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Bour's Theorem in Minkowski Geometry

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Abstract. In three dimensional Minkowski space, we show that a generalized helicoid is isometric to a rotation surface so that helices on the helicoid correspond to parallel circles on the rotation surface. Moreover, if these surfaces have the same Gauss map, we can determine them.

1. Introduction.

In classical surface geometry in Euclidean space, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotation) surface which is minimal. Moreover, a pair of these two surfaces has interesting properties. That is, they are both members of a one-parameter family of isometric minimal surfaces and have the same Gauss map. This pair is a typical example of a minimal surface and its conjugate one on the Weierstrass-Enneper representation for minimal surfaces. On the other hand, the pair of the right helicoid and the catenoid has following generalization.

BOUR'S THEOREM [1], [6]. *A generalized helicoid is isometric to a rotation surface so that helices on the helicoid correspond to parallel circles on the rotation surface.*

In this generalization, original properties that they are minimal and preserve the Gauss map are not generally kept. In [3], the author find pairs of a generalized helicoid and a rotation surface that are isometric by Bour's theorem and also have the same Gauss map.

In surface theory in Minkowski space, parallel to Euclidean geometry, surfaces of vanishing mean curvature in rotation surfaces or ruled surfaces, Weierstrass-Enneper representation, etc. are studied very much. However, Bour's theorem in Minkowski space is not known.

The purpose of this paper is to give Bour's theorem in Minkowski space and to determine pairs of surfaces under an additional condition that the pair has the same Gauss map.

In Section 2, we recall some formulas to study surfaces in Minkowski geometry and give the definition of the rotation surface and the generalized helicoid. Section 3 is devoted to list rotation surfaces of vanishing mean curvature. These examples are used to find a pair of surfaces that have the same Gauss map. In Section 4, we give Bour's theorem on surfaces in

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Minkowski space whose axes are timelike or spacelike. In the last section, we give Bour's theorem on surfaces of lightlike axis.

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2. Preliminaries.

First of all, we recall elementary properties in a 3-dimensional Lorentz vector space. Let V be a 3-dimensional vector space with scalar product $\langle\ ,\ \rangle$ of index 1. Then V is called a Lorentz vector space. In the rest of this paper, we shall identify a vector X with a transporse ${}^tX$ of X. For any vectors $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ in a Lorentz vector space V the scalar product of X and Y is defined by $\langle X, Y\rangle = X_1Y_1 + X_2Y_2 - X_3Y_3$, which is called a Lorentz product. Let V be a 3-dimensional Lorentz vector space with Lorentz product $\langle\ ,\ \rangle$. A vector X in V is called *space-like* (resp. *time-like*) if $\langle X, X\rangle > 0$ or $X = 0$ (resp. $\langle X, X\rangle < 0$). If X ($\neq 0$) satisfies $\langle X, X\rangle = 0$, then X is called *light-like*.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected, unless otherwise mentioned. Let R^3_1 be a Minkowski space, namely, flat 3-dimensional Lorentz manifold. We consider a surface $S(u, v)$ with a coordinate system $\{u, v\}$ in R^3_1. The coefficients E, F and G of the first fundamental form are defined by

$$E = \langle S_u, S_u\rangle\,, \quad F = \langle S_u, S_v\rangle\,, \quad G = \langle S_v, S_v\rangle\,, \tag{2.1}$$

for the natural basis $\{S_u, S_v\}$ along the coordinate curves.

The line element is thus

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2\,.$$

The *Gauss map* e of $S(u, v)$ is defined by

$$e = \frac{S_u \times S_v}{\sqrt{|S_u \times S_v|}}\,. \tag{2.2}$$

The coefficients L, M and N of the second fundamental form are defined by

$$L = \langle S_{uu}, e\rangle\,, \quad M = \langle S_{uv}, e\rangle\,, \quad N = \langle S_{vv}, e\rangle\,, \tag{2.3}$$

and the *mean curvature* H is given by

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}\,. \tag{2.4}$$

Now we define a non-degenerate rotation surface and generalized helicoid in R^3_1. For an open interval $I \subset R$, let $\gamma : I \to \Pi$ be a curve in a plane Π in R^3_1 and let l be a straight line in Π which does not intersect the curve γ. A *rotation surface* in R^3_1 is defined as a non-degenerate surface rotating a curve γ around a line l (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis l, it simultaneously displaces parallel to l so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the *generalized helicoid*. If the profile curve of a generalized helicoid is a straight line perpendicular to the axis, then the surface is called a *right helicoid.*

Since we are concerned on R_1^3, the axis l may be space-like, time-like or light-like. As the surface is non-degenerate, it suffices to consider the case that the profile curve is space-like or time-like. We classify a surface by types of axis and profile curve and write as (Axis's type, Profile curve's type)-type; for example, by (S, T)-type we mean that the surface has a space-like axis and a time-like profile curve.

3. Rotation surface.

In this section, we give specific presentation of rotation surfaces and recall some rotation surfaces of zero mean curvature that are used to have main theorems in the latter half of this paper. For details of this section, see [4].

When the axis l is space-like, there is a Lorentz transformation by which the axis l is transformed to the x_1-axis of R_1^3. Since we consider non-degenerate surfaces, we may suppose that Π is the x_1x_2-plane or the x_1x_3-plane without loss of generality. If the profile curve is on the x_1x_2-plane, then it is space-like and parametrized as $\gamma = (\varphi(u), u, 0)$. If the profile curve is on the x_1x_3-plane, it is parametrized as $\gamma = (\varphi(u), 0, u)$. In this case, it may be space-like or time-like. Hence the rotation surface $R(u, v)$ can be written as

$$R(u,v) = \begin{bmatrix} \varphi(u) \\ u\cosh v \\ u\sinh v \end{bmatrix}, \tag{3.1}$$

or

$$R(u,v) = \begin{bmatrix} \varphi(u) \\ u\sinh v \\ u\cosh v \end{bmatrix}. \tag{3.2}$$

From (3.1) or (3.2), we can have the mean curvature H by virtue of (2.1), (2.2), (2.3) and (2.4). Solving a differential equation $H = 0$, we can get the profile curve $\varphi(u)$ of the rotation surface.

PROPOSITION 3.1. *If an* (S, S)*-type rotation surface is of zero mean curvature, then the surface is*

$$R(u,v) = \begin{bmatrix} b\cosh^{-1}(u/b) \\ u\cosh v \\ u\sinh v \end{bmatrix} \tag{3.3}$$

or

$$R(u,v) = \begin{bmatrix} b\sin^{-1}(u/b) \\ u\sinh v \\ u\cosh v \end{bmatrix}, \tag{3.4}$$

where b is a constant.

PROPOSITION 3.2. *If an* (S, T)-*type rotation surface is of zero mean curvature, then the surface is*

$$R(u,v)=\begin{bmatrix} b\sinh^{-1}(u/b) \\ u\sinh v \\ u\cosh v \end{bmatrix}, \tag{3.5}$$

where b is a constant.

When the axis l is time-like, then we may suppose that l is the x_3-axis, Π is the x_1x_3-plane and the profile curve γ is parametrized as $\gamma=(u,0,\varphi(u))$, without loss of generality. Hence the rotation surface can be written as

$$R(u,v)=\begin{bmatrix} u\cos v \\ u\sin v \\ \varphi(u) \end{bmatrix}. \tag{3.6}$$

PROPOSITION 3.3. *If a* (T, S)-*type rotation surface is of zero mean curvature, then the surface is*

$$R(u,v)=\begin{bmatrix} u\cos v \\ u\sin v \\ b\sinh^{-1}(u/v) \end{bmatrix}, \tag{3.7}$$

where b is a constant.

PROPOSITION 3.4. *If a* (T, T)-*type rotation surface is of zero mean curvature, then the surface is*

$$R(u,v)=\begin{bmatrix} u\cos v \\ u\sin v \\ b\sin^{-1}(u/v) \end{bmatrix}, \tag{3.8}$$

where b is a constant.

Last of all, if the axis l is light-like, then we may suppose that l is the line spanned by the vector $(0,1,1)$. Since the surface is non-degenerate, it suffices to consider the case that the plane Π is the x_2x_3-plane and the profile curve γ is parametrized as

$$\gamma(u)=(0,\varphi(u)+u,\varphi(u)-u)$$

without loss of generality. Hence the rotation surface can be written as

$$R(u,v)=\begin{bmatrix} 2uv \\ \varphi(u)+u-uv^2 \\ \varphi(u)-u-uv^2 \end{bmatrix}. \tag{3.9}$$

PROPOSITION 3.5. *If a rotation surface with light-like axis is of zero mean curvature, then the surface is*

$$R(u,v)=\begin{bmatrix} 2uv \\ (b/3)u^3+u-uv^2 \\ (b/3)u^3-u-uv^2 \end{bmatrix}. \tag{3.10}$$

4. Bour's theorems of surfaces of space-like or time-like axis.

In this section, we study an isometric relation between a generalized helicoid and a rotation surface of space-like or time-like axis.

Case 1. (S, S)-type, (I).

First of all, we consider the (S, S)-type surfaces, namely, the axis and the profile curve are both space-like. Moreover, first, we assume that the profile curve is on the x_1x_2-plane. Since a generalized helicoid is given by rotating the profile curve around the axis and simultaneously displacing parallel to the axis, so that the speed of displacement is proportional to the speed of rotation, from (3.1), we have the following representation of a generalized helicoid $H(u_H, v_H)$

$$H(u_H, v_H) = \begin{bmatrix} \varphi_H(u_H) + av_H \\ u_H \cosh v_H \\ u_H \sinh v_H \end{bmatrix}, \tag{4.1}$$

where a is a constant. For a moment, we assume that $\varphi'_H \neq 0$.

The coefficients E_H, F_H, and G_H of the first fundamental form and the line element ds_H^2 of the generalized helicoid (4.1) are given by

$$E_H = \varphi'^2_H + 1, \quad F_H = a\varphi'_H, \quad G_H = a^2 - u_H^2, \tag{4.2}$$

$$ds_H^2 = (\varphi'^2_H + 1)ds_H^2 + 2a\varphi'_H du_H dv_H + (a^2 - u_H^2)dv_H^2, \tag{4.3}$$

by virtue of (2.1). Since

$$E_H G_H - F_H^2 = -u^2(1 + \varphi'^2_H) + a^2,$$

if $0 < u_H^2 < a^2/(1 + \varphi'^2_H)$ (resp. $u_H^2 > a^2/(1 + \varphi'^2_H)$) then $H(u_H, v_H)$ is space-like (resp. time-like).

Helices in $H(u_H, v_H)$ are curves defined by $u_H =$ const., so curves in $H(u_H, v_H)$ that are orthogonal to helices satisfy the orthogonal condition

$$a\varphi'_H du_H + (a^2 - u_H^2)dv_H = 0.$$

Hence it follows that

$$dv_H = -\frac{a\varphi'_H}{a^2 - u_H^2} du_H$$

so

$$v_H = -\int \frac{a\varphi'_H}{a^2 - u_H^2} du_H + \text{const.}.$$

Therefore, if we put

$$\bar{v}_H = v_H + \int \frac{a\varphi'_H}{a^2 - u_H^2} du_H, \tag{4.4}$$

then curves that are orthogonal to helices are given by $\bar{v}_H =$ const.. Substituting the equation

$$dv_H = d\bar{v}_H + \frac{a\varphi'_H}{u_H^2 - a^2} du_H$$

into (4.3), we have

$$ds_H^2 = \left(1 + \frac{u_H^2 \varphi_H'^2}{u_H{}^2 - a^2}\right) d_H^2 - (u_H^2 - a^2) d\bar{v}_H^2 . \tag{4.5}$$

Case (i). $u_H^2 - a^2 > 0$.

First we assume $u_H^2 - a^2 > 0$. By putting

$$\bar{u}_H = \int \sqrt{1 + \frac{u_H^2 \varphi_H'^2}{u_H{}^2 - a^2}} du_H , \quad f_H(\bar{u}_H) = \sqrt{u_H^2 - a^2} , \tag{4.6}$$

(4.5) reduces to

$$ds_H^2 = d\bar{u}_H^2 - f_H^2(\bar{u}_H) d\bar{v}^2 . \tag{4.7}$$

On the other hand, an (S, S)-type rotation surface

$$R(u_R, v_R) = \begin{bmatrix} \varphi_R(u_R) \\ u_R \cosh v_R \\ u_R \sinh v_R \end{bmatrix}$$

has the line element

$$ds_R^2 = (\varphi_R'^2 + 1) du_R^2 - u_R^2 dv_R^2 , \tag{4.8}$$

by virtue of (2.1). Hence, if we put

$$\bar{u}_R = \int \sqrt{\varphi_R'^2 + 1} du_R , \quad u_R = f_R(\bar{u}_R) , \quad \bar{v}_R = v_R , \tag{4.9}$$

then (4.8) reduces to

$$ds_R^2 = d\bar{u}_R^2 - f_R^2(\bar{u}_R) d\bar{v}_R^2 . \tag{4.10}$$

Comparing (4.7) with (4.10), if

$$\bar{u}_H = \bar{u}_R , \quad \bar{v}_H = \bar{v}_R , \quad f_H(\bar{u}_H) = f_R(\bar{u}_R) , \tag{4.11}$$

then we have an isometry between $H(u_H, v_H)$ and $R(u_R, v_R)$. Therefore it follows that

$$\int \sqrt{1 + \frac{u_H^2 \varphi_H'^2}{u_H^2 - a^2}} du_H = \int \sqrt{\varphi_R'^2 + 1} du_R = \int \sqrt{\varphi_R'^2 + 1} \cdot \frac{u_H}{\sqrt{u_H^2 - a^2}} du_H ,$$

and

$$\varphi_R'^2 = \frac{\sqrt{u_H^2 \varphi_H'^2 - a^2}}{u_H} .$$

Hence

$$\int \varphi_R' du_R = \int \sqrt{\frac{u_H{}^2 \varphi_H'^2 - a^2}{u_H^2 - a^2}} du_H .$$

Therefore we have

THEOREM 4.1. *A generalized helicoid*

$$\begin{bmatrix} \varphi(u)+av \\ u\cosh v \\ u\sinh v \end{bmatrix} \qquad (a=\text{const.},\ u^2>a^2)$$

is isometric to the rotation surface

$$\begin{bmatrix} \int\sqrt{\frac{u^2\varphi'^2-a^2}{u^2-a^2}}\,du \\ \sqrt{u^2-a^2}\cosh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sinh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix}, \tag{4.12}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

Next we study Gauss maps of (4.1) and (4.12). From (4.1), we obtain

$$H_{uu}=\begin{bmatrix}\varphi''\\0\\0\end{bmatrix},\quad H_{vv}=\begin{bmatrix}0\\u\cosh v\\u\sinh v\end{bmatrix},\quad H_{uv}=\begin{bmatrix}0\\\sinh v\\\cosh v\end{bmatrix}. \tag{4.13}$$

Hence the Gauss map e_H of the generalized helicoid is

$$e_H=\frac{1}{\sqrt{u^2-a^2+u^2\varphi'^2}}\begin{bmatrix}u\\a\sinh v-u\varphi'\cosh v\\-u\varphi'\sinh v+a\cosh v\end{bmatrix}. \tag{4.14}$$

From (4.13) and (4.14), the coefficients L_H, M_H and N_H of the second fundamental form are given as

$$L_H=\frac{u\varphi''}{\sqrt{u^2-a^2+u^2\varphi'^2}},\quad M_H=\frac{-a}{\sqrt{u^2-a^2+u^2\varphi'^2}},\quad N_H=-\frac{u^2\varphi'}{\sqrt{u^2-a^2+u^2\varphi'^2}}.$$

Hence the mean curvature H_H of the generalized helicoid is

$$H_H=\frac{-u^2\varphi'^3-u^2\varphi'+a^2u\varphi''+2a^2\varphi'}{2(u^2-a^2+u^2\varphi'^2)^{3/2}}. \tag{4.15}$$

Next we calculate geometrical objects of the rotation surface (4.12). Since

$$R_u=\begin{bmatrix}\sqrt{\frac{u^2\varphi'^2-a^2}{u^2-a^2}}\\ \frac{u}{\sqrt{u^2-a^2}}\cosh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right)-\frac{a\varphi'}{\sqrt{u^2-a^2}}\sinh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right)\\ \frac{u}{\sqrt{u^2-a^2}}\sinh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right)-\frac{a\varphi'}{\sqrt{u^2-a^2}}\cosh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right)\end{bmatrix},$$

$$R_v=\begin{bmatrix}0\\\sqrt{u^2-a^2}\sinh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right)\\\sqrt{u^2-a^2}\cosh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right)\end{bmatrix},$$

the Gauss map e_R of the rotation surface is

$$e_R = \frac{1}{\sqrt{u^2\varphi'^2 + u^2 - a^2}} \begin{bmatrix} u \\ -\sqrt{u^2\varphi'^2 - a^2}\cosh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \\ -\sqrt{u^2\varphi'^2 - a^2}\sinh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix}. \tag{4.16}$$

Moreover, we have

$$R_{uu} = \begin{bmatrix} \frac{-a^2u\varphi'^2+u^4\varphi'\varphi''-a^2u^2\varphi'\varphi''+a^2u}{(u^2-a^2)^{3/2}\sqrt{u^2\varphi'^2-a^2}} \\ \frac{a^2\varphi'^2-a^2}{(u^2-a^2)^{3/2}}\cosh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) - \frac{a\varphi''}{\sqrt{u^2-a^2}}\sinh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \\ \frac{-a\varphi''}{\sqrt{u^2-a^2}}\cosh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) + \frac{a^2\varphi'^2-a^2}{(u^2-a^2)^{3/2}}\sinh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix},$$

$$R_{vv} = \begin{bmatrix} 0 \\ \sqrt{u^2-a^2}\cosh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sinh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix},$$

$$R_{uv} = \begin{bmatrix} 0 \\ \frac{u}{\sqrt{u^2-a^2}}\sinh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) - \frac{a\varphi'}{\sqrt{u^2-a^2}}\cosh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \\ \frac{u}{\sqrt{u^2-a^2}}\cosh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) - \frac{a\varphi'}{\sqrt{u^2-a^2}}\sinh\left(v - \int \frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix}.$$

Then the coefficients L_R, M_R and N_R of the second fundamental form are

$$L_R = \frac{-a^2u^2\varphi'^2 + u^5\varphi'\varphi'' - a^2u^3\varphi'\varphi'' + a^2u^2 - (a^2\varphi'^2 - a^2)(u^2\varphi'^2 - a^2)}{(u^2-a^2)^{3/2}\sqrt{u^2\varphi'^2 - a^2}\sqrt{u^2\varphi'^2 + u^2 - a^2}},$$

$$M_R = \frac{a\varphi'\sqrt{u^2\varphi'^2 - a^2}}{\sqrt{u^2\varphi'^2 - a^2}\sqrt{u^2\varphi'^2 + u^2 - a^2}}, \qquad N_R = \frac{-\sqrt{u^2-a^2}\sqrt{u^2\varphi'^2 - a^2}}{\sqrt{u^2\varphi'^2 + u^2 - a^2}}.$$

Hence the mean curvature H_R is

$$H_R = \frac{-u^2\varphi'(2a^2\varphi' - u^2\varphi'^3 - u^2\varphi' - u^3\varphi'' + a^2u\varphi'')}{2\sqrt{u^2-a^2}\sqrt{u^2\varphi'^2 - a^2}(u^2 - \varphi'^2 + u^2 - a^2)^{3/2}}. \tag{4.17}$$

If the generalized helicoid and the rotation surface have the same Gauss map, comparing (4.14) with (4.16), we obtain

$$a = \sqrt{u^2\varphi'^2 - a^2}\sinh\left(\int \frac{a\varphi'}{u^2-a^2}du\right), \tag{4.18}$$

$$u\varphi' = \sqrt{u^2\varphi'^2 - a^2}\cosh\left(\int \frac{a\varphi'}{u^2-a^2}du\right). \tag{4.19}$$

By differentiating (4.18), it follows that

$$0 = (u\varphi'^2 + u^2\varphi'\varphi'')\frac{a}{u^2\varphi'^2 - a^2} + \frac{au\varphi'^2}{u^2 - a^2},$$

by virtue of (4.18) and (4.19). Therefore we have

$$u^2\varphi' - 2a^2\varphi' + u^3\varphi'' - a^2u\varphi'' + u^2\varphi'^3 = 0. \tag{4.20}$$

From (4.17), this means that the mean curvature of the rotation surface is zero. Hence, from (3.3) and (4.12), it follows that

$$b\cosh^{-1}\left(\frac{\sqrt{u^2 - a^2}}{b}\right) = \int\sqrt{\frac{u^2\varphi'^2 - a^2}{u^2 - a^2}}du,$$

by virtue of (4.6), (4.9) and (4.11). Differentiating this equation, we have

$$u^2\varphi'^2 = \frac{(a^2 + b^2)u^2 - a^4 - a^2b^2}{u^2 - a^2 - b^2}. \tag{4.21}$$

By substituting (4.21) into (4.20), it follows that

$$\left(u^2 - a^2 + \frac{b^2u^2}{u^2 - a^2 - b^2}\right)\varphi' + (u^3 - a^2u)\varphi'' = 0,$$

and

$$-\frac{\varphi''}{\varphi'} = -\frac{u^2 - a^2 + \frac{b^2u^2}{u^2-a^2-b^2}}{u^3 - a^2u}.$$

Hence it follows that

$$-\log\varphi' = \log c\frac{u\sqrt{u^2 - a^2 - b^2}}{\sqrt{u^2 - a^2}},$$

where c is a constant. Hence

$$\varphi' = c\frac{\sqrt{u^2 - a^2}}{u\sqrt{u^2 - a^2 - b^2}}.$$

By comparing this equation with (4.21), it follows that

$$\varphi' = \sqrt{a^2 + b^2}\frac{\sqrt{u^2 - a^2}}{u\sqrt{u^2 - a^2 - b^2}}.$$

To integrate this equation, we put

$$t = \sqrt{\frac{u^2 - a^2}{u^2 - a^2 - b^2}}.$$

Then

$$\varphi = \int\sqrt{a^2 + b^2}\frac{\sqrt{u^2 - a^2}}{u\sqrt{u^2 - a^2 - b^2}}du = \int\frac{-b^2t^2}{((a^2 + b^2)t^2 - a^2)(t^2 - 1)}dt$$
$$= \frac{a}{\sqrt{a^2 + b^2}}\log\sqrt{\frac{a - \sqrt{a^2 + b^2}t}{a + \sqrt{a^2 + b^2}t}} + \log\sqrt{\frac{t + 1}{t - 1}}.$$

Therefore we have the following.

THEOREM 4.2. *In Theorem* 4.1, *if two surfaces have the same Gauss map, then they are*

$$\begin{bmatrix} \varphi(u)+au \\ u\cosh v \\ u\sinh v \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \sqrt{1-a^2}\cosh^{-1}\left(\frac{\sqrt{u^2-a^2}}{\sqrt{1-a^2}}\right) \\ \sqrt{u^2-a^2}\cosh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sinh\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix},$$

where

$$\varphi(u) = a\log\sqrt{\frac{a\sqrt{u^2-a^2-b^2}-\sqrt{a^2+b^2}\sqrt{u^2-a^2}}{a\sqrt{u^2-a^2-b^2}+\sqrt{a^2+b^2}\sqrt{u^2-a^2}}} + \sqrt{a^2+b^2}\log\sqrt{\frac{\sqrt{u^2-a^2}+\sqrt{u^2-a^2-b^2}}{\sqrt{u^2-a^2}-\sqrt{u^2-a^2-b^2}}}.$$

Case (ii). $a^2/(1+{\varphi'_H}^2) < u_H^2 < a^2$.
Next, assume $a^2/(1+{\varphi'_H}^2) < u_H^2 < a^2$. In this case, by putting

$$\bar{u}_H = \int\sqrt{\frac{u_H^2{\varphi'_H}^2}{a^2-u_H^2}-1}\,du_H\,, \quad f_H(\bar{u}_H) = \sqrt{a^2-u_H^2}\,,$$

we have from (4.5)

$$ds_H^2 = -d\bar{u}_H^2 + f_H^2(\bar{u})d\bar{v}^2\,.$$

Recall an (S, T)-type rotation surface is of the form

$$R(u_R, v_R) = \begin{bmatrix} \varphi_R(u_R) \\ u_R\sinh v_R \\ u_R\cosh v_R \end{bmatrix}.$$

So we have the following.

THEOREM 4.3. *A generalized helicoid*

$$\begin{bmatrix} \varphi(u)+av \\ u\cosh v \\ u\sinh v \end{bmatrix} \quad \left(a = (const.),\ \frac{a^2}{1+{\varphi'_H}^2} < u_H^2 < a^2\right)$$

is isometric to the rotation surface

$$\begin{bmatrix} -\int\sqrt{\frac{a^2-u^2\varphi'^2}{a^2-u^2}}du \\ \sqrt{a^2-u^2}\sinh\left(v+\int\frac{a\varphi'}{a^2-u^2}du\right) \\ \sqrt{a^2-u^2}\cosh\left(v+\int\frac{a\varphi'}{a^2-u^2}du\right) \end{bmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

THEOREM 4.4. *In Theorem 4.3, if these two surfaces have the same Gauss map, then they are*

$$\begin{bmatrix} \varphi(u)+av \\ u\cosh v \\ u\sinh v \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{1-a^2}\sinh^{-1}\left(\frac{\sqrt{u^2-a^2}}{\sqrt{1-a^2}}\right) \\ \sqrt{a^2-u^2}\sinh\left(v+\int \frac{a\varphi'}{a^2-u^2}du\right) \\ \sqrt{a^2-u^2}\cosh\left(v+\int \frac{a\varphi'}{a^2-u^2}du\right) \end{bmatrix},$$

where

$$\varphi(u) = a\log\sqrt{\frac{a\sqrt{a^2+b^2-u^2}+\sqrt{a^2+b^2}\sqrt{a^2-u^2}}{a\sqrt{a^2+b^2-u^2}-\sqrt{a^2+b^2}\sqrt{a^2-u^2}}} + \sqrt{a^2+b^2}\log\sqrt{\frac{\sqrt{a^2-u^2}-\sqrt{a^2+b^2-u^2}}{\sqrt{a^2-u^2}+\sqrt{a^2+b^2-u^2}}}\,.$$

Case (iii). $0 < u_H^2 < a^2/(1+(\varphi_H')^2)$.

Finally, we consider the case $0 < u_H^2 < a^2/(1+(\varphi_H')^2)$. In this case, (4.5) reduces to

$$ds_H^2 = du_H^2 + f_H^2(\bar{u})d\bar{v}^2\,.$$

Recall an (S, T)-type rotation surface is of the form

$$R(u_R, v_R) = \begin{bmatrix} \varphi_R(u_R) \\ u_R\sinh v_R \\ u_R\cosh v_R \end{bmatrix}.$$

So, by similar calculations as above, we have the following.

THEOREM 4.5. *A generalized helicoid*

$$\begin{bmatrix} \varphi(u)+av \\ u\cosh v \\ u\sinh v \end{bmatrix} \quad \left(a = const.,\ 0 < u^2 < \frac{a^2}{1+{\varphi_H'}^2}\right)$$

is isometric to the rotation surface

$$\begin{bmatrix} -\int\sqrt{\frac{a^2-u^2\varphi'^2}{a^2-u^2}}du \\ \sqrt{a^2-u^2}\sinh\left(v+\int \frac{a\varphi'}{a^2-u^2}du\right) \\ \sqrt{a^2-u^2}\cosh\left(v+\int \frac{a\varphi'}{a^2-u^2}du\right) \end{bmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

THEOREM 4.6. *In Theorem 4.5, if these two surfaces have the same Gauss map, then they are*

$$\begin{bmatrix} \varphi(u)+av \\ u\cosh v \\ u\sinh v \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{a^2+1}\sin^{-1}\left(\frac{u}{\sqrt{a^2+1}}\right) \\ \sqrt{a^2-u^2}\sinh\left(v+\int\frac{a\varphi'}{a^2-u^2}du\right) \\ \sqrt{a^2-u^2}\cosh\left(v+\int\frac{a\varphi'}{a^2-u^2}du\right) \end{bmatrix},$$

where

$$\varphi(u) = a\sqrt{b^2-a^2}\log\sqrt{\frac{a\sqrt{u^2+b^2-a^2}-\sqrt{b^2-a^2}\sqrt{a^2-u^2}}{a\sqrt{u^2+b^2-a^2}+\sqrt{b^2-a^2}\sqrt{a^2-u^2}}} + \sqrt{b^2-a^2}\tan^{-1}\left(\sqrt{\frac{a^2-u^2}{u^2+b^2-a^2}}\right).$$

COROLLARY 4.1. *The right helicoid*

$$\begin{bmatrix} av \\ u\cosh v \\ u\sinh v \end{bmatrix}$$

is isometric to the rotation surface

$$\begin{bmatrix} a\cosh^{-1}(\sqrt{u^2-a^2}/a) \\ \sqrt{u^2-a^2}\cosh v \\ \sqrt{u^2-a^2}\sinh v \end{bmatrix} \quad (\text{when } u^2 > a^2)$$

or

$$\begin{bmatrix} a\sin^{-1}(\sqrt{a^2-u^2}/a) \\ \sqrt{a^2-u^2}\sinh v \\ \sqrt{a^2-u^2}\cosh v \end{bmatrix} \quad (\text{when } 0 < u^2 < a^2).$$

Moreover the mean curvatures of two surfaces are zero and the Gauss maps of them are identical.

We continue to study the other cases; the rest of (S, T)-type, (S, T)-type, (T, S)-type and (T, T)-type. But the techniques of proofs are similar, so we only sketch proofs of them.

Case 2. (S, S)-type, (II).

We assume that the profile curve is on the x_1x_3-plane.

THEOREM 4.7. *A generalized helicoid*

$$\begin{bmatrix} \varphi(u)+av \\ u\sinh v \\ u\cosh v \end{bmatrix}$$

is isometric to the rotation surface

$$\begin{bmatrix} \int \sqrt{\frac{u^2\varphi'^2-a^2}{u^2+a^2}}du \\ \sqrt{u^2+a^2}\sinh\left(v+\int \frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\cosh\left(v+\int \frac{a\varphi'}{u^2+a^2}du\right) \end{bmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

In this theorem, if $a^2/(\varphi'-1) > u^2$ (resp. $a^2/(\varphi'-1) < u^2$) then the surfaces are time-like (resp. space-like) with $ds^2 = -du^2 + f^2(u)dv^2$ (resp. $ds^2 = du^2 + f^2(u)dv^2$) for a suitable coordinate $\{u, v\}$ and a function $f(u)$. In both cases we get the same representation.

THEOREM 4.8. *In Theorem* 4.7, *if these two surfaces have the same Gauss map, then they are*

(i) *if* $u^2 < a^2/(\varphi'-1)$,

$$\begin{bmatrix} \varphi(u)+av \\ u\sinh v \\ u\cosh v \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{1-a^2}\sinh^{-1}\left(\frac{\sqrt{u^2+a^2}}{\sqrt{1-a^2}}\right) \\ \sqrt{u^2+a^2}\sinh\left(v+\int \frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\cosh\left(v+\int \frac{a\varphi'}{u^2+a^2}du\right) \end{bmatrix},$$

where

$$\varphi(u) = a\log\sqrt{\frac{a\sqrt{u^2+a^2-b^2}-\sqrt{a^2-b^2}\sqrt{u^2+a^2}}{a\sqrt{u^2+a^2}+\sqrt{a^2-b^2}\sqrt{u^2+a^2}}} + \sqrt{a^2-b^2}\log\sqrt{\frac{\sqrt{u^2+a^2}+\sqrt{u^2+a^2-b^2}}{\sqrt{u^2+a^2}-\sqrt{u^2+a^2-b^2}}}.$$

(ii) *if* $a^2/(\varphi'^2-1) < u^2$,

$$\begin{bmatrix} \varphi(u)+av \\ u\sinh v \\ u\cosh v \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{a^2+1}\sin^{-1}\left(\frac{\sqrt{u^2+a^2}}{\sqrt{a^2+1}}\right) \\ \sqrt{u^2+a^2}\sinh\left(v+\int \frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\cosh\left(v+\int \frac{a\varphi'}{u^2+a^2}du\right) \end{bmatrix},$$

where

$$\varphi(u) = a\log\sqrt{\frac{a\sqrt{u^2+a^2-b^2}-\sqrt{a^2-b^2}\sqrt{u^2+a^2}}{a\sqrt{u^2+a^2}+\sqrt{a^2-b^2}\sqrt{u^2+a^2}}} + \sqrt{a^2-b^2}\log\sqrt{\frac{\sqrt{u^2+a^2}+\sqrt{u^2+a^2-b^2}}{\sqrt{u^2+a^2}-\sqrt{u^2+a^2-b^2}}}.$$

Case 3. (S, T)-type.

THEOREM 4.9. *A generalized helicoid*

$$\begin{bmatrix} \varphi(u)+av \\ u\sinh v \\ u\cosh v \end{bmatrix}$$

is isometric to the rotation surface

$$\begin{bmatrix} \int\sqrt{\frac{u^2\varphi'^2-a^2}{u^2+a^2}}du \\ \sqrt{u^2+a^2}\sinh\left(v+\int\frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\cosh\left(v+\int\frac{a\varphi'}{u^2+a^2}du\right) \end{bmatrix}$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

In this case, the surfaces are time-like with $ds^2 = -du^2 + f^2(u)dv^2$ for a suitable coordinate $\{u, v\}$ and a function $f(u)$.

THEOREM 4.10. *In Theorem 4.9, if these two surfaces have the same Gauss map, then they are*

$$\begin{bmatrix} \varphi(u)+av \\ u\sinh v \\ u\cosh v \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \sqrt{1-a^2}\sinh^{-1}\left(\frac{\sqrt{u^2+a^2}}{\sqrt{1-a^2}}\right) \\ \sqrt{u^2+a^2}\sinh\left(v+\int\frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{u^2+a^2}\cosh\left(v+\int\frac{a\varphi'}{u^2+a^2}du\right) \end{bmatrix},$$

where

$$\varphi(u) = a\log\sqrt{\frac{a\sqrt{u^2+a^2+b^2}-\sqrt{a^2+b^2}\sqrt{u^2+a^2}}{a\sqrt{u^2+a^2+b^2}+\sqrt{a^2+b^2}\sqrt{u^2+a^2}}} + \sqrt{a^2+b^2}\log\sqrt{\frac{\sqrt{u^2+a^2}+\sqrt{u^2+a^2+b^2}}{\sqrt{u^2+a^2}-\sqrt{u^2+a^2+b^2}}}.$$

Case 4. (T, S)-type.

THEOREM 4.11. *A generalized helicoid*

$$\begin{bmatrix} u\cos v \\ u\sin v \\ \varphi(u)+av \end{bmatrix}$$

is isometric to the rotation surface

$$\begin{bmatrix} \sqrt{u^2-a^2}\cos\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sin\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \int\sqrt{\frac{u^2\varphi'^2+a^2}{u^2-a^2}}du \end{bmatrix} \quad (\textit{when } a^2 < u^2)$$

or

$$\begin{bmatrix} -\int\sqrt{\frac{(a^2-u^2)-u^2(1-\varphi'^2)}{a^2-u^2}}du \\ \sqrt{a^2-u^2}\cosh\left(v+\int\frac{a\varphi'}{a^2-u^2}du\right) \\ \sqrt{a^2-u^2}\sinh\left(v+\int\frac{a\varphi'}{a^2-u^2}du\right) \end{bmatrix} \quad (\textit{when } u^2 < a^2)$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

In the case $a^2 < u^2$ (resp. $u^2 < a^2$) of this theorem, the surfaces are space-like (resp. time-like) of $ds^2 = du^2 + f^2(u)dv^2$ (resp. $ds^2 = -du^2 + f^2(u)dv^2$) for a suitable coordinate $\{u, v\}$ and a function $f(u)$.

THEOREM 4.12. *In the case $a^2 < u^2$ of Theorem 4.11, if these two surfaces have the same Gauss map, then they are*

(i) *If $a^2/(1-\varphi'^2) < u^2$,*

$$\begin{bmatrix} u\cos v \\ u\sin v \\ \varphi(u)+av \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \sqrt{u^2-a^2}\cos\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sin\left(v-\int\frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{a^2+1}\sinh^{-1}\left(\frac{\sqrt{a^2-u^2}}{\sqrt{a^2+1}}\right) \end{bmatrix},$$

where

$$\varphi(u) = -a\tan^{-1}\left(\frac{\sqrt{b^2-a^2}}{a}\frac{\sqrt{u^2-a^2}}{\sqrt{u^2+b^2-a^2}}\right) + \sqrt{b^2-a^2}\log\sqrt{\frac{\sqrt{u^2-a^2}-\sqrt{u^2+b^2-a^2}}{\sqrt{u^2-a^2}+\sqrt{u^2+b^2-a^2}}};$$

(ii) *If $a^2 < u^2 < a^2/(1-\varphi'^2)$,*

$$\begin{bmatrix} u\cos v \\ u\sin v \\ \varphi(u)+av \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \sqrt{u^2-a^2}\cos\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sin\left(v-\int\frac{a\varphi'}{u^2+a^2}du\right) \\ \sqrt{1-a^2}\sin^{-1}\left(\frac{\sqrt{u^2-a^2}}{\sqrt{1-a^2}}\right) \end{bmatrix},$$

where

$$\varphi(u) = a\log\sqrt{\frac{a\sqrt{u^2-a^2-b^2}-\sqrt{a^2+b^2}\sqrt{u^2-a^2}}{a\sqrt{u^2-a^2-b^2}+\sqrt{a^2+b^2}\sqrt{u^2-a^2}}} + \sqrt{a^2-b^2}\log\sqrt{\frac{\sqrt{u^2-a^2}-\sqrt{u^2-a^2-b^2}}{\sqrt{u^2-a^2}+\sqrt{u^2-a^2-b^2}}}.$$

In the case $u^2 < a^2$ of Theorem 4.11, the generalized helicoid and the rotation surface have different axes and these Gauss maps are definitely different.

Case 5. (T, T)-type.

THEOREM 4.13. *A generalized helicoid*

$$\begin{bmatrix} u\cos v \\ u\sin v \\ \varphi(u)+av \end{bmatrix}$$

is isometric to the rotation surface

$$\begin{bmatrix} \sqrt{u^2-a^2}\cos\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sin\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \int\sqrt{\frac{u^2\varphi'^2+a^2}{u^2-a^2}}du \end{bmatrix} \quad (\text{when } a^2<u^2)$$

or

$$\begin{bmatrix} -\int\sqrt{\frac{(a^2-u^2)+u^2(\varphi'^2-1)}{a^2-u^2}}du \\ \sqrt{u^2-a^2}\cos\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sin\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \end{bmatrix} \quad (\text{when } u^2<a^2)$$

so that helices on the generalized helicoid correspond to parallel circles on the rotation surface.

In this theorem the surfaces are time-like with $ds^2=-du^2+f^2(u)dv^2$ (when $a^2<u^2$) or $ds^2=du^2-f^2(u)dv^2$ (when $u^2<a^2$) for a suitable coordinate $\{u,v\}$ and a function $f(u)$.

THEOREM 4.14. *In the case $u^2>a^2$ of Theorem* 4.13, *if these two surfaces have the same Gauss map, then they are*

$$\begin{bmatrix} u\cos v \\ u\sin v \\ \varphi(u)+av \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{u^2-a^2}\cos\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{u^2-a^2}\sin\left(v-\int\frac{a\varphi'}{u^2-a^2}du\right) \\ \sqrt{a^2+1}\sinh^{-1}\left(\frac{\sqrt{u^2-a^2}}{\sqrt{a^2+1}}\right) \end{bmatrix},$$

when

$$\varphi(u)=-a\tan^{-1}\left(\frac{\sqrt{b^2-a^2}}{a}\frac{\sqrt{u^2-a^2}}{\sqrt{u^2+b^2-a^2}}\right) + \sqrt{b^2-a^2}\log\sqrt{\frac{\sqrt{u^2-a^2}+\sqrt{u^2+b^2-a^2}}{\sqrt{u^2-a^2}-\sqrt{u^2+b^2-a^2}}}.$$

In the case $u^2<a^2$ of Theorem 4.13, the generalized helicoid and the rotation surface have different axes and these Gauss maps are definitely different.

5. Bour's theorem on surfaces with light-like axies.

By the argument of the last part of Section 3, we have the following representation of a generalized helicoid

$$H(u_H, v_H) = \begin{bmatrix} 2u_H v_H \\ \varphi_H(u_H) + u_H - u_H v_H^2 + a v_H \\ \varphi_H(u_H) - u_H - u_H v_H^2 + a v_H \end{bmatrix}. \tag{5.1}$$

Differentiating (5.1), we have

$$H_u = \begin{bmatrix} 2v_H \\ \varphi'_H + 1 - v_H^2 \\ \varphi'_H - 1 - v_H^2 \end{bmatrix}, \quad H_v = \begin{bmatrix} 2u_H \\ -2u_H v_H + a \\ -2u_H v_H + a \end{bmatrix}. \tag{5.2}$$

So, it follows that

$$E_H = 4u_H^2, \quad F_H = 2a, \quad G_H = 4u_H^2,$$

and

$$ds^2 = 4\varphi'_H du_H^2 + 4a du_H dv_H + 4u_H^2 dv_H^2. \tag{5.3}$$

Helices in $H(u_H, v_H)$ are defined by $u_H =$ const.. From the orthogonality condition, it follows that

$$2u_H^2 dv_H + a dv_H = 0,$$

so

$$v_H = \frac{a}{2u} + (\text{const.}).$$

Hence, if we put

$$\bar{v}_H = v_H - \frac{a}{2u_H},$$

then the curves that are orthogonal to helices are given by $\bar{v}_H =$ const.. Substituting the equation

$$dv_H = d\bar{v}_H - \frac{a}{2u^2} du_H$$

into (5.3), we obtain

$$ds_H^2 = \left(4\varphi'_H - \frac{a^2}{u_H^2}\right) du_H^2 + 4u_H^2 d\bar{v}_H^2. \tag{5.4}$$

A rotation surface of (0, 1, 1)-axis is given as

$$R(u_R, v_R) = \begin{bmatrix} 2u_R v_R \\ \varphi_R(u_R) + u_R - u_R v_R^2 \\ \varphi_R(u_R) - u_R - u_R v_R^2 \end{bmatrix}. \tag{5.5}$$

Hence, from

$$R_{u_R} = \begin{bmatrix} 2v_R \\ \varphi'_R + 1 - v_R^2 \\ \varphi'_R - 1 - v_R^2 \end{bmatrix}, \quad R_{v_R} = \begin{bmatrix} 2u_R \\ -2u_R v_R \\ -2u_R v_R \end{bmatrix}, \tag{5.6}$$

the line element of $R(u_R, v_R)$ is obtained as

$$ds_R^2 = 4\varphi'_R du_R^2 + 2u_R^2 dv_R^2 .$$

Comparing this equation with (5.3), we obtain

$$u_H = u_R , \quad v_R = \bar{v}_R , \quad \varphi_R = \varphi_H + \frac{a^2}{4u_H} .$$

Therefore we obtain

THEOREM 5.1. *A generalized helicoid*

$$\begin{bmatrix} 2uv \\ \varphi(u) + u - uv^2 + av \\ \varphi(u) - u - uv^2 + av \end{bmatrix}$$

is isometric to the rotation surface

$$\begin{bmatrix} 2uv - a \\ \varphi + u - uv^2 + av \\ \varphi - u - uv^2 + av \end{bmatrix} .$$

Differentiating (5.2) and (5.6), we can easily see that coefficients of the second fundamental forms of the generalized helicoid and the rotation surface are equal to each other. Therefore, we have

COROLLARY 5.1. *Two surfaces of Theorem* 5.1 *have the same Gauss map.*

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