)(g)(\rho_m(g^{-1})v \otimes X), \omega_{m+2}^k\rangle v_{m+2}^k \\ &= \sum \langle \rho_m(g^{-1})v \otimes X, (\rho_m \otimes \rho_2)(g^{-1})\omega_{m+2}^k\rangle v_{m+2}^k \\ &= \sum \langle \rho_m(g^{-1})v \otimes X, \omega_{m+2}^k\rangle \rho_{m+2}(g)v_{m+2}^k \\ &= \rho_{m+2}(g)\rho_m^+(X)\rho_m(g^{-1})v.\end{aligned}$$

Thus we have proved the lemma. □

The infinitesimal version of this lemma is given as follows.

LEMMA 3.4. *For X, Y in $\mathbf{R}^3 \simeq \mathfrak{su}(2)$, it holds that*

$$\rho_m^0([X, Y]) = [\rho_m^0(X), \rho_m^0(Y)], \tag{3.15}$$

$$\rho_m^\pm([X, Y]) = \rho_{m\pm2}^0(X)\rho_m^\pm(Y) - \rho_m^\pm(Y)\rho_m^0(X), \tag{3.16}$$

where $[\cdot, \cdot]$ denotes the Lie bracket in $\mathfrak{su}(2)$.

Now, we know that the usual Clifford actions $\{\sigma_i\}_i = \{e_i \cdot\}_i$ satisfy the relations

$$\sigma_i\sigma_j + \sigma_i\sigma_j = -2\delta_{ij} \quad (0 \le i, j \le 3). \tag{3.17}$$

We should find what relations the Clifford homomorphisms satisfy.

LEMMA 3.5. *The Clifford homomorphisms have the following relations: for X, Y in $\mathbf{R}^3 \simeq \mathfrak{su}(2)$,*

$$\rho^0_{m+2}(X)\rho^+_m(Y) - \rho^+_m(X)\rho^0_m(Y) = \frac{m+2}{2}\rho^+_m([X, Y]), \tag{3.18}$$

$$\rho^0_{m-2}(X)\rho^-_m(Y) - \rho^-_m(X)\rho^0_m(Y) = -\frac{m}{2}\rho^-_m([X, Y]), \tag{3.19}$$

$$\rho^0_m(X)\rho^0_m(Y) + \rho^+_{m-2}(X)\rho^-_m(Y) = \frac{m}{2}\rho^0_m([X, Y]) - m^2\langle X, Y\rangle, \tag{3.20}$$

$$\rho^0_m(X)\rho^0_m(Y) + \rho^-_{m+2}(X)\rho^+_m(Y) = -\frac{m+2}{2}\rho^0_m([X, Y]) - (m+2)^2\langle X, Y\rangle, \tag{3.21}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbf{R}^3$.

PROOF. By direct calculations. □

We remark that, for $m = 1$, the relation (3.20) is the usual Clifford relation (3.17).

4. The higher spin bundles and the higher spin Dirac operators.

Let M be a 3-dimensional spin manifold without boundary and $\mathbf{Spin}(M)$ be a spin structure, which is a principal bundle on M with the structure group $Spin(3) = SU(2)$. We remark that, if M is a 3-dimensional closed oriented Riemannian manifold, then M is automatically a spin manifold. Now, all the associated complex vector bundles are induced from the representations of $SU(2)$. For any $m \geq 0$, we define the spin-$\frac{m}{2}$ bundle $\mathbf{S}_m$ by

$$\mathbf{S}_m = \mathbf{S}_m(M) := \mathbf{Spin}(M) \times_{\rho_m} V_m. \tag{4.1}$$

The inner product on V_m induces the one on each fiber of $\mathbf{S}_m$ naturally, which we denote by $\langle\cdot,\cdot\rangle$ on $(\mathbf{S}_m)_x$. For example, the spin-0 bundle $\mathbf{S}_0$ is the trivial rank 1 bundle $M \times \mathbf{C} \simeq \Lambda^0(M)\otimes\mathbf{C}$, the spin-$\frac{1}{2}$ bundle $\mathbf{S}_1$ is the spinor bundle, and the spin-1 bundle $\mathbf{S}_2$ is $T(M)\otimes\mathbf{C} \simeq \Lambda^1(M)\otimes\mathbf{C}$.

The spinor bundle $\mathbf{S}_1$ is known as a bundle of modules over the Clifford bundle $\mathbf{C}l(M)$ and the action of $T(M)$ on $\mathbf{S}_1$ is given by

$$T(M) \times \mathbf{S}_1 \ni ([p, e_i], [p, v]) \mapsto [p, e_i \cdot v] \in \mathbf{S}_1, \tag{4.2}$$

where p is in $\mathbf{Spin}(M)$, e_i in $\mathbf{R}^3$, and v in V_1. In the same way, we define the Clifford homomorphisms of $T(M)$ on the higher spin bundle $\mathbf{S}_m$ as follows:

$$T(M) \times \mathbf{S}_m \ni ([p, e_i], [p, v]) \mapsto [p, \rho^0_m(e_i)v] \in \mathbf{S}_m, \tag{4.3}$$

$$T(M) \times \mathbf{S}_m \ni ([p, e_i], [p, v]) \mapsto [p, \rho^\pm_m(e_i)v] \in \mathbf{S}_{m\pm 2}. \tag{4.4}$$

We can easily check from Lemma 3.3 that these bundle homomorphisms are well-defined.

Before considering the higher spin Dirac operators on $\Gamma(M, \mathbf{S}_m)$, we recall the definition of the Dirac operator D on $\Gamma(M, \mathbf{S}_1)$. Let ∇ be the covariant derivative associated to the spin connection. The Dirac operator D has the following (local) formula:

$$D = \sum_{i=1}^{3} e_i \cdot \nabla_{e_i} . \tag{4.5}$$

On the other hand, we know another description of D as follows: the Dirac operator D is the composed mapping $\mathrm{pr} \circ \nabla$,

$$\Gamma(M, \mathbf{S}_1) \xrightarrow{\nabla} \Gamma(M, \mathbf{S}_1 \otimes T^*(M)) \simeq \Gamma(M, \mathbf{S}_1 \otimes T(M)) \xrightarrow{\mathrm{pr}} \Gamma(M, \mathbf{S}_1) , \tag{4.6}$$

where we use $\mathbf{S}_1 \otimes T(M) \simeq \mathbf{S}_1 \otimes \mathbf{S}_2 \simeq \mathbf{S}_3 \oplus \mathbf{S}_1$.

We generalize this composed mapping to give the higher spin Dirac operator (see [3], [4] and [6]). Since the tensor bundle $\mathbf{S}_m \otimes \mathbf{S}_2$ is isomorphic to $\mathbf{S}_{m+2} \oplus \mathbf{S}_m \oplus \mathbf{S}_{m-2}$, we have three composed mappings for each bundle:

$$D_m^0 : \Gamma(M, \mathbf{S}_m) \xrightarrow{\nabla} \Gamma(M, \mathbf{S}_m \otimes T^*M) \xrightarrow{\mathrm{pr}^0} \Gamma(M, \mathbf{S}_m) , \tag{4.7}$$

$$D_m^{\pm} : \Gamma(M, \mathbf{S}_m) \xrightarrow{\nabla} \Gamma(M, \mathbf{S}_m \otimes T^*M) \xrightarrow{\mathrm{pr}^{\pm}} \Gamma(M, \mathbf{S}_{m\pm 2}) . \tag{4.8}$$

We call these first order differential operators *the higher spin Dirac operators*. In [6], Fegan shows that these operators are conformally invariant first order differential operators and all such operators are given in this way. The Clifford homomorphisms in section 3 lead us to represent the higher spin Dirac operators by local formulas such as (4.5).

PROPOSITION 4.1. *Let M be the 3-dimensional spin manifold, $\{e_i\}_{1\le i\le 3}$ a local orthonormal frame of $T(M)$, and ∇ the covariant derivative associated to the spin connection on $\mathbf{S}_m$. Then we have the following conformally invariant first order differential operators*:

$$D_m^0 = \sum_{1\le i\le 3} \rho_m^0(e_i)\nabla_{e_i} : \Gamma(M, \mathbf{S}_m) \to \Gamma(M, \mathbf{S}_m) , \tag{4.9}$$

$$D_m^{\pm} = \sum_{1\le i\le 3} \rho_m^{\pm}(e_i)\nabla_{e_i} : \Gamma(M, \mathbf{S}_m) \to \Gamma(M, \mathbf{S}_{m\pm 2}) . \tag{4.10}$$

EXAMPLE 4.1. Some higher spin Dirac operators are well-known differential operators.

1. D_0^+ is $2d$ on $\Gamma(M, \mathbf{S}_0) = \Gamma(M, \Lambda^0(M) \otimes \mathbf{C})$.
2. D_1^0 is the Dirac operator D and D_1^+ is the twistor operator on $\Gamma(M, \mathbf{S}_1)$.
3. D_2^0 is $2*d$ and D_2^- is $2d^*$ on $\Gamma(M, \mathbf{S}_2) = \Gamma(M, \Lambda^1(M)\otimes\mathbf{C})$, where $*$ is the Hodge star operator from $\Lambda^1(M)$ to $\Lambda^2(M)$.

From the discussion in section 3, we can derive some properties of the higher spin Dirac operators.

First, we discuss the adjointness of the operators. On $\Gamma(M, \mathbf{S}_m)$, we set the inner product by

$$(\phi_1, \phi_2) := \int_M \langle \phi_1(x), \phi_2(x) \rangle dx , \tag{4.11}$$

where dx denotes the volume element of M.

PROPOSITION 4.2. *We denote the formal adjoint of a differential operator A by A^*. Then we have*

$$(D_m^0)^* = D_m^0 , \tag{4.12}$$
$$(D_m^{\pm})^* = D_{m\pm 2}^{\mp} . \tag{4.13}$$

In particular, D_m^0 is formally self-adjoint.

PROOF. We can easily show that the Dirac operator is formally self-adjoint (for example, see [9]). In the same way, we can prove (4.12) and (4.13) by using Lemma 3.2. □

Next, we shall discuss the commutativity among the operators. So we have to introduce some curvature homomorphisms. For vector fields X and Y, the curvature R_m for $\mathbf{S}_m$ is given by

$$R_m(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \in \Gamma(M, \mathrm{End}(\mathbf{S}_m)) \tag{4.14}$$
$$= \frac{1}{4} \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \langle R(X, Y)(e_{\sigma(1)}), e_{\sigma(2)} \rangle \rho_m^0(e_{\sigma(3)}) , \tag{4.15}$$

where $R(\cdot, \cdot)$ is the curvature transformation for $T(M)$ and $\{e_i\}_{1 \le i \le 3}$ is a local orthonormal frame on $T(M)$. Then we obtain the following curvature homomorphisms from $\mathbf{S}_m$ to $\mathbf{S}_m$ or $\mathbf{S}_{m\pm 2}$:

$$R_m^0 := \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \rho_m^0(e_{\sigma(1)}) R_m(e_{\sigma(2)}, e_{\sigma(3)}) \in \Gamma(M, \mathrm{End}(\mathbf{S}_m)) , \tag{4.16}$$
$$R_m^{\pm} := \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \rho_m^{\pm}(e_{\sigma(1)}) R_m(e_{\sigma(2)}, e_{\sigma(3)}) \in \Gamma(M, \mathrm{Hom}(\mathbf{S}_m, \mathbf{S}_{m\pm 2})) . \tag{4.17}$$

Here, we show that $(R_m^0)^* = R_m^0$ and $(R_m^{\pm})^* = R_{m\pm 2}^{\mp}$. In particular, $(R_m^0)_x$ has real eigenvalues for each x in M.

EXAMPLE 4.2. Let Ric be the Ricci curvature and κ the scalar curvature. Then we have

$$R_1^0 = \frac{1}{2}\kappa , \quad R_2^0 = 4\mathrm{Ric} , \quad R_0^0 = R_2^- = R_0^+ = 0 . \tag{4.18}$$

The commutativity among the higher spin Dirac operators follows from Lemma 3.5. The important fact is that we have two Laplace type operators on $\Gamma(M, \mathbf{S}_m)$ for each $m \ge 1$.

THEOREM 4.3. *Let $\nabla^*\nabla$ be the connection Laplacian on $\mathbf{S}_m$. Then the higher spin Dirac operators satisfy the following Bochner type identities*:

$$D^0_m D^0_m + D^+_{m-2} D^-_m = m^2 \nabla^*\nabla + \frac{m}{2} R^0_m , \tag{4.19}$$

$$D^0_m D^0_m + D^-_{m+2} D^+_m = (m+2)^2 \nabla^*\nabla - \frac{m+2}{2} R^0_m , \tag{4.20}$$

$$D^0_{m+2} D^+_m - D^+_m D^0_m = \frac{m+2}{2} R^+_m , \tag{4.21}$$

$$D^0_{m-2} D^-_m - D^-_m D^0_m = -\frac{m}{2} R^-_m . \tag{4.22}$$

PROOF. We shall prove (4.19). We fix x in M and choose an orthonormal frame $\{e_i\}$ in a neighborhood of x such that $(\nabla_{e_i} e_j)_x = 0$ for all i, j. Hence, we have $(\nabla_{e_i} \rho^0_m(e_j))_x = 0$ for all i, j. Then it holds from Lemma 3.5 that

$$\begin{aligned}
&D^0_m D^0_m + D^+_{m-2} D^-_m \\
&= \sum_{i,j} \left(\rho^0_m(e_i) \nabla_{e_i} \rho^0_m(e_j) \nabla_{e_j} + \rho^+_{m-2}(e_i) \nabla_{e_i} \rho^-_m(e_j) \nabla_{e_j} \right) \\
&= \sum_{i} \left(\rho^0_m(e_i) \rho^0_m(e_i) + \rho^+_{m-2}(e_i) \rho^-_m(e_i) \right) \nabla_{e_i} \nabla_{e_i} \\
&\quad + \sum_{i \neq j} \left(\rho^0_m(e_i) \rho^0_m(e_j) + \rho^+_{m-2}(e_i) \rho^-_m(e_j) \right) \nabla_{e_i} \nabla_{e_j} \\
&= -m^2 \sum_i \nabla_{e_i} \nabla_{e_i} + \frac{m}{2} \sum_{i<j} \rho^0_m(e_i e_j - e_j e_i)(\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) \\
&= m^2 \nabla^*\nabla + \frac{m}{2} R^0_m .
\end{aligned}$$

□

EXAMPLE 4.3. 1. (the case of $m = 0$) The relation (4.20) means $d^*d = \nabla^*\nabla$ and the relation (4.21) does $dd = 0$.

2. (the case of $m = 1$) The relation (4.19) means

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa . \tag{4.23}$$

3. (the case of $m = 2$) The relation (4.19) means

$$d^*d + dd^* = \nabla^*\nabla + \mathrm{Ric} , \tag{4.24}$$

and (4.22) does $dd = 0$.

Now, we denote the Laplace type operators in (4.19) and (4.20) by

$$\Delta_m := D^0_m D^0_m + D^+_{m-2} D^-_m , \tag{4.25}$$

$$\tilde{\Delta}_m := D^0_m D^0_m + D^-_{m+2} D^+_m . \tag{4.26}$$

If M is compact, these Laplace type operators are non-negative operators and satisfy that

$$\ker \Delta_m = \ker D_m^0 \cap \ker D_m^- , \tag{4.27}$$

$$\ker \tilde{\Delta}_m = \ker D_m^0 \cap \ker D_m^+ , \tag{4.28}$$

$$\ker \nabla = \ker \Delta_m \cap \ker \tilde{\Delta}_m = \ker D_m^0 \cap \ker D_m^- \cap \ker D_m^+ . \tag{4.29}$$

The following corollary is the key to give lower bounds for the first eigenvalues of Δ_m and $\tilde{\Delta}_m$.

COROLLARY 4.4. *The Laplace type operators Δ_m and $\tilde{\Delta}_m$ satisfy that*

$$(m+2)^2\Delta_m - m^2\tilde{\Delta}_m = m(m+1)(m+2)R_m^0 . \tag{4.30}$$

PROOF. We eliminate the connection Laplacian $\nabla^*\nabla$ from (4.19) and (4.20). □

Finally, we discuss the ellipticity of the operators. Of course, it is clear that Δ_m and $\tilde{\Delta}_m$ are elliptic.

PROPOSITION 4.5. 1. *The second order differential operator $D_{m+2}^- D_m^+ = (D_m^+)^* D_m^+$ is elliptic for each m.*

2. *If m is odd, then the first order differential operator D_m^0 is elliptic. Hence D_m^0 is an elliptic self adjoint operator.*

PROOF. We investigate the ellipticity of D_m^0. The principal symbol of D_m^0 is

$$\sigma_\xi(D_m^0) = \rho_m^0(\xi) , \tag{4.31}$$

where $\xi = \sum \xi_i e_i$ is in $T_x^*(M) \simeq T_x(M)$. There exists g in $SU(2)$ such that

$$g\xi g^{-1} = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}} e_1 . \tag{4.32}$$

Then we have

$$\begin{aligned} \det \sigma_\xi(D_m^0) &= \det \rho_m(g)\rho_m^0(\xi)\rho_m(g^{-1}) \\ &= \det \rho_m^0(g\xi g^{-1}) \\ &= \det \rho_m^0((\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}} e_1) \\ &= (\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{m+1}{2}} \det \rho_m^0(e_1) \\ &= (\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{m+1}{2}} \prod_{k=0}^{m} i(2k-m) . \end{aligned} \tag{4.33}$$

It follows that, if m is odd, then $\det \sigma_\xi(D_m^0)$ is not zero for $\xi \neq 0$. Hence D_m^0 is elliptic. In the same way, we verify that $D_{m+2}^- D_m^+$ is elliptic. □

COROLLARY 4.6. *We assume that the spin manifold M is compact. Then* ker D_m^+ *and* ker D_{2p+1}^0 *are finite dimensional vector spaces for any m and p.*

5. Lower bounds for the first eigenvalues of the higher spin Dirac operators.

In this section, we assume that M is a 3-dimensional closed spin manifold . From Corollary 4.4, we have

$$(m+2)^2(\Delta_m\phi,\phi) - m^2(\tilde{\Delta}_m\phi,\phi) = m(m+1)(m+2)(R_m^0(\phi),\phi)\,, \tag{5.1}$$

where ϕ is a section of $\mathbf{S}_m$ and

$$(R_m^0(\phi),\phi) := \int_M \langle (R_m^0)_x\phi(x),\phi(x)\rangle dx\,. \tag{5.2}$$

From the above equation (5.1), we can obtain lower bound estimations for the eigenvalues of Δ_m and $\tilde{\Delta}_m$ depending on the curvature transformation R_m^0.

First, we consider a lower bound for the first eigenvalue of the Dirac operator $D = D_1^0$. It follows from (5.1) that, for a spinor ϕ in $\Gamma(M,\mathbf{S}_1)$,

$$\begin{aligned} &9\|D\phi\|^2 - (\|D\phi\|^2 + \|D_1^+\phi\|^2) \\ =&8\|D\phi\|^2 - \|D_1^+\phi\|^2 \\ =&6(R_1^0(\phi),\phi) = 3(\kappa\phi,\phi)\,. \end{aligned} \tag{5.3}$$

Because of $\|D_1^+\phi\| \geq 0$, we have

$$\|D\phi\|^2 \geq \frac{3}{8}(\kappa\phi,\phi)\,. \tag{5.4}$$

If ϕ_1 is an eigenspinor with the first eigenvalue λ_1 of D, then $(\lambda_1)^2$ has a lower bound,

$$(\lambda_1)^2 \geq \frac{3(\kappa\phi_1,\phi_1)}{8\|\phi_1\|^2} \geq \frac{3}{8}\kappa_-. \tag{5.5}$$

where

$$\kappa_- := \min_{x\in M}\kappa(x)\,. \tag{5.6}$$

If the equality holds in (5.5), then ϕ_1 is in ker D_1^+, that is, ϕ_1 is a twistor spinor. This inequality coincides with the one given by Friedrich (see [2]).

Next, we investigate the case of the elliptic operator $D_3^- D_1^+ = (D_1^+)^* D_1^+$. It holds that

$$\begin{aligned} (D_3^- D_1^+\phi,\phi) &= 8\|D\phi\|^2 - 3(\kappa\phi,\phi) \\ &\geq -3(\kappa\phi,\phi). \end{aligned} \tag{5.7}$$

If we denote the first eigenvalue of $D_3^- D_1^+$ by μ_1, then we have

$$\mu_1 \geq -3\kappa_+\,, \tag{5.8}$$

where

$$\kappa_+ = \max_{x\in M}\kappa(x)\,. \tag{5.9}$$

In general case ($m \geq 2$), we have the inequalities

$$(\Delta_m \phi, \phi) = \|D_m^0 \phi\|^2 + \|D_m^- \phi\|^2 \geq \frac{m(m+1)}{m+2}(\mathrm{R}_m^0(\phi), \phi), \tag{5.10}$$

$$(\tilde{\Delta}_m \phi, \phi) = \|D_m^0 \phi\|^2 + \|D_m^+ \phi\|^2 \geq -\frac{(m+2)(m+1)}{m}(\mathrm{R}_m^0(\phi), \phi). \tag{5.11}$$

Then we give lower bounds for the first eigenvalues of Δ_m and $\tilde{\Delta}_m$.

THEOREM 5.1. *We assume that there exist constants r_{m-} and r_{m+} such that*

$$r_{m-}\|\phi\|^2 \leq (R_m^0(\phi), \phi) \leq r_{m+}\|\phi\|^2 \tag{5.12}$$

for any ϕ in $\Gamma(M, \mathbf{S}_m)$.

1. *Let λ_1 be the first eigenvalue of Δ_m. Then we have the inequality*

$$\lambda_1 \geq \frac{m(m+1)}{m+2} r_{m-}\,. \tag{5.13}$$

If the equality holds in (5.13), *the eigenvectors with the eigenvalue λ_1 are in* $\ker \tilde{\Delta}_m$.

2. *Let μ_1 be the first eigenvalue of $\tilde{\Delta}_m$. Then we have the inequality*

$$\mu_1 \geq -\frac{(m+2)(m+1)}{m} r_{m+}\,. \tag{5.14}$$

If the equality holds in (5.14), *then the eigenvectors with the eigenvalue μ_1 are in* $\ker \Delta_m$.

COROLLARY 5.2 ([7]). *We assume that there exists a constant* ric_- *such that*

$$(\mathrm{Ric}(\phi), \phi) \geq \mathrm{ric}_- \|\phi\|^2 \tag{5.15}$$

for any ϕ in $\Gamma(M, \Lambda^1(M))$. Let λ_1 be the first eigenvalue of the Laplace-Beltrami operator $dd^* + d^*d$ on $\Gamma(M, \Lambda^1(M))$. Then we have

$$\lambda_1 \geq \frac{3}{2}\mathrm{ric}_-\,. \tag{5.16}$$

If the equality holds in (5.16), *the eigenforms with the eigenvalue λ_1 are in* $\ker \tilde{\Delta}_2 = \ker d \cap \ker D_2^+$.

6. On the 3-dimensional manifold of constant curvature.

In this section, we shall discuss the higher spin Dirac operators on the 3-dimensional manifold of constant curvature.

LEMMA 6.1. *On the 3-dimensional spin manifold M of constant curvature c, the curvature homomorphism R_m^0 is $m(m+2)c$ and $R_m^{\pm}$ is zero.*

PROOF. Since M has constant curvature, it holds that, for vector fields X, Y, and Z,

$$R(X, Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}.$$

Then we have $\langle R(e_i, e_j)e_k, e_l\rangle = c(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})$. Hence,

$$R^0_m = -c\sum \rho^0_m(e_i)\rho^0_m(e_i) = m(m+2)c\,, \quad R^{\pm}_m = 0\,.$$

Here, we use that $-\sum \rho^0_m(\sigma_i)\rho^0_m(\sigma_i)$ is the Casimir operator on V_m. □

PROPOSITION 6.2. *On the 3-dimensional spin manifold of constant curvature c, it holds that*

$$D^0_m D^0_m + D^+_{m-2}D^-_m = m^2\nabla^*\nabla + \frac{m^2(m+2)}{2}c\,, \tag{6.1}$$

$$D^0_m D^0_m + D^-_{m+2}D^+_m = (m+2)^2\nabla^*\nabla - \frac{m(m+2)^2}{2}c\,, \tag{6.2}$$

$$D^0_{m+2}D^+_m - D^+_m D^0_m = 0\,, \tag{6.3}$$

$$D^0_{m-2}D^-_m - D^-_m D^0_m = 0\,. \tag{6.4}$$

In particular, we have

$$\Delta_m D^0_m = D^0_m\Delta_m\,, \quad \tilde{\Delta}_m D^0_m = D^0_m\tilde{\Delta}_m\,, \tag{6.5}$$

$$\Delta_m(D^+_{m-2}D^-_m) = (D^+_{m-2}D^-_m)\Delta_m\,, \quad \tilde{\Delta}_m(D^-_{m+2}D^+_m) = (D^-_{m+2}D^+_m)\tilde{\Delta}_m\,. \tag{6.6}$$

We conclude from this proposition that Δ_m, D^0_m, and $D^+_{m-2}D^-_m$ are simultaneously diagonalizable. As an example, we will calculate the eigenvalues of these operators on S^3 in the next section.

7. The spectra of the higher spin Dirac operators on S^3.

In this section, we calculate all the eigenvalues of the higher spin Dirac operators on the symmetric space S^3 with constant curvature 1. In [8], the author gives a method for calculating of the eigenvalues and the eigenspinors for the Dirac operator on S^3. We can use the same method in our situation and calculate the eigenvalues. So we refer to the paper [8] for details.

First, we shall explain the 3-dimensional sphere S^3 as the symmetric space $Spin(4)/Spin(3)$. It is well-known that $Spin(4)$ and $Spin(3)$ are isomorphic to $SU(2)\times SU(2)$ and $SU(2)$, respectively. We realize S^3 as $SU(2)$,

$$S^3 \ni x = (x_1, x_2, x_3, x_4) \mapsto h = \begin{pmatrix} x_4 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_4 - ix_1 \end{pmatrix} \in SU(2)\,. \tag{7.1}$$

Therefore, the action of $SU(2)\times SU(2)$ on S^3 is represented by

$$(SU(2)\times SU(2))\times S^3 \ni (g, h) \mapsto phq^{-1} \in S^3, \tag{7.2}$$

where $g = (p, q)$ is in $SU(2)\times SU(2)$. Since the isotropy subgroup of $e = (0,0,0,1)$ is the subgroup $SU(2)$ in $SU(2)\times SU(2)$, we have the symmetric space S^3,

$$S^3 = Spin(4)/Spin(3) = SU(2)\times SU(2)/\mathrm{diag}SU(2)\,. \tag{7.3}$$

Here, the map 'diag' is given by

$$\mathrm{diag} : SU(2) \ni h \mapsto (h, h) \in SU(2)\times SU(2)\,. \tag{7.4}$$

The principal spin bundle $\mathbf{Spin}(S^3)$ is the Lie group $Spin(4)$, whose projection from the total space to the base space is

$$\mathbf{Spin}(S^3) = Spin(4) \ni g \mapsto pq^{-1} \in S^3 . \tag{7.5}$$

This principal spin bundle induces the spin $-\frac{m}{2}$ bundle $\mathbf{S}_m$ as a homogeneous vector bundle:

$$\mathbf{S}_m := Spin(4) \times_{\rho_m} V_m . \tag{7.6}$$

Hence the space of sections $L^2(S^3, \mathbf{S}_m)$ is a representation space of $Spin(4)$.

Now, we trivialize the vector bundle $\mathbf{S}_m$ as follows:

$$\mathbf{S}_m = Spin(4) \times_{\rho_m} V_m \ni [g, v] \mapsto (pq^{-1}, \rho_m(p)v) \in S^3 \times V_m . \tag{7.7}$$

So the sections of $\mathbf{S}_m$ are represented as the $\mathbf{C}^{m+1}$-valued or the V_m-valued functions on S^3. In this situation, we can present explicit formulas of the higher spin Dirac operators on S^3, where the operators act on the V_m-valued functions.

PROPOSITION 7.1. *For the trivialization* (7.7), *the higher spin Dirac operators on* S^3 *are the following*:

$$D_m^0 = \frac{m(m+2)}{2} + \sum \rho_m^0(e_i) Z_i , \tag{7.8}$$

$$D_m^{\pm} = \sum \rho_m^{\pm}(e_i) Z_i . \tag{7.9}$$

Here Z_i *is the right invariant vector field on* $S^3 = SU(2)$ *corresponding to* σ_i *in* $\mathfrak{su}(2)$, *which is given by*

$$\begin{aligned} Z_1 &= -x_1 \frac{\partial}{\partial x_4} + x_4 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} , \\ Z_2 &= -x_2 \frac{\partial}{\partial x_4} + x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} , \\ Z_3 &= -x_3 \frac{\partial}{\partial x_4} - x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} . \end{aligned} \tag{7.10}$$

COROLLARY 7.2. *The Laplace type operators* Δ_m *and* $\tilde{\Delta}_m$ *are realized as*

$$\Delta_m = -m^2 \sum Z_i^2 + m^2 D_m^0 - \frac{m^2(m+2)(m-2)}{4} , \tag{7.11}$$

$$\tilde{\Delta}_m = -(m+2)^2 \sum Z_i^2 + (m+2)^2 D_m^0 - \frac{m(m+2)^2(m+4)}{4} . \tag{7.12}$$

Since the higher spin Dirac operators on S^3 are homogeneous differential operators, the eigenspaces are representation spaces of $Spin(4)$. So we have to decompose $L^2(S^3, \mathbf{S}_m)$ into its irreducible components. By the Frobenius reciprocity, we have the following lemma.

LEMMA 7.3. *The representation space* $L^2(S^3, \mathbf{S}_m)$ *decomposes into its irreducible components as follows*:

1. (*the case of* $m = 2p+1$)

$$L^2(S^3, \mathbf{S}_{2p+1}) \simeq \bigoplus_{\substack{0 \le s \le p \\ k \ge p-s}} E_{k,k+2s+1} \oplus E_{k+2s+1,k}\,. \tag{7.13}$$

2. (*the case of* $m = 2p$)

$$L^2(S^3, \mathbf{S}_{2p}) \simeq \bigoplus_{\substack{1 \le s \le p \\ k \ge p-s}} E_{k,k+2s} \oplus E_{k+2s,k} \bigoplus_{k \ge p} E_{k,k}\,. \tag{7.14}$$

Here $E_{k,l}$ is the representation space for the outer tensor product representation $\rho_k \hat{\otimes} \rho_l$ of $Spin(4) = SU(2) \times SU(2)$ and $\dim E_{k,l} = (k+1)(l+1)$.

We calculate the action of the higher spin Dirac operators on $E_{k,l}$ by the method given in [8]. Then we have the following propositions.

PROPOSITION 7.4. 1. *The eigenvalues of the self adjoint operator D_m^0 on S^3 are given as follows*:

(a) (*the case of* $m = 2p+1$)

$$\begin{cases} (2s+1)\left(k + \frac{2s+3}{2}\right) & on\ E_{k+2s+1,k}\,, \\ -(2s+1)\left(k + \frac{2s+3}{2}\right) & on\ E_{k,k+2s+1}\,. \end{cases} \tag{7.15}$$

In particular, $\ker D_{2p+1}^0$ *is zero for each* p.

(b) (*the case of* $m = 2p$)

$$\begin{cases} 2s(k+s+1) & on\ E_{k+2s,k}\,, \\ -2s(k+s+1) & on\ E_{k,k+2s}\,, \\ 0 & on\ E_{k,k}\,. \end{cases} \tag{7.16}$$

2. *The eigenvalues of the second order operator $D_{m-2}^+ D_m^-$ on S^3 are given as follows*:

(a) (*the case of* $m = 2p+1$)

$$4(p-s)(k+1-(p-s))(p+s+1)(k+p+s+2) \quad on\ E_{k+2s+1,k}\ or\ E_{k,k+2s+1}\,. \tag{7.17}$$

(b) (*the case of* $m = 2p$)

$$4(p-s)(k+1-(p-s))(p+s)(k+p+s+1) \quad on\ E_{k+2s,k}\ or\ E_{k,k+2s}\,. \tag{7.18}$$

3. *The eigenvalues of the second order elliptic operator $D_{m+2}^- D_m^+$ on S^3 are given as follows*:

(a) (*the case of* $m = 2p+1$)

$$4(p-s+1)(k-(p-s))(p+s+2)(k+p+s+3) \quad on\ E_{k+2s+1,k}\ or\ E_{k,k+2s+1}\,. \tag{7.19}$$

(b) (*the case of* $m = 2p$)

$$4(p - s + 1)(k - (p - s))(p + s + 1)(k + p + s + 2) \quad \text{on } E_{k+2s,k} \text{ or } E_{k,k+2s}\,. \tag{7.20}$$

In particular,

$$\dim\ker D_m^+ = \frac{1}{6}(m+1)(m+2)(m+3)\,. \tag{7.21}$$

PROPOSITION 7.5. 1. *The eigenvalues of the Laplace type operator* Δ_m *are as follows*:

(a) (*the case of* $m = 2p + 1$)

$$(2s+1)^2\left(k + \frac{2s+3}{2}\right)^2 + 4(p-s)(k+1-(p-s))(p+s+1)(k+p+s+2) \quad \text{on } E_{k+2s+1,k} \text{ or } E_{k,k+2s+1}\,. \tag{7.22}$$

(b) (*the case of* $m = 2p$)

$$(2s)^2(k+s+1)^2 + 4(p-s)(k+1-(p-s))(p+s)(k+p+s+1) \quad \text{on } E_{k+2s,k} \text{ or } E_{k,k+2s}\,. \tag{7.23}$$

In particular, $\ker \Delta_m$ *is zero.*

2. *The eigenvalues of the Laplace type operator* $\tilde{\Delta}_m$ *are given as follows*:

(a) (*the case of* $m = 2p + 1$)

$$(2s+1)^2\left(k + \frac{2s+3}{2}\right)^2 + 4(p-s+1)(k-(p-s))(p+s+2)(k+p+s+3) \quad \text{on } E_{k+2s+1,k} \text{ or } E_{k,k+2s+1}\,. \tag{7.24}$$

In particular, $\ker \tilde{\Delta}_{2p+1}$ *is zero.*

(b) (*the case of* $m = 2p$)

$$(2s)^2(k+s+1)^2 + 4(p-s+1)(k-(p-s))(p+s+1)(k+p+s+2) \quad \text{on } E_{k+2s,k} \text{ or } E_{k,k+2s}\,. \tag{7.25}$$

In particular, $\dim\ker \tilde{\Delta}_{2p} = (p+1)^2$.

ACKNOWLEDGEMENTS. The author would like to thank Prof. Tosiaki Kori for his encouragement.

References

[1] H. BAUM, Th. FRIEDRICH, R. GRUNEWALD and I. KATH, *Twistors and Killing spinors on Riemannian manifolds*, Teubner Texts in Mathematics **124** (1991), Teubner-Verlag.

[2] H. BAUM and Th. FRIEDRICH, Eigenvalues of the Dirac operator, Twistors and Killing spinors on Riemannian manifold, *Clifford Algebras and Spinor Structures*, Kluwer Academic Publisher (1995), 243–256.

[3] J. BUREŠ and V. SOUČEK, Eigenvalues of conformally invariant operators on spheres, *The 18th Winter School "Geometry and Physics"*, Rend. Circ. Mat. Palermo (2) Suppl. **59** (1999), 109–122.

[4] J. BUREŠ, The higher spin Dirac operators, *Differential geometry and applications*, Masaryk Univ. (1999), 319–334.

[5] C. BÄR, The Dirac operator on homogeneous spaces and its spectrum on 3-dimensional lens spaces, Arch. Math., **59** (1992), 65–79.

[6] H. D. FEGAN, Conformally invariant first order differential operators, Quart. J. Math. Oxford **27** (1976), 371–378.

[7] S. GALLOT and D. MEYER, Opérateur de courbure et laplacien des formes différentielles d'une variété Riemannienne, J. Math. Pures Appl. **54** (1975), 259–284.

[8] Y. HOMMA, A representation of $Spin(4)$ on the eigenspinors of the Dirac operator on 3-dimensional sphere, Tokyo J. Math. **23** (2000), 453–472.

[9] H. B. LAWSON and M. L. MICHELSOHN, *Spin Geometry*, Princeton Univ. Press (1989).

[10] D. PERRONE, On the minimal eigenvalue of the Laplacian operator for p-forms in conformally flat Riemannian manifolds, Proc. A.M.S. **86** (1982), 103–108.

Present Address:
DEPARTMENT OF MATHEMATICAL SCIENCES,
WASEDA UNIVERSITY,
OHKUBO, SHINJUKU-KU, TOKYO, 169–8555 JAPAN.
e-mail: homma@gm.math.waseda.ac.jp