TOKYO J. MATH.
VOL. 24, NO. 2, 2001

Representations of Nevanlinna-type Spaces by Weighted Hardy Spaces

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(Communicated by K. Kobayasi)

Abstract. In this paper, we shall show some representations of Nevanlinna-type spaces N^p, $1 \leqq p < \infty$, as unions of weighted H^q-spaces, $0 < q < \infty$. Moreover, we shall prove that the usual metric topology on N^p is equivalent to an inductive limit topology on N^p.

0. Introduction.

Let U be the unit disk in the complex plane and T the unit circle. The Nevanlinna class N is the class of all holomorphic functions f on U which satisfy

$$\sup_{0<r<1} \int_0^{2\pi} \log(1+|f(re^{i\theta})|)d\theta < +\infty .$$

It is well-known that each function f in N has the nontangential limit $f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ (a.e. $e^{i\theta} \in T$).

The Smirnov class N_* consists of all $f \in N$ for which

$$\lim_{r\to 1^-} \int_0^{2\pi} \log(1+|f(re^{i\theta})|)d\theta = \int_0^{2\pi} \log(1+|f^*(e^{i\theta})|)d\theta .$$

The class N^p, $p > 1$, is the class of all holomorphic functions f on U which satisfy

$$\sup_{0<r<1} \int_0^{2\pi} (\log(1+|f(re^{i\theta})|))^p d\theta < +\infty .$$

The class N^p, $p > 1$, lies between Hardy spaces H^q ($0 < q \leqq \infty$) and N_*; i.e., we have $H^q \subset N^p \subset N_* \subset N$ ($0 < q \leqq \infty$, $p > 1$). These including relations are proper. The notion of N^p was introduced by Stoll [9] and has been explored by several authors (see [1], [2] and [7]). N and its subspaces (N_*, N^p and H^q) are called *Nevanlinna-type spaces*. In this note, the symbol N^1 is used to denote the Smirnov class N_*.

Helson [3, 4] and Eoff [2] represented N^p, $1 \leqq p < \infty$, as a union of weighted H^2-spaces respectively. In this paper, we show some extensions of their result of N^p. Moreover,

Received March 23, 2000
Revised April 24, 2001

by using our representations, we shall prove that the usual metric topology on N^p is equivalent to an inductive limit topology on N^p.

The author wishes to express his sincere gratitude to his thesis advisor Professor N. Mochizuki, who introduced this subject and kindly directed him. Thanks go also to Professor Y. Ohno, who has encouraged him and suggested many improvements to this note.

1. Preliminaries.

Recall that an outer function F for the class N is of the form

$$F(z) = a \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \psi(e^{i\theta})d\theta\right), \tag{1.1}$$

where $\psi \geqq 0$, $\log \psi \in L^1(T)$ and $a \in T$.

It is well-known that $f \in N^1$ is factored as $f = BSF$, where B is the Blaschke product determined by the zeros of f, S is a singular inner function and F is an outer function for N.

Mochizuki [7] introduced outer functions for the class N^p, $p > 1$, of the form (1.1) with $\log^+ \psi \in L^p(T)$. After that Eoff [2] proved that $f \in N^p$ if and only if $f = BSF$, where F is an outer function for the class N^p.

Note that f is in N^1 if and only if it can be expressed as the quotient g/h, where g and h are in H^q $(0 < q \leqq \infty)$, and h is an outer function for N.

Let $(N^p)^{-1}$ denote the class of all invertible elements of N^p. When $q = 2$ and $q = \infty$, Eoff [2] proved $N^p = \{g/h : g, h \in H^q, h \in (N^p)^{-1}\}$ for $p > 1$.

From Eoff's result, we easily have the following:

LEMMA 1.1. *Let $1 \leqq p < \infty$ and $0 < q \leqq \infty$. Then*

$$N^p = \left\{\frac{g}{h} : g, h \in H^q, h \in (N^p)^{-1}\right\}.$$

2. Union of weighted Hardy spaces.

In this section, we shall show that N^p may be expressed as a union of certain weighted Hardy spaces.

Let w be a weight (i.e., nonnegative L^1-function on T) and denote by W_p the class of weights w satisfying $\log w \in L^p(T)$ for $1 \leqq p < \infty$. We also denote by $H^q(w)$, $0 < q < \infty$, the closure of the polynomials in $L^q(w d\theta)$.

Using these weighted Hardy spaces, we can characterize N^p as follows:

THEOREM 2.1. *Let $1 \leqq p < \infty$ and $0 < q < \infty$. Then $H^q(|h|^q) = H^q(w)$ for $h \in H^q \cap (N^p)^{-1}$ and $w \in W_p$. Moreover, we have*

$$N^p = \bigcup_{h \in H^q \cap (N^p)^{-1}} H^q(|h|^q) = \bigcup_{w \in W_p} H^q(w). \tag{2.1}$$

The proof requires a well-known result (see [8, Theorem 7]).

LEMMA 2.2. *For $f \in N^1$, f is invertible if and only if f is an outer function for the class N.*

LEMMA 2.3. *Let $1 \leqq p < \infty$, $0 < q < \infty$ and $h \in H^q \cap (N^p)^{-1}$. Then $f \in H^q(|h|^q)$ if and only if $f \in N^p$ and $f^* \in L^q(|h^*(e^{i\theta})|^q d\theta)$.*

PROOF. From Lemma 2.2, h is an outer function for the class N.

Let $f \in H^q(|h|^q)$, then $f^* \in L^q(|h^*(e^{i\theta})|^q d\theta)$. Therefore if $g = fh$, then $g \in H^q \subset N^p$. Since N^p is an algebra, so $f = g \cdot 1/h \in N^p$.

Conversely, if $f \in N^p$ and $f^* \in L^q(|h^*(e^{i\theta})|^q d\theta)$, then $f^*h^* \in L^q(T)$, so that $fh \in H^q$. And rh is in H^q for any polynomial r.

Since

$$\int_0^{2\pi} |f^*(e^{i\theta})h^*(e^{i\theta}) - r^*(e^{i\theta})h^*(e^{i\theta})|^q d\theta = \int_0^{2\pi} |f^*(e^{i\theta}) - r^*(e^{i\theta})|^q |h^*(e^{i\theta})|^q d\theta$$

and $\{rh : r \text{ is a polynomial}\}$ is dense in H^q ([5, p. 79]), we observe that f belongs to the $L^q(|h^*(e^{i\theta})|^q d\theta)$-closure of the polynomials, i.e., $f \in H^q(|h|^q)$. q.e.d.

PROOF OF THEOREM 2.1. If $h \in H^q \cap (N^p)^{-1}$, then we have $|h^*(e^{i\theta})|^q \in W_p$. Therefore we observe one inclusion. On the other hand, let

$$h(z) = \exp\left(\frac{1}{2\pi q}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log w(e^{i\theta}) d\theta\right),$$

where $w \in W_p$. Then we see that $h \in H^q \cap (N^p)^{-1}$ and $|h^*(e^{i\theta})|^q \in W_p$. It follows that the reverse inclusion is also true.

To show the first equality in (2.1), let $f \in N^p$. From Lemma 1.1, $f = g/h$ with $g, h \in H^q$ and $h \in (N^p)^{-1}$, so that $fh = g \in H^q$.

Since

$$\int_0^{2\pi} |g^*(e^{i\theta})|^q d\theta = \int_0^{2\pi} |f^*(e^{i\theta})|^q |h^*(e^{i\theta})|^q d\theta,$$

we have $f \in L^q(|h^*(e^{i\theta})|^q d\theta)$. By Lemma 2.3, we get $f \in H^q(|h|^q)$. The converse inclusion is clear.

The second equality in (2.1) is the consequence of $H^q(|h|^q) = H^q(w)$. q.e.d.

3. Equivalent topologies.

In the rest of this paper, we show that the metric topology on N^p, $1 \leqq p < \infty$, is equivalent to another topology on N^p.

Let $1 \leqq p < \infty$. Recall that the metric d_p on N^p is defined by

$$d_p(f, g) = \left\{\frac{1}{2\pi}\int_0^{2\pi} [\log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}} \quad (f, g \in N^p).$$

We usually deal with the topological structure on N^p of the metric topology τ_p induced by d_p.

By virtue of Theorem 2.1, we can induce an inductive limit topology on N^p. We define V_λ, the neighborhood of zero in N^p, as follows:

$$\{V_\lambda \mid V_\lambda \cap H^q(w) \text{ is a neighborhood of zero in } H^q(w) \text{ for any } w \in W_p\}.$$

This inductive limit topology is denoted by $I_{p,q}$. We are inspired to generalize the result of McCarthy [6] and Eoff [2].

THEOREM 3.1. *Let* $1 \leqq p < \infty$ *and* $0 < q < \infty$. *Then* $I_{p,q}$ *and* τ_p *are equivalent on* N^p.

The proof of this theorem requires the following result, which is proved in [9, Theorem 4.4].

LEMMA 3.2. *A function* $f \in N^p$, $p > 1$, *is invertible if and only if* $f(z) = \exp g(z)$, *where* $g(z) \in H^p$.

PROOF OF THEOREM 3.1 (cf. [2, 6]). We restrict our attention to the case where $1 \leqq q < \infty$, because the proof is similar for $0 < q < 1$.

Let $V \in \tau_p$ be the neighborhood of zero given by

$$V = \{g \in N^p \mid d_p(g, 0) < 4\varepsilon\},$$

for an $\varepsilon > 0$. We have to show that $V \cap H^q(|h|^q)$ is a neighborhood of zero in $H^q(|h|^q)$ for any $h \in H^q \cap (N^p)^{-1}$. Since $h \in (N^p)^{-1}$, there exists a $\delta_1 > 0$ such that

$$\frac{1}{2\pi}\int_E \left[\log^+\left|\frac{1}{h^*(e^{i\theta})}\right|\right]^p d\theta < \varepsilon^p$$

whenever $|E| < \delta_1$. Let us define ε_1, β, δ_2, δ and U_h as follows:

$$\varepsilon_1 = \min\{\varepsilon, \delta_1\}, \quad \beta^q = \inf_{|E|=\varepsilon_1}\left\{\frac{1}{2\pi}\int_E |h^*(e^{i\theta})|^q d\theta\right\}, \quad \delta_2 = \varepsilon_1\beta,$$

$$\delta = \min\left\{\delta_2, \left(\frac{e\varepsilon q}{p}\right)^{\frac{p}{q}}\right\}, \quad \text{and} \quad U_h = \{g \in N^p \mid \|gh\|_q < \delta\}.$$

Let $g \in U_h$. Since

$$\frac{1}{2\pi}\int_0^{2\pi} |g^*(e^{i\theta})|^q |h^*(e^{i\theta})|^q d\theta < \delta^q \leqq \varepsilon_1^q \beta^q,$$

we obtain that $|g| < \varepsilon_1$ except on a set of measure less than ε_1.

Let us define E_1 and E_2 by

$$E_1 = \{e^{i\theta} \in T \mid |g^*(e^{i\theta})| < \varepsilon_1\} \quad \text{and} \quad E_2 = \{e^{i\theta} \in T \mid |g^*(e^{i\theta})| \geqq \varepsilon_1\}.$$

We may assume $T = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$. In order to show $U_h \subset V$, we utilize the following elementary inequalities

$$\log(1+x) \leqq x\,, \quad \log(1+x) \leqq \log 2 + \log^+ x\,, \quad \log^+ x \leqq \frac{1}{qe}x^q\,,$$

$$\log^+ xy \leqq \log^+ x + \log^+ y\,, \quad \text{and} \quad (x+y)^{\frac{1}{p}} \leqq x^{\frac{1}{p}} + y^{\frac{1}{p}}$$

for $x, y \geqq 0$, $q > 0$, $p \geqq 1$.

If $g \in U_h$, then we obtain

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} [\log(1+|g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}}$$

$$\leqq \left\{\frac{1}{2\pi}\int_{E_1} [\log(1+|g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}} + \left\{\frac{1}{2\pi}\int_{E_2} [\log(1+|g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}}.$$

It is easy to see that the first integral on the right-hand side satisfies

$$\left\{\frac{1}{2\pi}\int_{E_1} [\log(1+|g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}} < \varepsilon\,.$$

Since $|E_2| < \varepsilon_1 \leqq \varepsilon$, we have

$$\left\{\frac{1}{2\pi}\int_{E_2} [\log(1+|g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}}$$

$$\leqq \left\{\frac{1}{2\pi}\int_{E_2}\left[\log 2 + \log^+|g^*(e^{i\theta})h^*(e^{i\theta})| + \log^+\left|\frac{1}{h^*(e^{i\theta})}\right|\right]^p d\theta\right\}^{\frac{1}{p}}$$

$$\leqq \left\{\frac{1}{2\pi}\int_{E_2}(\log 2)^p d\theta\right\}^{\frac{1}{p}} + \left\{\frac{1}{2\pi}\int_{E_2}[\log^+|g^*(e^{i\theta})h^*(e^{i\theta})|]^p d\theta\right\}^{\frac{1}{p}}$$

$$+ \left\{\frac{1}{2\pi}\int_{E_2}\left[\log^+\left|\frac{1}{h^*(e^{i\theta})}\right|\right]^p d\theta\right\}^{\frac{1}{p}}$$

$$< \varepsilon + \left\{\frac{1}{2\pi}\int_{E_2}\left(\frac{p}{qe}|g^*(e^{i\theta})h^*(e^{i\theta})|^{\frac{q}{p}}\right)^p d\theta\right\}^{\frac{1}{p}} + \varepsilon$$

$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon\,.$$

Consequently, we have

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} [\log(1+|g^*(e^{i\theta})|)]^p d\theta\right\}^{\frac{1}{p}} < \varepsilon + 3\varepsilon = 4\varepsilon\,.$$

Therefore, $U_h \subset V$; that is, $V \cap H^q(|h|^q)$ is a neighborhood of zero in $H^q(|h|^q)$, and thus $V \in I_{p,q}$.

Conversely, let $W \subset I_{p,q}$. We shall show that W contains a set V of the form

$$V = \{g \in N^p \mid d_p(g, 0) < \delta\}$$

for some $\delta > 0$. Suppose to a contrary that there exists a sequence $\{f_n\} \subset N^p$ such that $d_p(f_n, 0) < 2^{-n}$ and $f_n \notin W$ for each n. We may assume, passing to a subsequence, if necessary, that $\lim_{n\to\infty} f_n^*(e^{i\theta}) = 0$ (a.e. $e^{i\theta} \in T$). Put $w_m = \prod_{n=1}^{m}(1 + |f_n^*(e^{i\theta})|)$. Now if $m > k$,

$$\|\log w_m - \log w_k\|_p = \left\| \log \prod_{n=k+1}^{m} (1 + |f_n^*|) \right\|_p = \left\| \sum_{n=k+1}^{m} \log(1 + |f_n^*|) \right\|_p$$

$$\leqq \sum_{n=k+1}^{m} \|\log(1 + |f_n^*|)\|_p \leqq \sum_{n=k+1}^{\infty} 2^{-n} < 2^{-k}$$

so that $\{\log w_k\}$ is a Cauchy sequence in $L^p(T)$, $p \geqq 1$. Therefore there exists some $\log w \in L^p(T)$ such that $\log w_k \to \log w$ $(k \to \infty)$ in $L^p(T)$.

Now set

$$h(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log w(e^{i\theta}) d\theta\right).$$

We obtain $|h^*(e^{i\theta})| = w(e^{i\theta})$ for a.e. $e^{i\theta} \in T$, thus $h \in (N^p)^{-1}$ by Lemma 2.2 and Lemma 3.2. Even more, since $w_m \geqq 1$ is clear, so $\log w \geqq 0$. Therefore $1/h$ is bounded, i.e., $1/h \in H^\infty$. Moreover it is true that $|h^*(e^{i\theta})| = \prod_{n=1}^{\infty}(1 + |f_n^*(e^{i\theta})|)$ with $|f_n^*(e^{i\theta})| \leqq |h^*(e^{i\theta})|$, so that $|f_n^*(e^{i\theta})/h^*(e^{i\theta})|^q \leqq 1$ holds. Set $h_1 = 1/h$. Then $h_1 \in H^\infty \subset H^q$.

By the bounded convergence theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} |h_1^*(e^{i\theta}) f_n^*(e^{i\theta})|^q d\theta \to 0 \quad (n \to \infty),$$

i.e., $f_n \to 0$ in $H^q(|h_1|^q)$. Since $W \cap H^q(|h_1|^q)$ is a neighborhood of zero, we have a contradiction. Thus W must contain a metric ball centered at zero, therefore $W \in \tau_p$. q.e.d.

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