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On the Embedded Eigenvalues for the Self-Adjoint Operators with Singular Perturbations

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1. Introduction and assumptions.

This paper is a continuation of [7]. That is, in the framework of the $\mathcal{H}_{-2}$-construction we consider a finite rank perturbation of a self-adjoint operator H_0 without assuming semi-boundedness for H_0. The $\mathcal{H}_{-2}$-construction has been developed by A. Kiselev and B. Simon [1], S. T. Kuroda and H. Nagatani [2], [3] and have been applied to Schrödinger operators with a singular perturbation by H. Nagatani [4] and S. Shimada [6].

In this paper we consider the embedded eigenvalues of H_T and the existence of the wave operator $W_{\pm}(H_0, H_T)$. We prepare some notations. Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$, H_0 a self-adjoint operator in $\mathcal{H}$ and $R_0(z) = (H_0 - z)^{-1}$ ($\mathrm{Im}\, z \neq 0$). We put $\mathcal{H}_s := \{u \in \mathcal{H}; \|(|H_0| + 1)^{s/2}u\| < \infty\}$ for $s \geq 0$, and $\mathcal{H}_s := (\mathcal{H}_{-s})^*$ for $s < 0$. Remark that $\mathcal{H}_s \subset \mathcal{H} \subset \mathcal{H}_{-s}$ for $s \geq 0$. For simplicity we use the same symbol $\langle\cdot,\cdot\rangle$ for the dual coupling $\langle\cdot,\cdot\rangle_{s,-s}$ of $\mathcal{H}_s$ and $\mathcal{H}_{-s}$ ($s \in \mathbf{R}$), and regard the operator $R_0(z)$ with $\mathrm{Im}\, z \neq 0$ as the element of $\mathcal{L}(\mathcal{H}, \mathcal{H}) \cap \mathcal{L}(\mathcal{H}_s, \mathcal{H}_{s+2})$ for $\mathrm{Im}\, z \neq 0$.

DEFINITION. Define

$$W(z) = W(z, i) = (z - i)R_0(z)R_0(i)$$

and the operator $R_T(z)$ in $\mathcal{H}$

$$R_T(z) = R_0(z) - R_0(z)(1 + TW(z))^{-1}TR_0(z), \quad \mathrm{Im}\, z \neq 0\,. \tag{1}$$

To define the self-adjoint operator H_T for $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ we use the following theorem (cf. [3]).

THEOREM 1.1 ([3]). *If $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ satisfies*

$$T - T^* = TW(-i, i)T^* = T^*W(-i, i)T\,, \tag{2}$$

$$u - TR_0(i)u = 0, \quad u \in \mathcal{H}_0 \Rightarrow u = 0\,, \tag{3}$$

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then the operator $R_T(z)$ *above is well-defined and satisfies the resolvent equation, i. e., for* $\mathrm{Im}z, \mathrm{Im}w \neq 0$

$$R_T(z) - R_T(w) = (z-w)R_T(z)R_T(w) = (z-w)R_T(w)R_T(z)\,.$$

Furthermore there exists a unique self-adjoint operator H_T *such that* $R_T(z) = (H_T - z)^{-1}$.

ASSUMPTION (H_0). H_0 has only absolutely continuous spectrum and satisfies

$$\sigma(H_0)(=\sigma_{ac}(H_0)) = \mathbf{R}\,. \tag{4}$$

ASSUMPTION (T). For $T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_{-2})$ with $\mathcal{R} = \mathrm{Range}T$ assume the conditions (2) and (3) and

(T1) For any $\lambda \in \mathbf{R}$ and for any $f, g \in \mathcal{R}$,

$$\lim_{\varepsilon\downarrow 0}\langle R_0(\lambda \pm i\varepsilon)R_0(-i)f, g\rangle\,,$$

exist, locally uniformly in $\mathbf{R}$.

(T2) There exists a dense subset $\mathcal{D}$ of $\mathcal{H}$ such that for any $\lambda \in \mathbf{R}$ and for any $f \in \mathcal{R}$,

$$\lim_{\varepsilon\downarrow 0}\langle R_0(\lambda \pm i\varepsilon)u, f\rangle, \quad u \in \mathcal{D}\,,$$

exist, locally uniformly in $\mathbf{R}$.

In this paper we always suppose Assumptions (H_0) and (T). We are mainly interested in the existence of the embedded eigenvalues of H_T, the explicit form of the eigenvectors corresponding to the eigenvalues and the asymptotic completeness of the wave operators $W_\pm(H_0, H_T)$. The organization of this paper is as follows. In section 2 we investigate the necessary and sufficient condition for the existence of the eigenvalue of H_T. In section 3 we prove the asymptotic completeness of the wave operators $W_\pm(H_0, H_T)$. In section 4 we investigate the case where a perturbation has rank one and compare with their results ([4], [6]) and ours.

2. Embedded eigenvalues (Finite rank case).

In this section we consider the case $\dim\mathcal{R} = N$. By the condition (2) we can easily obtain the following lemma.

LEMMA 2.1. *There exist a basis* $[f_1, \cdots, f_N]$ *of* $\mathcal{R}$ *and* $\mu_j\ (\neq 0) \in \mathbf{C}\ (j = 1, \cdots, N)$ *such that*

$$\langle R_0(i)f_j, R_0(i)f_k\rangle = \delta_{jk}\,,$$

$$Tu = \sum_{j=1}^{N} \mu_j\langle u, f_j\rangle f_j, \quad u \in \mathcal{H}_2\,.$$

PROOF. Putting $T_1 := R_0(i)TR_0(-i)$ we multiply the equation (2) by $R_0(i)$ (from left) and $R_0(-i)$ (from right). Then we have

$$T_1 - {T_1}^* = -2iT_1{T_1}^* = -2i{T_1}^*T_1$$

Hence T_1 is a normal operator. Therefore T can be decomposed as above. □

We fix a basis $[f_1, \cdots, f_N]$ of $\mathcal{R}$ as in Lemma 2.1. We use the following notations:

NOTATIONS.

$$g_j = \bar{\mu}_j f_j, \quad v_{jk}(z) = \langle W(z) f_k, g_j \rangle \quad (1 \le j, k \le N),$$
$$V(z) = (v_{jk}(z))_{1 \le j,k \le N} (N \times N \text{matrix}), \quad \Delta(z) = \det(I + V(z)),$$
$\Delta_{jk}(z)$ is a cofactor of $I + V(z)$.

Then we have

$$(I + V(z))^{-1} = \frac{1}{\Delta(z)} \begin{pmatrix} \Delta_{11}(z) & \Delta_{21}(z) & \cdots & \Delta_{N1}(z) \\ \Delta_{12}(z) & \Delta_{22}(z) & \cdots & \Delta_{N2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1N}(z) & \Delta_{2N}(z) & \cdots & \Delta_{NN}(z) \end{pmatrix}. \tag{5}$$

LEMMA 2.2. *For $z \in \rho(H_T) \cap \rho(H_0)$ and for $u \in \mathcal{H}$ we have*

$$R_T(z)u = R_0(z)u - \Delta(z)^{-1} \sum_{j,k=1}^{N} \Delta_{jk}(z) \langle R_0(z)u, g_k \rangle R_0(z) f_j. \tag{6}$$

Furthermore we have

$$\begin{aligned} &R_T(z) R_0(i) f_m \\ &\quad = R_0(z) R_0(i) f_m - \frac{1}{(z-i)} R_0(z) f_m + \frac{1}{(z-i)\Delta(z)} \sum_{j=1}^{N} \Delta_{jm}(z) R_0(z) f i, \end{aligned} \tag{7}$$

$$\begin{aligned} &\langle R_T(z) R_0(i) f_m, R_0(-i) g_n \rangle \\ &\quad = \langle R_0(z) R_0(i) f_m, R_0(-i) g_n \rangle - (z-i)^{-2} v_{nm}(z) + \frac{\Delta_{mn}(z)}{(z-i)^2 \Delta(z)}. \end{aligned} \tag{8}$$

PROOF. For simplicity we write $W(z) = W$ and $v_{jk}(z) = v_{jk}$. We calculate $(I + TW)^{-1} T_u$ ($u \in \mathcal{H}_2$). Since $(I+TW)^{-1} T = T(I+TW)^{-1}$ (cf. [3]), we put $(I+TW)^{-1} T_u = \sum_{j=1}^{N} c_j f_j$ and determine c_j. Since $Tu = \sum_{j=1}^{N} c_j (I + TW) f_j$, we have

$$\begin{aligned} \sum_{l=1}^{N} \langle u, g_l \rangle f_l = Tu = \sum_{j=1}^{N} c_j (I + TW) f_j = \sum_{j=1}^{N} c_j \left(f_j + \sum_{k=1}^{N} \langle W f_j, g_k \rangle f_k \right) \\ = \sum_{j=1}^{N} c_j \left(f_j + \sum_{k=1}^{N} v_{kj} f_k \right). \end{aligned}$$

Comparing the coefficients of f_j of each hand side, we have

$$\begin{pmatrix} \langle u, g_1\rangle \\ \langle u, g_2\rangle \\ \vdots \\ \langle u, g_N\rangle \end{pmatrix} = \left(I + \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1N} \\ v_{21} & v_{22} & \cdots & v_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ v_{N1} & v_{N2} & \cdots & v_{NN} \end{pmatrix} \right) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} .$$

By Crammer's formula we have

$$\begin{aligned} c_j &= \begin{vmatrix} 1+v_{11} & v_{12} & \cdots & \langle u, g_1\rangle & \cdots & v_{1N} \\ v_{21} & 1+v_{22} & \cdots & \langle u, g_2\rangle & \cdots & v_{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{N1} & v_{N2} & \cdots & \langle u, g_N\rangle & \cdots & 1+v_{NN} \end{vmatrix} / \Delta(z) \\ &= \sum_{k=1}^{N} \Delta_{jk}\langle u, g_k\rangle / \Delta(z) \end{aligned}$$

where we used (5). Hence we have (6).

By $\langle R_0(z)R_0(i)f_m, g_j\rangle = (z-i)^{-1}\langle W(z)f_m, g_j\rangle$ and the cofactor expansion of the matrix $I + V(z)$ we obtain

$$\begin{aligned} &\Delta(z)^{-1} \sum_{j,k=1}^{N} \Delta_{jk}(z)\langle R_0(z)R_0(i)f_m, g_k\rangle R_0(z)f_j \\ &\qquad = \frac{1}{(z-i)\Delta(z)} \sum_{j,k=1}^{N} \Delta_{jk}(z)v_{mk}R_0(z)f_j . \end{aligned}$$

We first calculate the sum with respect to k.

$$\begin{aligned} \sum_{k=1}^{N} \Delta_{jk}v_{km} &= \sum_{k=1}^{N} \begin{vmatrix} 1+v_{11} & \cdots & 0 & \cdots & v_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k1} & \cdots & 1 & \cdots & v_{kN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{N1} & \cdots & 0 & \cdots & v_{NN} \end{vmatrix} v_{km} \\ &= \begin{vmatrix} 1+v_{11} & \cdots & v_{1m} & \cdots & v_{1N} \\ \vdots & \cdots & v_{km} & \cdots & v_{kN} \\ v_{N1} & \cdots & v_{Nm} & \cdots & v_{NN} \end{vmatrix} \\ &= \begin{vmatrix} 1+v_{11} & \cdots & v_{1m} & \cdots & v_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{m1} & \cdots & 1+v_{mm} & \cdots & v_{mN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{N1} & \cdots & v_{Nm} & \cdots & v_{NN} \end{vmatrix} - \begin{vmatrix} 1+v_{11} & \cdots & 0 & \cdots & v_{1N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{m1} & \cdots & 1 & \cdots & v_{mN} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{N1} & \cdots & 0 & \cdots & v_{NN} \end{vmatrix} \\ &= \delta_{jm}\Delta(z) - \Delta_{jm}(z) . \end{aligned}$$

Hence we obtain (7). Similarly we have (8). □

To the end of this section we fix $\lambda \in \mathbf{R}$.

LEMMA 2.3. *Let*

$$c_{mn}(z) = \frac{(\lambda - z)\Delta_{mn}(z)}{(z-i)\Delta(z)} .$$

Then the following limit exists:

$$c_{mn}(\lambda + i0) := \lim_{\varepsilon\downarrow 0} c_{mn}(\lambda + i\varepsilon) .$$

PROOF. Remark that by Assumption (T1) $\lim_{\varepsilon\downarrow 0} v_{nm}(\lambda+i\varepsilon)$ exists and that $E_0(\{\lambda\}) = 0$. Using (8), we have

$$\begin{aligned}
\langle E_T(\{\lambda\})R_0(i)f_m, R_0(-i)g_n\rangle &= \lim_{\varepsilon\downarrow 0}(-i\varepsilon\langle R_T(\lambda+i\varepsilon)R_0(i)f_m, R_0(-i)g_n\rangle) \\
&= \lim_{\varepsilon\downarrow 0}\Bigg(-i\varepsilon\langle R_0(\lambda+i\varepsilon)R_0(i)f_m, R_0(-i)g_n\rangle + i\varepsilon(\lambda+i\varepsilon-i)^{-2}v_{nm}(\lambda+i\varepsilon) \\
&\quad + \frac{-i\varepsilon\Delta_{mn}(\lambda+i\varepsilon)}{(\lambda+i\varepsilon-i)^2\Delta(\lambda+i\varepsilon)}\Bigg) \\
&= \langle E_0(\{\lambda\})R_0(i)f_m, R_0(-i)g_n\rangle + \lim_{\varepsilon\downarrow 0}\frac{-i\varepsilon\Delta_{mn}(\lambda+i\varepsilon)}{(\lambda+i\varepsilon-i)^2\Delta(\lambda+i\varepsilon)} \\
&= \lim_{\varepsilon\downarrow 0}\frac{-i\varepsilon\Delta_{mn}(\lambda+i\varepsilon)}{(\lambda+i\varepsilon-i)^2\Delta(\lambda+i\varepsilon)} = \frac{1}{\lambda - i}\lim_{\varepsilon\downarrow 0} c_{mn}(\lambda+i\varepsilon) .
\end{aligned}$$

Hence $\lim_{\varepsilon\downarrow 0} c_{mn}(\lambda + i\varepsilon)$ exists. □

We put

$$\begin{aligned}
&C(\lambda+i0) = (c_{mn}(\lambda+i0))_{1\le m,n\le N} , \\
&h_m(z) = \sum_{j=1}^{N} c_{mj}(z)R_0(z)f_j \quad (\operatorname{Im} z \ne 0, 1 \le m \le N) .
\end{aligned} \tag{9}$$

THEOREM 2.4. $\lambda \in \sigma_{pp}(H_T)$ *if and only if the following condition is satisfied*:

$$\mathrm{Rank} C(\lambda + i0) \ne 0 .$$

If $\lambda \in \sigma_{pp}(H_T)$, *then* w-$\lim_{\varepsilon\downarrow 0} h_m(\lambda+i\varepsilon)$ $(1 \le m \le N)$ *exists and satisfies*

$$R_T(i)h_m(\lambda+i0) = \frac{1}{\lambda - i} h_m(\lambda + i0) , \tag{10}$$

and $\dim E_T(\{\lambda\})\mathcal{H} = \mathrm{Rank} C(\lambda+i0)$.

REMARK. (i) $\Delta(\lambda+i0) = 0$ follows from $\mathrm{Rank} C(\lambda+i0) \ne 0$. In fact, if $\Delta(\lambda+i0) \ne 0$, then $\lim_{\varepsilon\downarrow 0} \varepsilon\Delta_{mn}(\lambda+i\varepsilon)/\Delta(\lambda+i\varepsilon) = 0$ $(1 \le m, n \le N)$.

(ii) The equality (10) is desirable, because by comparing with Theorem 4.1 we expect that there exists $f(\ne 0) \in \mathcal{R}$ such that $H_T R_0(\lambda+i0)f = \lambda R_0(\lambda+i0)f$, i.e., $R_0(\lambda+i0)f$ is an eigenvector of H_T corresponding to λ.

To prove Theorem 2.4 we prove some lemmas.

LEMMA 2.5. *For $f \in \mathcal{R}$ we have*

(i)
$$\sup_{0<\varepsilon<1} \|R_0(\lambda + i\varepsilon)f\| < \infty$$

if and only if

$$\sup_{0<\varepsilon<1} \|R_0(i)R_0(\lambda + i\varepsilon)f\| < \infty\,,$$

(ii) $\lim_{\varepsilon\downarrow 0} \|R_0(\lambda + i\varepsilon)f\|$ *exists.* (*The value may be infinity.*)

PROOF. (i) By the resolvent equation we have

$$\begin{aligned}
&|z-i|^2\|R_0(z)R_0(i)f\|^2 \\
&\quad = \langle (R_0(z) - R_0(i))(R_0(\bar{z}) - R_0(-i))f, f\rangle \\
&\quad = \|R_0(z)f\|^2 - 2\,\mathrm{Re}\langle R_0(z)R_0(i)f, f\rangle + \|R_0(i)f\|^2\,.
\end{aligned}$$

Since the second term $\langle R_0(z)R_0(i)f, f\rangle$ converges as $\varepsilon \downarrow 0$ ($z = \lambda + i\varepsilon$) by Assumption $(T1)$, we obtain (i).

(ii) By the spectral representation of H_0 we see that $\|R_0(\lambda+i\varepsilon)R_0(i)f\|^2$ is monotonuously increasing as $\varepsilon \downarrow 0$. Hence we have (ii). □

LEMMA 2.6. *For any $u \in D(H_T)$ and for any $f \in \mathcal{R}$ we have*

$$\lim_{\varepsilon\downarrow 0} \varepsilon\langle u, R_0(\lambda + i\varepsilon)f\rangle = 0\,.$$

PROOF. It is sufficient to prove that $\lim_{\varepsilon\downarrow 0} \varepsilon\langle R_T(i)u, R_0(\lambda+i\varepsilon)f\rangle = 0$ for any $u \in \mathcal{H}$ and for any $f \in \mathcal{R}$. By (7) we can easily obtain

$$\begin{aligned}
&\langle R_T(i)u, R_0(\lambda + i\varepsilon)f\rangle \\
&\quad = \langle R_0(i)u, R_0(\lambda + i\varepsilon)f\rangle - \sum_{j=1}^{N}\langle R_0(i)u, g_j\rangle\langle R_0(i)f_j, R_0(\lambda + i\varepsilon)f\rangle \\
&\quad = \langle u, R_0(\lambda + i\varepsilon)R_0(-i)f\rangle - \sum_{j=1}^{N}\langle u, R_0(-i)g_j\rangle\langle R_0(i)f_j, R_0(\lambda + i\varepsilon)f\rangle\,.
\end{aligned}$$

Multiplying each side by ε, we have, by Assumption (T1) and $E_0(\{\lambda\}) = 0$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon\langle R_T(i)u, R_0(\lambda + i\varepsilon)f\rangle = 0\,.$$ □

LEMMA 2.7. *Assume that $u \in D(H_T)$ satisfies $H_T u = \lambda u$. If $\langle u, R_0(-i)f\rangle = 0$ for any $f \in \mathcal{R}$, then $u = 0$.*

PROOF. Since u is an eigenvector of H_T and by (7), we can easily see that

$$\frac{1}{\lambda - i}u = R_T(i)u = R_0(i)u\,.$$

Hence we have $u \in D(H_0)$ and $H_0 u = \lambda u$. By $\sigma_{pp}(H_0) = \emptyset$ we conclude $u = 0$. $\square$

LEMMA 2.8. *For $h_m(z)$ in* (9), *we have*

(i) $\text{w-}\lim_{\varepsilon\downarrow 0} h_m(\lambda + i\varepsilon)$ $(1 \le m \le N)$ *exists*,

(ii) $\langle R_T(i) h_m(\lambda + i0), u\rangle = \frac{1}{\lambda - i}\langle h_m(\lambda + i0), u\rangle$ *for any* $u \in \mathcal{H}$,

(iii) $\dim L.h.[h_1(\lambda + i0), \cdots, h_N(\lambda + i0)] = \text{Rank} C(\lambda + i0)$.

PROOF. (i) By (7) and $E_0(\{\lambda\}) = 0$ for $u \in \mathcal{H}$ we have

$$\begin{aligned}
&\langle R_0(-i) E_T(\{\lambda\}) R_0(i) f_m, u\rangle = \lim_{\varepsilon\downarrow 0}(-i\varepsilon\langle R_0(-i) R_T(\lambda + i\varepsilon) R_0(i) f_m, u\rangle) \\
&\quad = \lim_{\varepsilon\downarrow 0}\{-i\varepsilon\langle R_0(-i) R_0(\lambda + i\varepsilon) R_0(i) f_m, u\} \\
&\qquad + i\varepsilon(\lambda + i\varepsilon - i)^{-1}\langle R_0(-i) R_0(\lambda + i\varepsilon) f_m, u\rangle + \langle R_0(-i) h_m(\lambda + i\varepsilon), u\rangle)\} \\
&\quad = \lim_{\varepsilon\downarrow 0}\langle R_0(-i) h_m(\lambda + i\varepsilon), u\rangle\,.
\end{aligned}$$

This means that $\text{w-}\lim_{\varepsilon\downarrow 0} R_0(-i) h_m(\lambda + i\varepsilon)$ exists and is equal to

$$R_0(-i) E_T(\{\lambda\}) R_0(i) f_m\,.$$

By Lemma 2.5 (i) $h_m(\lambda + i\varepsilon)$ is bounded. Since $\mathcal{H}_2$ is dense in $\mathcal{H}$, by the standard argument we conclude that $\text{w-}\lim_{\varepsilon\downarrow 0} h_m(\lambda + i\varepsilon) = E_T(\{\lambda\}) R_0(i) f_m$.

(ii) By (i) it is suffcient to prove that

$$R_T(i) h_m(z) = \frac{1}{z - i} h_m(z) - \frac{\lambda - z}{(z - i)^2} R_0(i) f_m, \quad (1 \le m \le N)\,.$$

Using (7) and $\Delta(i) = 1$ we have

$$\begin{aligned}
&R_T(i) h_m(z) \\
&\quad = R_0(i)\sum_{j=1}^N c_{mj}(z) R_0(z) f_j - \sum_{k=1}^N \left\langle R_0(i)\sum_{j=1}^N c_{mj}(z) R_0(z) f_j, g_k\right\rangle R_0(i) f_k \\
&\quad = \frac{1}{z - i}\sum_{j=1}^N c_{mj}(z) R_0(z) f_j - \frac{1}{z - i}\sum_{j=1}^N c_{mj}(z) R_0(i) f_j \\
&\qquad - \sum_{j=1}^N\sum_{k=1}^N c_{mj}(z)\langle R_0(i) R_0(z) f_j, g_k\rangle R_0(i) f_k \\
&\quad = \frac{1}{z - i} h_m(z) \\
&\qquad - \frac{1}{z - i}\left(\sum_{j=1}^N c_{mj}(z) R_0(i) f_j + \sum_{k=1}^N\sum_{j=1}^N c_{mj}(z)\langle (z - i) R_0(i) R_0(z) f_j, g_k\rangle R_0(i) f_k\right).
\end{aligned}$$

Calculating the sum of j of the third term in the right hand side, we see that

$$\sum_{j=1}^{N} c_{mj}(z)\langle W(z)f_j, g_k\rangle = \sum_{j=1}^{N} c_{mj}(z)v_{kj}(z) = \frac{\lambda - z}{z - i}\Delta_{mk} - c_{mk}(z)\,.$$

Hence we obtain (ii).

(iii) For simplicity we write

$$h_m = h_m(\lambda + i0), \quad c_{mj} = c_{mj}(\lambda + i0), \quad C = C(\lambda + i0), \quad v_{mj} = v_{mj}(\lambda + i0)\,.$$

Putting $A = \{(\alpha_1, \cdots, \alpha_N) \in \mathbf{C}^N; \sum_{m=1}^{N} \alpha_m h_m = 0\}$, we calculate dim A. By (ii) and Lemma 2.7 we see that

$$\dim A = \dim \left\{(\alpha_1, \cdots, \alpha_N) \in \mathbf{C}^N; \sum_{m=1}^{N} \alpha_m \langle h_m, R_0(-i)g_k\rangle = 0, 1 \le k \le N\right\}.$$

By (ii) we can justify the following calculation: for $1 \le k \le N$

$$\begin{aligned} 0 &= \left\langle R_T(i)\sum_{m=1}^{N} \alpha_m h_m, R_0(-i)g_k\right\rangle = \frac{1}{\lambda - i}\sum_{m=1}^{N} \alpha_m \langle h_m, R_0(-i)g_k\rangle \\ &= \frac{1}{\lambda - i}\sum_{m=1}^{N} \alpha_m \sum_{j=1}^{N} c_{mj}\langle R_0(\lambda + i0)f_j, R_0(-i)g_k\rangle \\ &= \frac{1}{(\lambda - i)^2}\sum_{m=1}^{N} \alpha_m \sum_{j=1}^{N} c_{mj}v_{kj} = \frac{1}{(\lambda - i)^2}\sum_{m=1}^{N} \alpha_m c_{mk}\,. \end{aligned}$$

Hence we have $\dim A = \dim\ker {}^tC$. Therefore we conclude that

$$\dim L.h.[h_1, \cdots, h_N] = \mathrm{Rank} C\,. \qquad \square$$

LEMMA 2.9. *If $u \in D(H_T)$ satisfies $H_T u = \lambda u$, then $\langle R_0(i)f_m, u\rangle = \langle h_m, u\rangle$ for $1 \le m \le N$.*

PROOF. Combining (7) and Lemma 2.6 we see that

$$\langle R_0(i)f_m, u\rangle = \lim_{\varepsilon\downarrow 0}(-i\varepsilon\langle R_T(\lambda + i\varepsilon)R_0(i)f_m, u\rangle) = \langle h_m, u\rangle\,. \qquad \square$$

LEMMA 2.10. *Let u_j satisfy $H_T u_j = \lambda u_j$ $(1 \le j \le N)$. Then*

$$\dim L.h.[u_1, \cdots, u_N] \le \mathrm{Rank} C\,.$$

PROOF. By Lemma 2.7 we see that

$$\begin{aligned} &\left\{(\alpha_1, \cdots, \alpha_N) \in \mathbf{C}^N; \sum_{j=1}^{N} \alpha_j u_j = 0\right\} \\ &\quad = \left\{(\alpha_1, \cdots, \alpha_N) \in \mathbf{C}^N; \sum_{j=1}^{N} \alpha_j \langle u_j, R_0(-i)f_m\rangle = 0, (1 \le m \le N)\right\}. \end{aligned}$$

Hence we have

$$\dim L.h.[u_1, \cdots, u_N] = \dim L.h.[\mathbf{a}_1, \cdots, \mathbf{a}_N] = \mathrm{Rank}[\mathbf{a}_1, \cdots, \mathbf{a}_N]$$

where $\mathbf{a}_j = {}^t(\langle u_j, R_0(-i) f_1\rangle, \cdots, \langle u_j, R_0(-i) f_N\rangle)$. By Lemma 2.9 (ii) we have

$$\mathrm{Rank}[\mathbf{a}_1, \cdots, \mathbf{a}_N] = \mathrm{Rank}[\mathbf{b}_1, \cdots, \mathbf{b}_N]$$

where $\mathbf{b}_j = {}^t(\langle u_j, h_1\rangle, \cdots, \langle u_j, h_N\rangle)$. Since $\dim L.h.[h_1, \cdots, h_N] = \mathrm{Rank}C$ by Lemma 2.8, we have proved this lemma. □

PROOF OF THEOREM 2.4. We have already obtained (10) by Lemma 2.8 (ii). So we prove the rest of the statements. Let $\lambda \in \sigma_{pp}(H_T)$ and u an eigenvector of H_T corresponding to λ. We prove $\mathrm{Rank}C \neq 0$. We assume that $\mathrm{Rank}C = 0$. Combining Lemma 2.8 (ii), (iii) and Lemma 2.9, we have $0 = \langle u, 0\rangle = \langle u, h_m\rangle = \langle u, R_0(i) f_m\rangle$, $(1 \leq m \leq N)$. By Lemma 2.7 we have $u = 0$, which is a contradiction.

Conversely we assume $\mathrm{Rank}C \neq 0$. Then there exists, at least, an (m, n) such that $c_{mn} \neq 0$. By (8) we see that

$$\begin{aligned}\langle E_T(\{\lambda\}) R_0(i) f_m, R_0(-i) g_n\rangle &= \lim_{\varepsilon \downarrow 0}(-i\varepsilon\langle R_T(\lambda + i\varepsilon) R_0(i) f_m, R_0(-i) g_n\rangle \\ &= c_{mn}(\lambda + i0)\,.\end{aligned}$$

Hence we obtain $E_T(\{\lambda\}) \neq 0$.

We prove that $\dim E_T(\{\lambda\}) = \mathrm{Rank}C$. In general, we remark that $N \geq \dim E_T(\{\lambda\})$. By Lemma 2.8 (ii) and (iii) $\dim E_T(\{\lambda\}) \geq \dim L.h.[h_1, \cdots, h_N]$ ($= \mathrm{Rank}C$). On the other hand, by Lemma 2.10 we have $\dim E_T(\{\lambda\})\mathcal{H} \leq \mathrm{Rank}C$. We have thus completed the proof of Theorem 2.4. □

3. Asymptotic completeness of wave operators.

In this section we consider the asymptotic completeness of the wave operators $W_\pm(H_0, H_T)$. (We use the same notations as in section 2.) In general, the wave operators $W_\pm(H_1, H_2)$ for self-adjoint operators H_1 and H_2 are defined by

$$W_\pm(H_1, H_2) := s - \lim_{t \to \pm\infty} e^{itH_2} e^{itH_1} P_{ac}(H_1)\,,$$

where $P_{ac}(H_1)$ is the projection for the absolutely continuous subspace of H_1. If $W_\pm(H_1, H_2)$ exists, then we say that $W_\pm(H_1, H_2)$ are *complete* if and only if Range $W_\pm = P_{ac}(H_2)$. And we say that $W_\pm(H_1, H_2)$ is *asymptotically complete* if and only if $W_\pm(H_1, H_2)$ is complete and $\sigma_{sing}(H_2) = \emptyset$.

THEOREM 3.1. *The wave operators $W_\pm(H_0, H_T)$ are asymptotically complete.*

REMARK. As for the scattering matrix, it inverstigated in [8] for more general T. And see [4] in the case of the usual Laplacian $H_0 = -\Delta$ and $RankT = 1$.

Using the following theorem we can easily see that $W_\pm(H_0, H_T)$ are complete.

THEOREM 3.2 (Kuroda-Birman theorem, [5, Theorem XI.9]). *Let H_1 and H_2 be self-adjoint operators such that $(H_1 - z)^{-1} - (H_2 - z)^{-1}$ is of trace class for some $z \in \rho(H_1) \cap \rho(H_2)$. Then $W_\pm(H_1, H_2)$ exist and are complete.*

Since

$$R_T(i)u - R_0(i)u = \sum_{j=1}^{N} \langle R_0(i)u, g_j\rangle R_0(i) f_j, \quad u \in \mathcal{H},$$

we see that $W_\pm(H_0, H_T)$ exist and are complete. Hence, in order to show the asymptotic completeness of $W_\pm(H_0, H_T)$ it remains only to verify $\sigma_{sing}(H_T) = \emptyset$.

LEMMA 3.3. *Put $\mathcal{N}_\pm := \{\lambda \in \mathbf{R}; \Delta(\lambda \pm i0) = 0\}$. Then $\mathcal{N}_+ = \mathcal{N}_-$ and $\mathcal{N}_\pm$ is discrete.*

PROOF. We prove $\mathcal{N}_+ = N_-$. Putting

$$A = (\mu_j \delta_{jk})_{1\le j,k\le N}, \quad B(z) = (w_{jk}(z))_{1\le j,k\le N},$$

where $w_{jk}(z) = \langle W(z) f_k, f_j\rangle$, we see that $V(z) = AB(z)$. Since $w_{jk}(\bar{z}) = \overline{w_{kj}(z)}$, we have

$$\begin{aligned}\det(I + V(\bar{z})) &= \det(I + AB^*(z)) = \det((I + B(z)A)^*) \\ &= \overline{\det(I + B(z)A)} = \overline{\det(A^{-1}(I + AB(z))A)} = \overline{\det(I + AB(z))} = \overline{\Delta(z)}.\end{aligned}$$

Hence $\mathcal{N}_+ = \mathcal{N}_-$.

We put $\mathcal{N} := \mathcal{N}_+ = \mathcal{N}_-$ and prove that $\mathcal{N}_\pm$ is discrete. Assume, for contradition, that $\mathcal{N}$ is dense in some open interval (a, b). Since $\Delta(\lambda + i0)$ is continuous in (a, b) by Assumption (T) and (8), we see that $\Delta(\lambda + i0) = 0$ in (a, b). Since $\Delta(z)$ is analytic in $\{z \in \mathbf{C}; \operatorname{Im} z > 0\}$, by the reflection principle of the analytic function there exists some $\varepsilon > 0$ such that $\Delta(z)$ has an analytic continuation $\tilde{\Delta}(z)$ in $(a, b) \times [-i\varepsilon, i\varepsilon]$. So by the identity theorem of the analytic function, we see that $\tilde{\Delta}(z) = 0$ in $(a, b) \times [-i\varepsilon, i\varepsilon]$. This is a contradiction.

Follwing [5, section XIII], we prove that $\sigma_{sing}(H_T) = \emptyset$. By Weyl's theorem we see that $\sigma_{ess}(H_T) = \sigma_{ess}(H_0) = \mathbf{R}$. So it is sufficient to prove $\sigma_{sing}(H_T) \cap [0, \infty) = \emptyset$. We put $\mathcal{N} := \mathcal{N}_\pm$. If we prove $\sigma_{sing}(H_T) \subset \mathcal{N} \cup \{0\}$, then $\sigma_{sing}(H_T)$ is a countable set and hence $\sigma_{sing}(H_T) = \emptyset$. Since $\mathcal{N}$ is discrete, we can take an open interval (a, b) such that $[a, b] \cap (\mathcal{N} \cup \{0\}) = \emptyset$. We remark that for $u \in \mathcal{D}\langle R_T(\lambda + i0)u, u\rangle - \langle R_0(\lambda + i0)u, u\rangle$ are continuous in (a, b) by Assumption (T2). By the continuity of $\langle R_T(\lambda + i0)u, u\rangle$ $(u \in \mathcal{D})$ and Stone's formula, i.e.,

$$\langle E_T((a, b))u, u\rangle = \frac{1}{2\pi i}\int_a^b (\langle R_T(\lambda + i0)u, u\rangle - \langle R_T(\lambda + i0)u, u\rangle)d\lambda,$$

we have

$$\langle E_T((a, b))u, u\rangle \in C_1(u)(b - a) + \langle E_0((a, b))u, u\rangle,$$

where $C_1(u)$ is a constant dependent on u. Hence we have $E_T((a, b))\mathcal{D} \subset P_{ac}(H_T)\mathcal{H}$. Since $P_{ac}(H_T)\mathcal{H}$ is closed in $\mathcal{H}$ and $\mathcal{D}$ is dense in $\mathcal{H}$, $E_T((a, b))\mathcal{H} \subset P_{ac}(H_T)\mathcal{H}$. □

4. Rank one perturbation.

We consider a rank one perturbation. We assume Assumption (H_0), (T) and $\dim \mathcal{R} = 1$, and put $f := f_1$, $\mu := 1$ and $\Delta(z) = 1 + (z-i)\langle R_0(z)R_0(i)f, f\rangle$.

THEOREM 4.1. *Let* $\lambda \in \mathbf{R}$. $\lambda \in \sigma_{pp}(H_T)$ *if and only if the following condition is satisfied*:

$$C(\lambda + i0) := \lim_{\varepsilon \downarrow 0} \frac{i\varepsilon}{\Delta(\lambda + i\varepsilon)} \neq 0\,.$$

Under the condition above, we have

$$\lim_{\varepsilon \downarrow 0} \|R_0(\lambda + i\varepsilon)f\|^2 = \mathrm{Re}\left(\frac{1}{C(\lambda + i0)}\right) < \infty\,,$$

and $R_0(\lambda + i0)f$ *is an eigenvector of* H_T *corresponding to* λ.

PROOF. By Theorem 2.4 we have already obtained the first and the last statements. So we prove the second statement. We can take $u_n \in \mathcal{H}$ such that $u_n \to f$ $(n \to \infty)$ in $\mathcal{H}_{-2}$. We have, by the resolvent equation,

$$\begin{aligned}
&1 + (\lambda + i\varepsilon - i)\langle R_0(\lambda + i\varepsilon)R_0(i)u_n, u_n\rangle - (1 + (\lambda - i\varepsilon - i)\langle R_0(\lambda - i\varepsilon)R_0(i)u_n, u_n\rangle) \\
&\quad = \langle (R_0(\lambda + i\varepsilon) - R_0(i))u_n, u_n\rangle - \langle (R_0(\lambda - i\varepsilon) - R_0(i))u_n, u_n\rangle \\
&\quad = 2i\varepsilon\langle R_0(\lambda + i\varepsilon)R_0(\lambda - i\varepsilon)u_n, u_n\rangle = 2i\varepsilon\|R_0(\lambda + i\varepsilon)u_n\|^2\,.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\Delta(\lambda + i\varepsilon) - \Delta(\lambda - i\varepsilon) = 2i\varepsilon\|R_0(\lambda + i\varepsilon)f\|^2\,.$$

Taking account of $\Delta(\bar{z}) = \overline{\Delta(z)}$, we have

$$\mathrm{Re}\left(\frac{\Delta(\lambda + i\varepsilon)}{i\varepsilon}\right) = \|R_0(\lambda + i\varepsilon)f\|^2\,.$$

Hence we have

$$\lim_{\varepsilon \downarrow 0} \|R_0(\lambda + i\varepsilon)f\|^2 = \mathrm{Re}\left(\frac{1}{C(\lambda + i0)}\right)\,.$$

The rest of the proof is to show $\mathrm{Re}\, C(\lambda + i0) \neq 0$. Assume that $\mathrm{Re}\, C(\lambda + i0) = 0$ and put $a(\varepsilon) := \mathrm{Re}(i\varepsilon/\Delta(\lambda + i\varepsilon))$ $(\lim_{\varepsilon\downarrow 0} a(\varepsilon) = 0)$. Then

$$\begin{aligned}
a(\varepsilon) &= \frac{i\varepsilon}{\Delta(\lambda + i\varepsilon)} + \frac{-i\varepsilon}{\Delta(\lambda - i\varepsilon)} = \frac{i\varepsilon(\Delta(\lambda - i\varepsilon) - \Delta(\lambda + i\varepsilon))}{|\Delta(\lambda + i\varepsilon)|^2} \\
&= \frac{2\varepsilon^2\|R_0(\lambda + i\varepsilon)f\|^2}{|\Delta(\lambda + i\varepsilon)|^2}\,.
\end{aligned}$$

Remark that $\lim_{\varepsilon\downarrow 0} \|R_0(\lambda + i\varepsilon)f\|$ exists by Lemma 2.5 (ii). If we assume that $\lim_{\varepsilon\downarrow 0} \|R_0(\lambda + i\varepsilon)f\| > 0$, then we have $\lim_{\varepsilon\downarrow 0} 2\varepsilon^2/|\Delta(\lambda + i\varepsilon)|^2 = 0$. This is a contradiction to $C(\lambda + i0) \neq 0$. Therefore $\lim_{\varepsilon\downarrow 0} \|R_0(\lambda + i\varepsilon)f\| = 0$. Now we consider $\|R_0(\lambda + i\varepsilon)R_0(i)f\|$. We see that $\lim_{\varepsilon\downarrow 0} \|R_0(\lambda + i\varepsilon)R_0(i)f\| = \lim_{\varepsilon\downarrow 0} \|R_0(i)R_0(\lambda + i\varepsilon)f\| = 0$. Hence $R_0(\lambda + i0)(R_0(i)f) = 0$, and so $R_0(i)f = 0$. We reach a contradiction to $\|R_0(i)f\| = 1$.

We quote two examples without the proofs (see [4, 6]).

EXAMPLE 4.1 (cf. [6]). Let $\mathcal{H} = L^2(\mathbf{R}^3)$ and $H_0 = -\Delta$ (the usual Laplacian) with the domain $D(H_0) = H^2(\mathbf{R}^3)$ (Sobolev space of order 2). And let $t(x_1) \in L^1(\mathbf{R})$, $f(x_1, x_2, x_3) = t(x_1)\delta(x_2, x_3)$ and $T_u = \alpha\langle u, f\rangle f$. Then we can take $\mathcal{D}$ in Assumption (T) as $L^{2,s}(\mathbf{R}^3) = \{u \in L^2(\mathbf{R}^3); (1 + |x|)^s u(x) \in L^2\}$ $(s > 3/2)$.

EXAMLE 4.2 (cf. [4]). Let $\mathcal{H}$ and H_0 be the same as above. And let $t(x_1, x_2) \in L^1(\mathbf{R}^2)$, $f(x_1, x_2, x_3) = t(x_1, x_2)\delta(x_3)$ and $T_u = \alpha\langle u, f\rangle f$. Then we can take $\mathcal{D}$ in Assumption (T) as $L^{2,s}(\mathbf{R}^3)$ $(s > 3/2)$.

We give a brief comment of the relation between their results ([4], [6]) and ours. In [4, 6], under a (stronger) assumption that t is almost in some weighted L^1-space, they showed the asymptotic completeness of the wave operators for H_0 and H_T. By using Theorem 3.1 we can show the asymptotic completeness under a (weaker) assumption that t is in L^1.

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