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Intrinsically n-linked Complete Graphs

Gabriel C. DRUMMOND-COLE and Danielle O'DONNOL

State University of New York at Stony Brook and University of California, Los Angeles

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Abstract. In this paper we examine the question: given $n > 1$, find a function $f : \mathbf{N} \to \mathbf{N}$ where $m = f(n)$ is the smallest integer such that K_m is intrinsically n-linked. We prove that for $n > 1$, every embedding of $K_{\lfloor \frac{7}{2}n \rfloor}$ in $\mathbf{R}^3$ contains a non-splittable link of n components. We also prove an asymptotic result, that there exists a function $f(n)$ such that $\lim_{n\to\infty} \frac{f(n)}{n} = 3$ and, for every n, $K_{f(n)}$ is intrinsically n-linked.

1. Introduction

A graph, G, is *intrinsically linked* if every embedding of G in $\mathbf{R}^3$ contains a nontrivial link. Conway and Gordon [3] and Sachs [8] first showed the existence of such graphs by proving that the complete graph on six vertices, K_6, is intrinsically linked. Sachs [8] proved that the graphs in the Petersen family are minor minimal, namely they are intrinsically linked and that no proper minor of them is intrinsically linked. Then Robertson, Seymour, and Thomas [7] proved that any intrinsically linked graph contains a graph in the Petersen family as a minor. Together these results fully characterize intrinsically linked graphs.

The concept of intrinsically linked graphs can be generalized to a graph that intrinsically contains a link of more than two components. A link L is *split* if there is an embedding of a 2-sphere F in $\mathbf{R}^3 \setminus L$ such that each component of $\mathbf{R}^3 \setminus F$ contains at least one component of L. A link that is not split is called *non-splittable* (or *non-split*). A graph G is *intrinsically n-linked* if every embedding of G in $\mathbf{R}^3$ contains a non-splittable n-component link. Flapan, Naimi, and Pommersheim investigate intrinsically 3-linked graphs (or intrinsically triple linked graphs) in [5]. They proved that K_{10} is the smallest complete graph to be intrinsically 3-linked. Bowlin and Foisy [2] also looked at intrinsically 3-linked graphs. They exhibited two different subgraphs of K_{10} that are also intrinsically 3-linked, thus proving that K_{10} is not minor minimal with respect to being intrinsically 3-linked. However, it is not known if either of these subgraphs is minor minimal. Flapan, Foisy, Naimi, and Pommersheim addressed the question of minor minimal intrinsically n-linked graphs in [4], where they constructed families of minor minimal intrinsically n-linked graphs.

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In this paper we examine the question: given $n > 1$ find a function $f : \mathbf{N} \to \mathbf{N}$ where $m = f(n)$ is the smallest integer such that K_m is intrinsically n-linked. The analogous question for complete bipartite graphs was considered in [6]. There it was shown that the complete bipartite graph $K_{2n+1,2n+1}$ (and the complete tripartite graph $K_{2n,2n,1}$) are intrinsically n-linked. It is not known if $K_{2n+1,2n+1}$ is the smallest intrinsically n-linked complete bipartite graph, however the smallest is one of the following three graphs, $K_{2n,2n}$, $K_{2n+1,2n}$, or $K_{2n+1,2n+1}$. Since $K_{2n,2n,1}$ is a subgraph of $K_{2n+2n+1}$ this gives an upper bound of $4n + 1$ for the number of vertices m needed for the smallest complete graph K_m which is intrinsically n-linked. On the other hand, based on the number of disjoint simple closed curves needed in K_m, we obtain a lower bound $m \geq 3n$. This lower bound is realized in the $n = 2$ case, but not for the $n = 3$ case, where $m = 10$ [5].

In this paper we present two bounds for m obtained through a variety of techniques. In Section 2 we prove that every embedding of K_{14} into $\mathbf{R}^3$ contains a 3-linked K_9. This is then used to prove, for $n > 1$, that $K_{\lfloor \frac{7}{2}n \rfloor}$, (where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x), is intrinsically n-linked. In Section 3 we use a combinatorial argument to prove an asymptotic bound: there exists a function $f(n)$ such that $\lim_{n\to\infty} \frac{f(n)}{n} = 3$ and, for every n, $K_{f(n)}$ is intrinsically n-linked.

These results are interesting for two reasons. For small n, the results of Section 2 together with the fact that $f(n) = m \geq 3n$ give a tight set of bounds for $f(n)$. For large n, Section 3 shows that if $f(n)$ has a coherent closed form presentation, the linear part of it must be $3n$, that is, $f(n) = 3n + o(n)$. This might also be susceptible to further refinement; at the least it indicates that "most" of the simple closed curves in a non-splittable n-component link in a minimal K_m must be triangles.

2. $K_{\lfloor \frac{7}{2}n \rfloor}$ is intrinsically n-linked

Let a simple closed curve containing exactly three vertices be called a *triangle*, and one containing exactly four vertices be called a *square*. Let a non-splittable n-component link be called an *n-link*. The edge between the vertices x and y will be denoted by $\overline{xy}$. The complete graph on n vertices with vertices labelled $v_1, v_2, \ldots, v_n$ will be denoted by $\langle v_1, v_2, \ldots, v_n \rangle$. All proofs will use mod (2) linking. So when two simple closed curves are said to link this should be taken to mean that they have non-zero linking number mod (2). Let the graph G be embedded in $\mathbf{R}^3$, and let γ be a simple closed curve in $\mathbf{R}^3 \setminus G$. We say γ *links* G if there exists a triangle J in G such that γ links J. Similarly, a simple closed curve γ *links a link* L if there is some component J of the link L such that γ links J.

In this section we prove a number of constructive lemmas and, the proposition that every embedding of K_{14} contains a 3-link of triangles. Together these results are used to prove the main result of this section, that $K_{\lfloor \frac{7}{2}n \rfloor}$ is intrinsically n-linked. The lemma below follows from the proofs in [5].

LEMMA 1. *Given an embedding of K_6 in $\mathbf{R}^3$ and a simple closed curve γ in $\mathbf{R}^3 \setminus K_6$, if γ links K_6 then one of the following holds*:

- *γ links four triangles of K_6 all of which contain a common edge*
- *γ links six triangles which are of the form pqx and prx where p, q and r are fixed vertices and x is a vertex such that $x \neq q$ and $x \neq r$*
- *γ forms a 3-link L with two of the triangles of K_6, where at least two pairs of the components of L link.*

As in [5], a simple close curve γ which links four triangles all containing the edge $\overline{pq}$ will be said to link in a *4-pattern* $\overline{pq}$. A simple closed curve γ that links six triangles which are of the form pqx and prx, where p, q and r are fixed vertices and the vertex x is such that $x \neq q$ and $x \neq r$, will be said to link in a *6-pattern* p_r^q. Though the form of the 3-link is not made explicit in [5], all proofs were done using mod (2) linking so the a 3-link where at least two of the components of the link are linked, is the only 3-link that is detectable.

LEMMA 2. *Given an embedding of K_7 in $\mathbf{R}^3$ and a simple closed curve γ in $\mathbf{R}^3 \setminus K_7$, if the curve γ links K_7 then one of the following holds:*

- *γ links five triangles of K_7 all of which contain a common edge*
- *γ links eight triangles which are of the form pqx and prx where p, q and r are fixed vertices and x is a vertex such that $x \neq q$ and $x \neq r$*
- *γ is in a 3-link L with two triangles in K_7, where at least two pairs of the components of L link.*

PROOF. The proof consists of examining all of the K_6 subgraphs of K_7 and using Lemma 1. Label the vertices of $K_7 = \langle 1, 2, 3, 4, 5, 6, 7\rangle$. If γ is in a 3-link L with two triangles, where at least two pairs of the components of L link, we are done. So suppose γ links some triangle J and γ is not in a 3-link with two triangles. Without loss of generality, J is in the complete graph on 6 vertices $\langle 1, 2, 3, 4, 5, 6\rangle$. By Lemma 1, the curve γ links $\langle 1, 2, 3, 4, 5, 6\rangle$ in a 4-pattern or a 6-pattern.

Suppose γ links $\langle 1, 2, 3, 4, 5, 6\rangle$ in a 6-pattern which, without loss of generality, we take to be the 6-pattern 1_3^2. So γ links 124, 125, 126, 134, 135, and 136. Now consider $\langle 1, 2, 3, 4, 5, 7\rangle$, then γ links 124, 125, 134 and 135. Since there is no edge that appears in all four of these triangles, γ must link $\langle 1, 2, 3, 4, 5, 7\rangle$ in a 6-pattern, either 1_3^2 or 1_5^4. Suppose that γ links $\langle 1, 2, 3, 4, 5, 7\rangle$ in the 6-pattern 1_3^2, so γ also links 127 and 137. Then γ links $\langle 1, 2, 4, 5, 6, 7\rangle$ in the 4-pattern $\overline{12}$, γ links $\langle 1, 3, 4, 5, 6, 7\rangle$ in the 4-pattern $\overline{13}$, γ links $\langle 1, 2, 3, x, y, z\rangle$ with $x, y, z \in \{4, 5, 6, 7\}$ in the 6-pattern 1_3^2, and γ does not link any triangles in $\langle 2, 3, 4, 5, 6, 7\rangle$ because all triangles in $\langle 2, 3, 4, 5, 6, 7\rangle$ appear in one of the before mentioned $K_6's$. This accounts for all of the $K_6's$ in K_7. So these eight triangles: 124, 125, 126, 127, 134, 135, 136, and 137 are all the triangles that γ links. Note that all of these triangles are of the form pqx and prx with $p = 1$, $q = 2$, $r = 3$ and $x \in \{4, 5, 6, 7\}$.

Next suppose that γ links $\langle 1, 2, 3, 4, 5, 6\rangle$ in the 6-pattern 1_3^2 and γ links $\langle 1, 2, 3, 4, 5, 7\rangle$ in the 6-pattern 1_5^4. So γ links 124, 125, 126, 134, 135, 136, 147 and 157. The simple closed curve γ links $\langle 1,2,4,5,6,7\rangle$ in the triangles 124, 125, 126, 147, and 157. So γ must link $\langle 1,2,4,5,6,7\rangle$ in the 6-pattern 1^2_7, and it links 167 in addition to the previously mentioned triangles. Then γ links $\langle 1,2,3,5,6,7\rangle$ in the 6-pattern 1^5_6, γ links $\langle 1,2,3,4,6,7\rangle$ in the 6-pattern 1^4_6, γ links $\langle 1,3,4,5,6,7\rangle$ in the 6-pattern 1^3_7,and γ does not link any of the triangles of $\langle 2,3,4,5,6,7\rangle$ because all of these triangles appear in one of the $K_6's$ mentioned above. This accounts for all of the $K_6's$ in K_7. So γ links the nine triangles: 124, 125, 126, 134, 135, 136, 147, 157, and 167. Notice that these are all of the triangles of the from $1xy$ with $x \in \{2,3,7\}$ and $y \in \{4,5,6\}$. Consider the subgraph $K_{3,3,1} \subset \langle 1,2,3,4,5,6,7\rangle$ with the three sets of vertices of the tripartite graph being $\{2,3,7\}$, $\{4,5,6\}$, and $\{1\}$. The graph $K_{3,3,1}$ is in the Petersen family, and it always contains a link of a square S and triangle T [8]. In this case $S = axby$ and $T = 1cz$ with $\{a,b,c\} = \{2,3,7\}$ and $\{x,y,z\} = \{4,5,6\}$. Thus T is one of the triangles that γ links. Since T links S, then T links one of the triangles axb or bya; call this linking triangle Q. Thus γ links T and T links Q, so $\gamma \cup T \cup Q$ forms a 3-link with two triangles. This contradicts the assumption that γ is not in a 3-link with two triangles.

Finally, suppose γ links $\langle 1,2,3,4,5,6\rangle$ in a 4-pattern, without loss of generality we may assume it is the 4-pattern $\overline{12}$, and γ does not link any K_6 in a 6-pattern. Then in $\langle 1,2,3,4,5,7\rangle$, the curve γ links 123, 124, 125, and by assumption must link in a 4-pattern. So γ links $\langle 1,2,3,4,5,7\rangle$ in the 4-pattern $\overline{12}$. Thus γ links $12x$ for all $x \in \{3,4,5,6,7\}$. There are five such triangles. As with the previous case by looking at all the $K_6's$ in K_7 we see these are the only triangles that link γ in K_7. □

If a simple closed curve γ, embedded in $\mathbf{R}^3 \setminus K_7$ links five triangles of K_7 all of which contain a common edge $\overline{pq}$, it will be said to link K_7 in a *5-pattern* $\overline{pq}$. If a simple closed curve γ links eight triangles which are of the form pqx and prx where p, q and r are fixed vertices and x is the vertex such that $x \neq q$ and $x \neq r$, the curve will be said to link K_7 in an *8-pattern* p^q_r. Lemma 2 enables us to prove the following key constructive lemma.

LEMMA 3. *Given an embedding of a complete graph G in $\mathbf{R}^3$, if G contains an n-link L where $L = L_1 \cup J$, L_1 is an $(n-1)$-link, J links L_1, and there are at least four vertices of G not in L, then G contains an $(n+1)$-link $L_2 = L_1 \cup T_0 \cup T_1$ where T_1 links $L_1 \cup T_0$ and $L_1 \cup T_0$ is an n-link.*

PROOF. Fix an arbitrary embedding of G. By assumption G contains an n-link L with a component J such that $L \setminus J$ is an $(n-1)$-link. Also by assumption J links L, so there is a simple closed curve γ in L that links J. The component J can be assumed to be a triangle. Suppose instead of linking a triangle J that γ links a square S. Since G is a complete graph there is a diagonal edge e of S. Let τ_0 and τ_1 be the triangles formed from the edges of S and the edge e. Since γ links S, then $[S]$ is nontrivial in $H_1(\mathbf{R}^3 \setminus \gamma; \mathbf{Z}_2)$. In $H_1(\mathbf{R}^3 \setminus \gamma; \mathbf{Z}_2)$ we have the following equation $[\tau_0] + [\tau_1] = [S]$. So one of $[\tau_0]$ or $[\tau_1]$ is nontrivial in $H_1(\mathbf{R}^3 \setminus \gamma; \mathbf{Z}_2)$. Thus one of these triangles links γ, call it J. So J contains three vertices and there are four vertices in G that are not in L. Label the vertices of $J \cup (G \setminus L)$ as

$\{1, 2, 3, 4, 5, 6, 7\}$. Let H be the complete graph $\langle 1, 2, 3, 4, 5, 6, 7\rangle$. The simple closed curve γ links J so, by Lemma 2, γ links H in a 5-pattern or an 8-pattern or it forms a 3-link with two triangles in H, where two pair of the components of the 3-link link.

Suppose γ forms a 3-link with two triangles in H say T_0 and T_1. There are two ways the 3-link can be formed: either γ links both of the triangles, T_0 and T_1, or γ links T_0 and T_1 links T_0. In both of these cases let $L_2 = (L \setminus J) \cup T_0 \cup T_1$ which forms an $(n+1)$-link in G, T_1 links $(L \setminus J) \cup T_0$, and $(L \setminus J) \cup T_0$ forms an n-link.

Suppose γ links H in a 5-pattern, then we may assume, without loss of generality that γ links H in the 5-pattern $\overline{12}$. So γ links all of the triangles in H containing the edge $\overline{12}$ and no other triangles in H. The graph H is a complete graph on seven vertices, so each edge of H is contained in a 2-link [2]. Thus $\overline{12}$ is in one component, say T_0, of a link $T_0 \cup T_1$ in H. So γ links T_0, and $L_2 = (L \setminus J) \cup T_0 \cup T_1$ forms an $(n+1)$-link in G, T_1 links $(L \setminus J) \cup T_0$, and $(L \setminus J) \cup T_0$ forms an n-link.

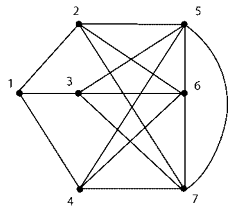

FIGURE 1. The graph G_7.

Suppose γ links H in an 8-pattern, then we may assume, without loss of generality that γ links H in the 8-pattern 1^2_3. So γ links $12x$ and $13x$ with $x \in \{4, 5, 6, 7\}$. Consider the subgraph $G_7 \subset H$ with 1 the vertex of valence three, $\{2, 3, 4\}$ the vertices of valence four and $\{5, 6, 7\}$ the vertices of valence five. The graph G_7 shown in Figure 1, is a graph in the Petersen family, and it always contains a link of a square S containing that vertex 1 and triangle T [8]. In this case $S = 1axb$ and $T = ycz$ with $\{a, b, c\} = \{2, 3, 4\}$ and $\{x, y, z\} = \{5, 6, 7\}$. The triangle T does not contain the vertex 1, so γ does not link T. If a or $b = 4$, then γ links $1ab$ and does not link xab, thus γ links S. So we see that we are done as follows, take S to be T_0 and T to be T_1 and $L_2 = (L \setminus J) \cup T_0 \cup T_1$ forms an $(n+1)$-link in G, T_1 links $(L \setminus J) \cup T_0$, and $(L \setminus J) \cup T_0$ forms an n-link. Otherwise $S = 12x3$ and $T = y4z$. So either $12x$ or $13x$ links T. Take the appropriate triangle to be T_0 and let $T = T_1$. So again, $L_2 = (L \setminus J) \cup T_0 \cup T_1$ forms an $(n+1)$-link in G, T_1 links $(L \setminus J) \cup T_0$, and $(L \setminus J) \cup T_0$ forms an n-link. □

Notice that Lemma 3 may be applied iteratively producing a link with an additional linking component for each set of four free vertices in a complete graph. The following more general version of Lemma 1 and Lemma 2 is needed to prove Proposition 1.

LEMMA 4. *Given an embedding of K_n in $\mathbf{R}^3$ where $n \geq 6$ and a simple closed curve γ in $\mathbf{R}^3 \setminus K_n$, if the curve γ links K_n then one of the following holds:*

- *γ links $n-2$ triangles of K_n all of which contain a common edge $\overline{pq}$, (called an $(n-2)$-pattern $\overline{pq}$)*
- *γ links $2(n-3)$ triangles which are of the form pqx and prx where p, q and r are fixed and $x \neq q$ and $x \neq r$, (called a $2(n-3)$-pattern p_r^q)*
- *γ is in a 3-link L with two triangles in K_n, where at least two pairs of the components of L link.*

PROOF. This proof is by induction on n. The case $n = 6$ is Lemma 1, proved in [5]. The case $n = 7$ is Lemma 2. Assume the lemma holds for some $n \geq 7$. Now consider an embedding of $K_{n+1} = \langle 1, 2, \ldots, n+1 \rangle$ in $\mathbf{R}^3$ and a simple closed curve γ in $\mathbf{R}^3 \setminus K_{n+1}$. For ease of notation let the complete graph on n vertices $\langle 1, 2, \ldots, j-1, j+1, \ldots, n+1 \rangle$ be denoted $\langle \hat{j} \rangle$. If γ is in a 3-link L with two triangles in K_{n+1}, where at least two pairs of the components of L link, we are done. So suppose γ links some triangle T in K_{n+1} but it is not in a 3-link with two triangles in K_{n+1}. Without loss of generality T is in $K_n = \langle \widehat{n+1} \rangle$. By the inductive assumption, γ links K_n in an $(n-2)$-pattern or a $2(n-3)$-pattern.

Suppose γ links $\langle \widehat{n+1} \rangle$ in a $2(n-3)$-pattern which, without loss of generality, we take to be the $2(n-3)$-pattern 1_3^2. So γ links $124, 125, \ldots, 12n, 134, 135, \ldots, 13n$. Now consider $K_n = \langle \hat{n} \rangle$. The curve γ links $124, 125, \ldots, 12(n-1), 134, 135, \ldots, 13(n-1)$. Since the triangles are all of the form $12x$ or $13x$, the curve γ must link $\langle \hat{n} \rangle$ in the $2(n-3)$-pattern 1_3^2. Then γ links $\langle \hat{3} \rangle$ in the $(n-2)$-pattern $\overline{12}$, γ links $\langle \hat{2} \rangle$ in the $(n-2)$-pattern $\overline{13}$, γ links $\langle \widehat{v_i} \rangle$ with $v_i \in \{4, 5, \ldots, n+1\}$ in the $2(n-3)$-pattern 1_3^2, and γ does not link any triangles in $\langle \hat{1} \rangle$ because all of the triangles in $\langle \hat{1} \rangle$ appear in one of the previously mentioned $K_n's$. This accounts for all of the $K_n's$ in K_{n+1}. So these $2(n+1-3)$ triangles: $124, 125, \ldots 12(n+1), 134, 135, \ldots, 13(n+1)$ are all of the triangles that γ links. Notice that they are all of the triangles of the form pqx and prx with $p = 1$, $q = 2$, $r = 3$ and $x \in \{4, 5, \ldots, n+1\}$. So γ links K_{n+1} in the $2((n+1)-3)$-pattern 1_3^2.

Finally, suppose γ links $\langle \widehat{n+1} \rangle$ in an $(n-2)$-pattern which, without loss of generality, we may assume is the $(n-2)$-pattern $\overline{12}$, γ does not link any K_n in a $2(n-3)$-pattern and γ is not in a 3-link of triangles in K_{n+1}. Then in $\langle \hat{n} \rangle$, the curve γ links $123, 124, \ldots, 12(n-1)$. So γ links $\langle \hat{n} \rangle$ in the $(n-2)$-pattern $\overline{12}$. Thus γ links $12x$ for all $x \in \{3, 4, \ldots, n+1\}$ (there are $(n+1)-2$ such triangles). As with the previous case these are the only triangles that link γ in K_n. So γ links K_{n+1} in the $((n+1)-2)$-pattern $\overline{12}$. □

PROPOSITION 1. *Every embedding of K_{14} in $\mathbf{R}^3$ contains a 3-link $L = T_0 \cup T_1 \cup T_2$ of triangles, where T_0 links T_1 and T_1 links T_2.*

PROOF. Since K_6 is intrinsically linked, in any embedding of K_{14} in $\mathbf{R}^3$ there must be at least $\binom{14}{6}$ distinct pairs of linked triangles. There are $\binom{14}{3}$ triangles in K_{14}. By Lemma 4, each triangle T in K_{14} either does not link any of the triangles in K_{11} (the graph defined by the vertices of $K_{14} \setminus T$), or links K_{11} in either a 9-pattern, or a 16-pattern, or is in a 3-link

L with two other triangles, where two pairs of the components of L link. So the greatest number of triangles a triangle T can link and not be in a 3-link $L = T_0 \cup T_1 \cup T_2$ of triangles, where T_0 links T_1 and T_1 links T_2, in K_{11} is 16. So the maximum total number of pairs of linked triangles there can be in K_{14} without the existence of a 3-link of triangles of this form is $\binom{14}{3}\frac{16}{2}$, where the second term is divided by two because each pair of linked triangles is counted twice, once for each triangle in the pair. Since $\binom{14}{6} > \binom{14}{3}\frac{16}{2}$, every embedding of K_{14} contains a 3-link $L = T_0 \cup T_1 \cup T_2$ of triangles, where T_0 links T_1 and T_1 links T_2. □

The following additional constructive lemmas are needed to prove the main result of this section.

LEMMA 5 ([5]). *Given an embedding of K_4 in $\mathbf{R}^3$ and an embedding of a simple closed curve γ in $\mathbf{R}^3 \setminus K_4$, γ links an even number of triangles of K_4.*

LEMMA 6. *Suppose we have an embedding of K_6 in $\mathbf{R}^3$, and two non-split disjoint links L of n components and J of m components, where $L = L_1 \cup T$, $J = J_1 \cup S$, L_1 and J_1 are non-splittable links, and T and S are disjoint triangles in K_6. Let γ be a component of L_1 that links T and let α be a component of J_1 that links S. Then there is a square (or triangle) R in K_6 such that $J_1 \cup R \cup L_1$ is an $(n+m-1)$-link and R links both γ and α.*

PROOF. Let $K_6 = \langle 1, 2, 3, 4, 5, 6\rangle$. Recall that γ is a component of L_1 that links T and α is a component of J_1 that links S. Without loss of generality, label the vertices of T with $\{1, 2, 3\}$ and the vertices of S with $\{4, 5, 6\}$. Note that by Lemma 1, γ and α will link triangles in K_6 other than T and S, respectively. If α and γ link the same triangle in K_6, take it to be R. Then $J_1 \cup R \cup L_1$ forms an $(n+m-1)$-link.

Suppose α and γ do not link the same triangle in K_6. In particular α does not link 123 and γ does not link 456. Consider the K_4 subgraph $G = \langle 3, 4, 5, 6\rangle$ of K_6. Since α links 456 then, by Lemma 5, α must link another triangle in G, and in particular one that contains the vertex 3 (as 456 is the only triangle that does not). Without loss of generality, suppose α links 345. Next consider $H = \langle 2, 3, 4, 5\rangle$ then, since α links 345, it must link at least one of 234, 235 or 245. Suppose α links 234 or 235, without loss of generality suppose α links 234. Since α does not link 123 and γ does not link 456, γ and α both link the square 1243. So take R to be 1243 and $J_1 \cup R \cup L_1$ forms an $(m+n-1)$-link. Finally, suppose that α links 345 and 245. The simple closed curve γ links 123 so it must link one of 124, 134, or 234 in the K_4 subgraph $\langle 1, 2, 3, 4\rangle$. All three of these triangles share an edge with one of the triangles that α links, either $\overline{24}$ with the triangle 245 or $\overline{34}$ with 345. Thus there is a square R that links both γ and α, so that $J_1 \cup R \cup L_1$ forms an $(m+n-1)$-link. □

Notice that the number of vertices of the resulting link $J_1 \cup R \cup L_1$ in Lemma 6 is at most two less than the sum of the number of vertices L and J. This is because the two triangles in K_6 are replaced with a square or a triangle. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, as usual.

THEOREM 1. *Given $n > 1$, every embedding of $K_{\lfloor \frac{7}{2}n \rfloor}$ contains an n-link.*

PROOF. For $n = 2$, $K_{\lfloor \frac{7}{2}n \rfloor} = K_7 \supset K_6$. Since K_6 is intrinsically linked [3], K_7 is as well. For $n = 3$, $K_{\lfloor \frac{7}{2}n \rfloor} = K_{10}$ which is known to be intrinsically 3-linked [5]. For $n = 4$, $K_{\lfloor \frac{7}{2}n \rfloor} = K_{14}$. Let K_{14} be embedded in $\mathbf{R}^3$. By Proposition 1, K_{14} contains a 3-link $L_1 = T_0 \cup T_1 \cup T_2$ of triangles where T_0 links T_1 and T_1 links T_2. So $L_1 \setminus T_2$ is a 2-link. There are five vertices of K_{14} that are not used in the link L_1 and T_2 links $T_0 \cup T_1$, so by Lemma 3 K_{14} contains a 4-link.

We will proceed by induction for $n \geq 5$. We will write n in the form $n = 2a+5$ or $2a+6$ for some integer $a \geq 0$, so that $\lfloor \frac{7}{2}n \rfloor = 7a + 17$ or $7a + 21$. The proof is by induction on a for n odd, then the case n even follows from Lemma 3 above. The inductive hypothesis for n odd is that every embedding of K_{7a+17} contains a $(2a + 5)$-link L, where $L = L_0 \cup R_0 \cup R_1$ and $L \setminus R_1$ is a $(2a+4)$-link. Also, the link L_0 is a $(2a+3)$-link that contains at most $7a+9$ vertices, and there is a triangle T in L_0 such that T links $L_0 \setminus T$ and $L_0 \setminus T$ is a $(2a+2)$-link.

Suppose n odd, and thus $\lfloor \frac{7}{2}n \rfloor = 7a+17$. First consider $a = 0$, so $n = 5$ and $\lfloor \frac{7}{2}n \rfloor = 17$. Fix an embedding of K_{17} in $\mathbf{R}^3$. Since $17 > 14$, by Proposition 1 the graph K_{17} contains a 3-link $L_0 = T_0 \cup T_1 \cup T_2$ of triangles where T_0 links T_1 and T_1 links T_2. So then both $L_0 \setminus T_0$ and $L_0 \setminus T_2$ are 2-links. There are nine vertices in L_0 so there are eight vertices in K_{17} which are not used in L_0. By Lemma 3, with each additional set of four vertices an additional linked component can be added, so we can find L, a 5-link in K_{17}, with a component that links at least one of the other components and when removed leaves a 4-link.

Fix an embedding of $K_{7(a+1)+17}$. Choose a $K_{7a+17} \subset K_{7(a+1)+17}$, then by the inductive assumption, K_{7a+17} contains a $(2a + 3)$-link L_0 that contains at most $7a + 9$ vertices, and there is a triangle T in L_0 such that $L_0 \setminus T = L_1$ is a $(2a + 2)$-link and T links L_1. So there are 15 vertices of $K_{7(a+1)+17}$ that are not in L_0. Let H be the complete graph defined by these 15 vertices. Since $15 > 14$, by Proposition 1, H contains a 3-link $J_1 = T_0 \cup T_1 \cup T_2$ of triangles where T_0 links T_1 and T_1 links T_2. So then $J_1 \setminus T_0$ and $J_1 \setminus T_2$ are both 2-links. By Lemma 6, given L_0 and J_1 there is a $(2(a + 1) + 3)$-link $L_1 = T_0 \cup T_1 \cup R \cup L_1$ that contains at most $(7a+9)+9-2 = 7(a+1)+9$ vertices. Given that $J_1 \setminus T_0$ is a 2-link with T_0 linking $J_1 \setminus T_0$, the link $L_1 \setminus T_0$ is a $(2(a + 1) + 2)$-link with T_0 linking $L_1 \setminus T_0$, since the construction given by Lemma 6 does not change the links outside of the triangles replaced with a square (or triangle). At this point there are at least $8 = 7(a + 1) + 17 - (7(a + 1) + 9)$ vertices of $K_{7(a+1)+17}$ that are not in L_1. By Lemma 3, with each additional set of four vertices an additional linked component can be added. So we can find L, a $(2(a+1)+5)$-link in $K_{7(a+1)+17}$, with a component that when removed leaves a $(2(a + 1) + 4)$-link.

Now suppose n is even and thus $\lfloor \frac{7}{2}n \rfloor = 7a + 21$. Fix an embedding of K_{7a+21}. Choose a subgraph $K_{7a+17} \subset K_{7a+21}$. The construction above produces L, a $(2a + 5)$-link in K_{7a+17} with a component that, when removed, leaves a $(2a+4)$-link. There are at least $4 = 7a+21-(7a + 17)$ vertices that are not in L. By Lemma 3, with each additional set of four vertices an additional linked component can be added, so we can find a $(2a + 6)$-link in K_{7a+21}. □

3. Asymptotics

We begin with the main result of this section:

THEOREM 2. *There exists a function $f(n)$ such that*

$$\lim_{n\to\infty} \frac{f(n)}{n} = 3$$

and for every n, $K_{f(n)}$ is intrinsically n-linked.

REMARK 1. The bounds in this theorem improve the linear bounds of Theorem 1 for sufficiently large n.

REMARK 2. Using the methods in this section, functions $f(n)$ as in Theorem 2 can be constructed explicitly. The function with the best asymptotic behavior that can be constructed using Proposition 2 directly is of the form $3n + A\log n + B$ for some constants A and B.

To prove this result, we will first introduce some terminology. Consider the triangles in K_m. There are two numbers which measure the size of a set S of triangles. One is the magnitude of the set S, that is, the number of triangles in the set. This we denote by $|S|$ as usual. Another notion is the *disjoint size* (S) of S which is the maximal magnitude of any subset S' of S so that all the triangles of S' are disjoint, that is, so that no two of them share a vertex.

LEMMA 7. *Let S be a non-empty set of triangles in K_m with $(S) = n$.*
Suppose $m > 3n + 8$, then

$$|S| \le 11nm^2 .$$

PROPOSITION 2. *For $n > 0$, every embedding of K_{660n} in $\mathbf{R}^3$ contains a triangle which is linked with $n - 1$ other triangles.*

A *key ring link* is a link where one of the components, the *ring* links all of the other components, the *keys*. So Proposition 2 implies that there is a key ring n-link of triangles. Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x, as usual.

REMARK 3. This proposition can be strengthened at the cost of a much lengthier proof. In particular, there exist numbers ρ, c with $\rho < 29$ and $-17 < c < -16$ such that every embedding of $K_{\lceil \rho n + c \rceil}$ in $\mathbf{R}^3$ contains a key ring n-link of triangles.

We will prove Lemma 7 and Proposition 2 after the proof of Theorem 2.

PROOF OF THEOREM 2. Each $n \in \mathbf{N}$ can be written as $n = x^2 - x + 1$ for some positive real number x. Let $f(n) = \lfloor 3x^2 + 663x \rfloor$. Since $3x^2 + 663x \le f(n) \le 3x^2 + 663x + 1$ and

$$\lim_{x\to\infty} \frac{3x^2 + 663x}{x^2 - x + 1} = \lim_{x\to\infty} \frac{3x^2 + 663x + 1}{x^2 - x + 1} = 3 .$$

So

$$\lim_{n\to\infty} \frac{f(n)}{n} = 3$$

as desired.

We will show that for every n, $K_{f(n)}$ is intrinsically n-linked. This is trivial for $n = 1$ and for $n = 2$ it follows from K_6 being intrinsically linked. So we shall assume that $n \geq 3$. Thus $\lfloor x \rfloor \geq 2$. For a given $n \in \mathbf{N}$, fix an embedding of $K_{f(n)}$ in $\mathbf{R}^3$. Notice that

$$3\lfloor x \rfloor^2 + 663\lfloor x \rfloor \leq \lfloor 3x^2 + 663x \rfloor = f(n)\,.$$

We will restrict our attention to a subgraph $H = K_{3\lfloor x \rfloor^2+663\lfloor x \rfloor} \subset K_{f(n)}$. By Proposition 2 and the fact that $\lfloor x \rfloor \geq 1$, since $3\lfloor x \rfloor^2 + 663\lfloor x \rfloor > 660\lfloor x \rfloor$, there is a key ring $\lfloor x \rfloor$-link of triangles in H, call the link L_1. There are $3\lfloor x \rfloor$ vertices in a $\lfloor x \rfloor$-link of triangles. Removing the vertices in L_1 we are left with an embedding of $K_{3\lfloor x \rfloor(\lfloor x \rfloor-1)+663\lfloor x \rfloor}$ in $\mathbf{R}^3$. Since

$$3\lfloor x \rfloor(\lfloor x \rfloor - 1) + 663\lfloor x \rfloor \geq 660\lfloor x \rfloor\,,$$

we can apply Proposition 2 again. Continuing in this way, let $g : \mathbf{N} \to \mathbf{Z}$ be the function $g(i) = 3\lfloor x \rfloor(\lfloor x \rfloor - i) + 663\lfloor x \rfloor$ which gives the number of vertices of H that are not contained in one of the $\lfloor x \rfloor$-links of triangles $L_1, \ldots, L_i$ after i applications of Proposition 2 to H. Since $g(i) \geq 660\lfloor x \rfloor$ for $i \leq \lfloor x \rfloor + 1$, we may apply Proposition 2 to H, $\lfloor x \rfloor + 2$ times. In this way we find that the given embedding of $H \subset K_{f(n)}$ contains $\lfloor x \rfloor + 2$ disjoint $\lfloor x \rfloor$-links of triangles $L_1, \ldots, L_{\lfloor x \rfloor+2}$. For $i \in 1, \ldots, \lfloor x \rfloor - 1$, label two of the keys of the ith key ring L_i as A_i and B_i and the ring C_i. Both $L_i \setminus A_i$ and $L_i \setminus B_i$ are $(\lfloor x \rfloor - 1)$-links. Then we perform $\lfloor x \rfloor + 1$ applications of Lemma 6 on the disjoint links $L_1, \ldots, L_{\lfloor x \rfloor+2}$ of triangles in H. In the i application of Lemma 6 the components B_i and A_{i+1} are replaced by a single component that links both C_i and C_{i+1}. So the resulting link N is a non-splittable link with $\lfloor x \rfloor(\lfloor x \rfloor + 2) - (\lfloor x \rfloor + 1) = \lfloor x \rfloor^2 + \lfloor x \rfloor - 1$ components, where A_1 links $N \setminus A_1$ and $N \setminus A_1$ is an $\lfloor x \rfloor^2 + \lfloor x \rfloor - 2$-link. Finally, $g(\lfloor x \rfloor + 2) = 657\lfloor x \rfloor > 4$, so by Lemma 3 we can add a component to the link, obtaining a $\lfloor x \rfloor^2 + \lfloor x \rfloor$-link L in H.

Since $\lfloor x \rfloor > x - 1$, we see $\lfloor x \rfloor^2 + \lfloor x \rfloor > x^2 - x$. Since $x^2 - x = n - 1$ it is an integer, so then $\lfloor x \rfloor^2 + \lfloor x \rfloor \geq x^2 - x + 1 = n$. Thus L is a non-splittable link of at least n components. □

PROOF OF PROPOSITION 2. Assume $n > 0$, and let $m = 660n$. Fix an embedding of K_m in $\mathbf{R}^3$. For each triangle T in K_m, let S_T denote the set of triangles linked to T in the given embedding. There are $\binom{m}{3}$ distinct triangles and $\binom{m}{6}$ distinct complete graphs on six vertices K_6, each of which contains a pair of linked triangles (each pair is distinct, though not necessarily disjoint, because each K_6 is distinct). Then

$$\binom{m}{3} \max_{T\subset K_m} |S_T| \geq \sum_{T\subset K_m} |S_T| \geq 2\binom{m}{6}\,.$$

Hence

$$\max_{T\subset K_m} |S_T| \geq \frac{2\binom{m}{6}}{\binom{m}{3}} .$$

Now we can calculate

$$\max_{T\subset K_m} |S_T| \geq \frac{2\binom{m}{6}}{\binom{m}{3}} = \frac{1}{60}(m-3)(m-4)(m-5) \geq \frac{1}{60}(m-3)^2(m-6) .$$

Continuing, substituting $m = 660n$,

$$\max_{T\subset K_m} |S_T| \geq (m-3)^2\left(11n - \frac{6}{60}\right) \geq 11(m-3)^2\left(n - \frac{6}{60}\right) > 11(m-3)^2(n-2) .$$

This counting argument shows that there is a triangle Δ in K_m such that $|S_\Delta| > 11(m-3)^2(n-2)$. The set of triangles that Δ links, S_Δ are in the subgraph $K_{m-3} = K_m \setminus \Delta$. So since $|S_\Delta| > 11(m-3)^2(n-2)$ by Lemma 7, $(S_\Delta) > n-2$. Thus Δ is linked to more than $n-2$ disjoint triangles in K_{m-3}. Consequently, there are $n-1$ disjoint triangles in S_Δ, that is, there is a key ring link of n disjoint triangles in this embedding of K_m, proving the proposition. □

PROOF OF LEMMA 7. Let $T_1, \ldots, T_n$ be a set of disjoint triangles in S. Let S_i be the subset of triangles in S which intersect T_i and no other T_j. Let S_{ij} be the subset of triangles in S which intersect T_i and T_j and no other T_k. Finally, let S_{ijk} be the subset of triangles in S which have vertices in T_i, T_j, and T_k.

Fix i. If S_i contains two disjoint triangles, then T_i could be replaced with those two triangles to make a larger set of disjoint triangles in S. This would imply that $(S) > n$, a contradiction. Therefore no two triangles in S_i are disjoint.

Now let us examine three cases.

1. Every triangle in S_i intersects T_i in at least two vertices.
2. There exists a triangle in S_i which intersects T_i in only one vertex, but not every triangle in S_i contains that vertex.
3. There exists a triangle in S_i which intersects T_i in only one vertex, and every triangle in S_i contains that vertex.

For ease of notation, let $R = m - 3n$. This is the number of the vertices of K_m excluding the union of those in the T_i. Recall that $R > 8$ from the hypotheses of the lemma.

In Case 1, S_i could contain at most the triangle T_i along with $3R$ triangles which each contain two vertices of T_i, for a total $|S_i| \leq 3R + 1 < R^2$ since $R > 8$.

In Case 2, there exists a triangle P in S_i which intersects T_i only in the vertex v and another triangle $Q \in S_i$ which misses that vertex. Now let U be a triangle in S_i which is not T_i, P or Q.

There are two possibilities. One is that U contains the vertex v and a vertex of Q. There are fewer than $3(R+1)$ such triangles, because we can choose the third vertex from the vertices in T_i or outside all T_j but not the vertex v or the vertex chosen from Q.

The other possibility is that U contains separate vertices other than v from both T_i and P. There are at most $2 \times 2 \times (R+1)$ such triangles, since there are two vertices other than v in each of T_i and P to pick from and the final vertex could come from T_i or outside all T_j. So the total number of such U is at most $3(R+1)+4(R+1)$. Thus $|S_i| \le 7R+10 < R^2$ since $R \ge 9$, counting the three triangles T_i, P, Q as well.

In Case 3, $|S_i| \le \binom{R}{2} + 2R + 1$, where the first term is for those triangles which intersect T_i in one vertex, the second term for those triangles that intersect T_i in two vertices, and the third term for T_i itself. This bound is $\frac{1}{2}R^2 + \frac{3}{2}R + 1 < R^2$ since $R \ge 9$.

Now note that

$$|S| = \sum_{1 \le i \le n} |S_i| + \sum_{1 \le i < j \le n} |S_{ij}| + \sum_{1 \le i < j < k \le n} |S_{ijk}| .$$

Now a triangle in S_{ij} either has all three vertices from T_i and T_j or has a vertex from outside all T_k. In the first case, there are nine ways of picking two vertices from T_i and one from T_j and nine ways of doing the reverse, for a total of 18 possible triangles. In the second case, there are $3 \times 3 \times R$, for a total bound of $|S_{ij}| \le 9R + 18 \le 11R$ since $R \ge 9$.

A triangle in S_{ijk} has one vertex each from the three disjoint triangles T_i, T_j, T_k so $|S_{ijk}| \le 27$. Combining these inequalities with $|S_i| \le R^2$ and $R, n \le m$ yields

$$|S| \le nR^2 + 11R\binom{n}{2} + 27\binom{n}{3} < nm^2 + \frac{11}{2}n^2m + \frac{27}{6}n^3 \le nm^2 + 10n^2m \le 11nm^2$$

as desired. □

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Present Addresses:
GABRIEL C. DRUMMOND-COLE
MATHEMATICS PH.D. PROGRAM,
THE CUNY GRADUATE CENTER,
365 FIFTY AVENUE, ROOM 4208, NEW YORK, NY 10016–4309, USA.
e-mail: gdrummond-cole@gc.cuny.edu

DANIELLE O'DONNOL
MATH DEPARTMENT—MS 136,
RICE UNIVERSITY,
6100 S. MAIN ST., HOUSTON TX 77005–1892, USA.
e-mail: odonnole@rice.edu