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Multiplicative SK Invariants for G-Manifolds with Boundary

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0. Introduction

Let G be a finite abelian group. In this paper, a G-manifold means an unoriented compact smooth manifold (which may have boundary) together with a smooth action of G. Let T be a map for m-dimensional G-manifolds which takes its values in the ring $\mathbf{Z}$ of rational integers and is additive with respect to the disjoint union of G-manifolds. We call T a G-SK invariant if it is invariant under equivariant cuttings and pastings (Schneiden und Kleben in German) [5, 6, 9]. For example, χ^H given by $\chi^H(M) = \chi(M^H)$ for G-manifolds M is a G-SK invariant, where χ is the Euler characteristic, H is a subgroup of G and $M^H = \{x \in M \mid hx = x \text{ for any } h \in H\}$. Further suppose that T is defined for all G-manifolds with various dimensions. Then it is said to be multiplicative if $T(M \times N) = T(M) \cdot T(N)$ for any G-manifolds M and N. For example, the above χ^H is multiplicative.

The main object of this paper is to characterize a form of multiplicative G-SK invariants. In [1, 3], the author has discussed such a question in case where G is a cyclic group of finite order.

In Section 1, we describe the irreducible G-modules and G-slice types. These notions are needed in order to proceed with our argument.

In Section 2, we first introduce an SK group $SK_*^G(\partial)$ resulting from equivariant cuttings and pastings of G-manifolds. In [4, 8], Koshikawa and the author have studied its SK_*-module structure, where SK_* is an SK ring of closed manifolds (Proposition 2.2). A G-SK invariant T induces an additive homomorphism $SK_*^G(\partial) \to \mathbf{Z}$. For a slice type σ, let χ_σ be a G-SK invariant defined by $\chi_\sigma(M) = \chi(M_\sigma)$, where M_σ is a G-submanifold of M with slice types containing σ (Definition 2.5). Then, using these χ_σ, we have a basis of a free $\mathbf{Z}$-module $\mathcal{T}_*^G$ consisting of all G-SK invariants [2] (Proposition 2.8). Next we study a multiplicative G-SK invariant, which is considered to be a ring homomorphism $SK_*^G(\partial) \to \mathbf{Z}$. Such an invariant T is said to be of type $\langle G/H \rangle$ if H is the minimum element (with respect to the inclusion of subgroups) in the set consisting of those subgroups K of G such that $T(G/K) \neq 0$ (Definition 2.11). For example, χ^H is of type $\langle G/H \rangle$. It is seen that T

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is determined by its values on the one-dimensional disk D^1 (with the trivial action) and G-manifolds $G \times_H D(V_i)$, where $\{V_i\}$ is the complete set of non-trivial irreducible H-modules and $D(V_i)$ is the associated H-disk (Theorem 2.17). Finally we give a typical example of such invariants (Example 2.21).

1. Preliminaries

A G-module means a finite-dimensional real vector space together with a linear action of G. For a subgroup H of G, let $C(H)$ consist of all subgroups J of H such that the quotient $H/J \cong \mathbf{Z}_d$, a cyclic group of order d (≥ 2). Then, for $J \in C(H)$ a non-trivial irreducible H-module $V(J, j)$ $(1 \leq j < \frac{1}{2}\phi(d) + 1)$, where ϕ is the Euler phi-function, is defined as follows.

(1) If $d = 2$, then the underlying space of $V(J, 1)$ is the set $\mathbf{R}$ of real numbers with a generator of $H/J \cong \mathbf{Z}_2$ acting by multiplication by -1.

(2) If $d \geq 3$, then the underlying space of $V(J, j_k)$ is the set $\mathbf{C}$ of complex numbers with a generator of $H/J \cong \mathbf{Z}_d$ acting by multiplication by $\exp(2\pi i j_k/d)$, where $\{j_k\}$ is the complete set of integers such that $0 < j_1 < j_2 < \cdots < j_{\phi(d)} < d$ and each j_k is prime to d (cf. [9 ; Theorem 1.6.1]).

If M is a G-manifold and $x \in M$, then there is a G_x-module U_x which is equivariantly diffeomorphic to a G_x-neighbourhood of x. Here $G_x = \{g \in G \mid gx = x\}$ is the isotropy subgroup at x. The module U_x decomposes as $U_x = \mathbf{R}^p \oplus V_x$, where G_x acts trivially on $\mathbf{R}^p$ and $V_x^{G_x} = \{0\}$. We refer to the pair $\sigma_x = [G_x; V_x]$ as a slice type of x. By a G-slice type in general, we mean a pair $\sigma = [H; V]$ of a subgroup H of G and an H-module V such that $V^H = \{0\}$. An H-module V is a product of non-trivial irreducible H-modules $V(J, j_k)$ with $J \in C(H)$. We denote by σ_0 the slice type $[\{1\}; \{0\}]$, where $\{1\}$ is the trivial group. Let $St[H]$ be the set of all slice types $\sigma = [H; V]$ and $St(G) = \bigcup_{H \subseteq G} St[H]$ the set of all G-slice types. There is a total ordering on $St(G)$ as follows. For any positive divisor k of $|G|$, the order of G, let $L(k)$ be the set consisting of all subgroups H of G such that $|H| = k$. First order the elements in $L(k)$ appropriately, then give an ordering $<$ on the set $\bigcup_k L(k)$ of all subgroups of G, preserving inclusion of subgroups, that is, if $H \subseteq K$ then $H \leq K$. Let us fix a subgroup H, then such an ordering leads to the one on the set of all non-trivial irreducible H-modules: $V(J_1, j_1) < V(J_2, j_2)$ if $J_2 < J_1$ or $J_1 = J_2$ and $j_1 < j_2$. Finally we order the elements in $St(G)$ as follows.

(1) $[H; V] < [K; W]$ if $\dim(V) < \dim(W)$.

(2) Suppose that $\dim(V) = \dim(W)$, then $[H; V] < [K; W]$ if $H < K$.

(3) Suppose that $\dim(V) = \dim(W)$ and $H = K$, then $[H; V] < [H; W]$ if $V < W$ in the ordering of H-modules induced lexicographically from the one of all non-trivial irreducible H-modules: $V_1 < V_2 < \cdots < V_{t_H}$ (cf. [9 ; Section 1.7]). Here, by the definition of the total ordering, there is an integer p_H such that the underlying space of each V_i $(1 \leq i \leq p_H)$ is $\mathbf{R}$, while the one of each V_i $(p_H < i \leq t_H)$ is $\mathbf{C}$. We regard $p_H = 0$ if H is an odd order

group. A slice type $\sigma \in St[H]$ is therefore of the form $\sigma = \sigma^H(a(1), \cdots, a(p_H); a(p_H + 1), \cdots, a(t_H)) = [H; \prod_i V_i^{a(i)}]$ for some integers $a(i)$ (≥ 0).

DEFINITION 1.1. Let W be a K-module and H a subgroup of K. Then denote by $W_{(H)}$ an H-module W induced from $H \subseteq K$. Let $\{W_j\} : W_1 < \cdots < W_{p_K} < W_{p_K+1} < \cdots < W_{t_K}$ be the set of all non-trivial irreducible K-modules. If $\tau = [K; W] \in St[K]$, $W = \prod_j W_j^{b(j)}$, is a slice type, then we define a slice type $\tau_{(H)} \in St[H]$ by $\tau_{(H)} = [H; V]$, $V = \prod_j (W_j)_{(H)}^{b(j)}$, where the product is taken over all j such that $(W_j)_{(H)}$ are non-trivial H-modules. Since $(W_j)_{(\{1\})} = \mathbf{R}$ or $\mathbf{R}^2$, a trivial H-module $(1 \leq j \leq t_K)$, we have that $\tau_{(\{1\})} = \sigma_0$. We call $\dim(W) = \sum_{1 \leq j \leq p_K} b(j) + 2 \sum_{p_K < j \leq t_K} b(j)$ the dimension of τ and denote it by $|\tau|$.

REMARK 1.2. (i) More precisely, let $W_j = V(L, m_k)$ for some $L \subset K$ with $K/L \cong \mathbf{Z}_a$ and an integer m_k such that $0 < m_k < a$, $(m_k, a) = 1$. Then $(W_j)_{(H)} = V(L \cap H, m')$ with $0 < m' < b$, $(m', b) = 1$, where $H/(L \cap H) = LH/L \cong \mathbf{Z}_b$. The integer m' is determined by the action LH/L on $(W_j)_{(H)}$ induced from the one of K/L on W_j. We see that $(W_j)_{(H)}$ is the trivial H-module $\mathbf{R}$ or $\mathbf{R}^2$ only if $H \subseteq L$. It follows that the difference $|\tau| - |\tau_{(H)}|$ is the sum of $\dim(W_j)$ $(= 1$ or $2)$ with $H \subseteq L$.

(ii) $W_{(H)} = \mathbf{R}^{|\tau|-|\tau_{(H)}|} \times V$ as an H-module and $W^H = (W_{(H)})^H = \mathbf{R}^{|\tau|-|\tau_{(H)}|} \times \{0\}$ has slice types $\tau_{(U)}$ $(H \subseteq U \subseteq K)$ as a K-invariant subspace of W. Note that $\tau_{(U)} \leq \tau$ because $|\tau_{(U)}| \leq |\tau|$.

(iii) Let us write $\tau = \tau^K(b(1), \cdots, b(p_K); b(p_K + 1), \cdots, b(t_K))$ for τ in Definition 1.1. Then we have $\tau_{(H)} = \sigma^H(a(1), \cdots, a(p_H); a(p_H + 1), \cdots, a(t_H))$, where

$$\begin{aligned} a(i) &= \sum_{j \in J'(i)} b(j) + 2 \sum_{j \in J''(i)} b(j) \ (1 \leq i \leq p_H)\,, \\ a(i) &= \sum_{j \in J(i)} b(j) \ (p_H < i \leq t_H)\,. \end{aligned} \tag{1.2.1}$$

The sets $J(i) = J(H, K; i)$, $J'(i)$ and $J''(i)$ are as follows: $J(i)$ $(0 \leq i \leq t_H)$ are subsets of $J(K) = \{j \mid 1 \leq j \leq t_K\}$ given by

$J(i) = J'(i) \cup J''(i)$ if $1 \leq i \leq p_H$, where

$J'(i) = \{j \mid (W_j)_{(H)} = V_i, 1 \leq j \leq p_K\}$ and $J''(i) = \{j \mid (W_j)_{(H)} = V_i^2, p_K < j \leq t_K\}$.

$J(i) = \{j \mid (W_j)_{(H)} = V_i, p_K < j \leq t_K\}$ if $p_H < i \leq t_H$.

$J(0) = J(K) \setminus \bigcup_{1 \leq i \leq t_H} J(i) = J'(0) \cup J''(0)$, where

$J'(0) = \{j \mid (W_j)_{(H)} = \mathbf{R}, 1 \leq j \leq p_K\}$ and $J''(0) = \{j \mid (W_j)_{(H)} = \mathbf{R}^2, p_K < j \leq t_K\}$.

The set $J(K)$ is a disjoint union of these $J(i) = J(H, K; i)$ $(0 \leq i \leq t_H)$.

(iv) It follows that

$$|\tau| - |\tau_{(H)}| = \sum_{j \in J'(0)} b(j) + 2 \sum_{j \in J''(0)} b(j) . \tag{1.2.2}$$

2. Multiplicative G-SK invariants

Let N_i $(i = 1, 2)$ be m-dimensional G-manifolds, L an G-invariant codimension zero submanifold of each boundary ∂N_i and $\varphi, \psi : L \to L$ G-equivariant diffeomorphisms. Pasting along L, we have G-manifolds $M_1 = N_1 \cup_\varphi N_2$ and $M_2 = N_1 \cup_\psi N_2$. Then M_1 and M_2 are said to be obtained from each other by an equivariant cutting and pasting (G-SK process). Let $\mathcal{M}_m^G(\partial)$ be the set of all m-dimensional G-manifolds with boundary, which is an abelian semigroup with respect to the disjoint union $+$ and has a zero given by the empty set $\emptyset$.

DEFINITION 2.1 (cf. [6 ; Chapter 1]). G-manifolds M_1 and $M_2 \in \mathcal{M}_m^G(\partial)$ are said to be G-SK equivalent, in symbols $M_1 \sim M_2$, if there is a G-manifold $P \in \mathcal{M}_m^G(\partial)$ such that $M_1 + P$ and $M_2 + P$ can be obtained from each other by a finite sequence of G-SK processes.

The G-SK equivalence $\sim$ is an equivalence relation on $\mathcal{M}_m^G(\partial)$ and the set $\mathcal{M}_m^G(\partial)/\sim$ of all equivalence classes is a cancellative abelian semigroup. Denote by $[M]$ the equivalence class containing a G-manifold M. Let $SK_m^G(\partial)$ be the Grothendieck group of $\mathcal{M}_m^G(\partial)/\sim$. We then have a graded SK_*-module $SK_*^G(\partial) = \bigoplus_{m \geq 0} SK_m^G(\partial)$ given by the cartesian product of manifolds. Here SK_* is an SK ring of closed manifolds, which is a polynomial ring over $\mathbf{Z}$ with a generator α represented by the real projective plane $\mathbf{R}P^2$ (cf. [9; Theorem 2.5.1 (i)]).

PROPOSITION 2.2 (cf. [4; Proposition 1.13]). *$SK_*^G(\partial)$ is a free SK_*-module with basis $\mathcal{B} = \{[G \times_H D(\sigma)], [G \times_H D(\sigma \times \mathbf{R})] \mid \sigma = [H; V] \in St(G)\}$, where $D(\sigma) = D(V)$ is the associated H-disk.*

Let $M \times N$ be the cartesian product of G-manifolds M and N (with straightening the angle). This product makes $SK_*^G(\partial)$ an SK_*-algebra.

For $\sigma = \sigma^H(a(1), \cdots, a(t_H))$ and $\tau = \sigma^H(b(1), \cdots, b(t_H)) \in St[H]$, we denote by $\sigma \times \tau$ the slice type $\sigma \times \tau = \sigma^H(a(1) + b(1), \cdots, a(t_H) + b(t_H))$.

LEMMA 2.3 (cf. [2; Lemma 3.6]). *A multiplicative relations for the basis elements in $\mathcal{B}$ are given by the following.*

(i) $[G \times_H D(\sigma)] \cdot [G \times_K D(\tau)] = a(H, K)[D^b][G \times_{H\cap K} D(\sigma_{(H\cap K)} \times \tau_{(H\cap K)})]$ *for any $\sigma \in St[H]$ and $\tau \in St[K]$, where $a(H, K) = (|G||H \cap K|)/(|H||K|)$ and $b = |\sigma| - |\sigma_{(H\cap K)}| + |\tau| - |\tau_{(H\cap K)}|$.*

(ii) *$\widehat{x} \cdot y = x \cdot \widehat{y} = \widehat{x \cdot y}$ and $\widehat{(\widehat{x})} = \alpha x$ for any elements x, y, where $\widehat{x} = [D^1] \cdot x$ in general.*

(ii-i) *In particular, $\alpha = [D^2]$ in $SK_2(\partial)$, where $SK_*(\partial) = SK_*^{\{1\}}(\partial)$ is an SK ring of manifolds with boundary.*

DEFINITION 2.4 (cf. [5], [6 ; Chapter 1] and [9 ; Definition 5.2.5]). Let $T : \mathcal{M}_m^G(\partial) \to \mathbf{Z}$ be an additive map, that is, if $M = M_1 + M_2$ then $T(M) = T(M_1) + T(M_2)$. We call T a G-SK invariant if T is invariant under G-SK process, that is $T(N_1 \cup_\varphi N_2) = T(N_1 \cup_\psi N_2)$ for any G-diffeomorphisms $\varphi, \psi : L \to L$ in the beginning of this section. The map T induces an additive homomorphism $T : SK_*^G(\partial) \to \mathbf{Z}$ naturally. Denote by $\mathcal{T}_m^G$ the set of all these G-SK invariants, which is a $\mathbf{Z}$-module under the natural addition.

From now on, T is simply called an invariant. We sometimes write $T(M)$ instead of $T(x)$ for $x = [M]$ if no confusion can arise.

DEFINITION 2.5. Let M be a G-manifold and $\sigma \in \mathrm{St}[H]$ with $H \subseteq G$. Then define $M_\sigma = \{x \in M_{(H)} \mid \sigma_x = \sigma\}$, where $M_{(H)} = M$ with the induced action of H and σ_x is the slice type of x in $M_{(H)}$.

REMARK 2.6. In other words, M_σ is the set consisting of those points $x \in M$ whose slice types σ_x satisfy that $(\sigma_x)_{(H)} = \sigma$. By the slice theorem, M_σ is a G-invariant submanifold of M with $\dim(M_\sigma) = \dim(M) - |\sigma|$ and $\partial(M_\sigma) = (\partial M)_\sigma$ (cf. [7 ; Theorem 4.14]). In case $\sigma = \sigma_0$, we have $M_{\sigma_0} = M$. As an example, let $M = G \times_K D(\tau)$ for $\tau \in St[K]$. Then $M_\sigma = |G/K| D^{|\tau| - |\tau_{(H)}|}$ in $M_{(H)} = G \times_K D(\mathbf{R}^{|\tau| - |\tau_{(H)}|} \times V)$ if $H \subseteq$K and $\sigma = \tau_{(H)} = [H; V]$, while $M_\sigma = \emptyset$ otherwise (cf. Definition 1.1 and Remark 1.2 (ii)). In general, a submanifold $M^H = (M_{(H)})^H$ decomposes as $M^H = \sum_{\sigma \in St[H]} M_\sigma$. For $\sigma \in St[H]$, we have $(M \times N)_\sigma = \sum_{(\sigma', \sigma'')} (M_{\sigma'} \times N_{\sigma''})$ summing over all pairs $(\sigma', \sigma'') \in St[H] \times St[H]$ such that $\sigma' \times \sigma'' = \sigma$. Hence the following product formula holds:

$$\chi((M \times N)_\sigma) = \sum_{(\sigma', \sigma'')} \chi(M_{\sigma'}) \cdot \chi(N_{\sigma''}) . \tag{2.6.1}$$

EXAMPLE 2.7. A map χ_σ defined by $\chi_\sigma(M) = \chi(M_\sigma)$ is an invariant because $M_1 \sim M_2$ implies $(M_1)_\sigma \sim (M_2)_\sigma$ naturally. Note that $\chi_{\sigma_0} = \chi$. Let $M = G \times_K D(\tau)$ for $\tau \in St[K]$. Then $\chi_\sigma(M) = |G/K|$ if $H \subseteq K$ and $\sigma = \tau_{(H)}$, while $\chi_\sigma(M) = 0$ otherwise. Furthermore, for a subgroup H of G, the map χ^H defined by $\chi^H(M) = \chi(M^H)$ is also an invariant and the equality $\chi^H = \sum_{\sigma \in St[H]} \chi_\sigma$ holds in $\mathcal{T}_m^G$ (cf. Remark 2.6).

Let H be a subgroup of G. Then, by using the total ordering on the set of all subgroups of G in Section 1, define inductively integers $n_H(K)$ for subgroups K with $H \subseteq K \subseteq G$ as follows:

$$n_H(H) = 1 , \quad n_H(K) = |K/H| - \sum_{H \subseteq L \subset K} n_H(L) .$$

Here $L \subset K$ means that L is a proper subgroup of K. If $H = \{1\}$, then the integers $n_{\{1\}}(K)$ coincide with those n_i in [7; Definition 5.3]. For $\sigma \in St[H]$ and a subgroup K with $H \subset K \subseteq G$, let $\mathcal{S}_K(\sigma)$ be the set consisting of those slice types $\tau \in St[K]$ such that $\tau_{(H)} = \sigma$.

PROPOSITION 2.8 (cf. [2; Theorem 2.6 and Remark 2.8]). *For $\sigma \in St[H]$, define an invariant θ_σ by*

$$\theta_\sigma = |G/H|^{-1}\left\{\chi_\sigma + \sum_{H\subset K\subseteq G} n_H(K)\left(\sum_{\tau\in\mathcal{S}_K(\sigma)} \chi_\tau\right)\right\}.$$

Then we have the following.

(i) *Let $\tau \in St[U]$ for a subgroup U of G. Then $\theta_\sigma(G \times_U D(\tau)) = 1$ if $U \supseteq H$ and $\sigma = \tau_{(H)}$, while $\theta_\sigma(G \times_U D(\tau)) = 0$ otherwise.*

(ii) *The set $\{\theta_\sigma \mid \sigma \in St(G), |\sigma| \leq m\}$ provides a basis for $\mathcal{T}_m^G$ as a free $\mathbf{Z}$-module.*

DEFINITION 2.9. Assume that an invariant T is defined for all G-manifolds. Then it is said to be multiplicative if $T(M \times N) = T(M)\cdot T(N)$ for any G-manifolds M and N. The map T induces a ring homomorphism $T : SK_*^G(\partial) \to \mathbf{Z}$.

Let pt be the one-point set. We see that $T(pt) = 0$ or 1 because $T(pt)^2 = T(pt)$. If $T(pt) = 0$, then T is trivial, that is $T \equiv 0$. From now on, we treat a non-trivial invariant T, which therefore takes the value $T(pt) = 1$.

The remainder of this paper are devoted to studying of a form of (non-trivial) multiplicative invariants.

In case of the trivial group $G = \{1\}$, we have the following.

PROPOSITION 2.10 (cf. [1; Proposition 3.4]). *A multiplicative invariant $T_0{:}SK_*(\partial) \to \mathbf{Z}$ is uniquely determined by the value $a = T_0(D^1)$ and has a form $T_0(M) = a^{\dim(M)}\chi(M)$. Here, if $a = 0$, then a^0 is regarded as* 1.

We next consider the case where $G \neq \{1\}$. Given a multiplicative invariant T, let $\mathcal{C}_T$ be the set consisting of all subgroups K of G such that $T(G/K) \neq 0$. Note that $\mathcal{C}_T \neq \emptyset$ because $T(pt) = T(G/G) = 1$ and $G \in \mathcal{C}_T$. It is seen that $\mathcal{C}_T$ has the minimum element $H = \cap_{K\in\mathcal{C}_T} K$ (with respect to the inclusion $\subseteq$ of subgroups). Indeed there is a non-zero integer k such that $k\cdot G/H = \prod_{K\in\mathcal{C}_T} G/K$ by Lemma 2.3 (i). This implies that $k\cdot T(G/H) = \prod_{K\in\mathcal{C}_T} T(G/K)$ and hence $T(G/H) \neq 0$. In other words, such an H is the subgroup which satisfies the condition that $T(G/H) \neq 0$ and $T(G/U) = 0$ for any proper subgroups U of H.

DEFINITION 2.11. The above T is said to be of type $\langle G/H\rangle$.

Note that $G/H = G \times_H D(\sigma^H(\mathbf{0}))$, where $\sigma^H(\mathbf{0}) = \sigma^H(0, \cdots, 0; 0, \cdots, 0)$.

PROPOSITION 2.12. *If T is of type $\langle G\rangle$, then $T(M) = T_0(M_0)$ for any G-manifold M, where T_0 is the invariant in Proposition* 2.10 *given by $a = T(D^1)$ and $M_0 = M$ with ignoring group action.*

PROOF. Since $M \times G \cong M_0 \times G$ as G-manifolds, we have $T(M) = T(M_0) = T_0(M_0)$ because $T(G) \neq 0$. q.e.d.

We next study an invariant of type $\langle G/H\rangle$ with $H \neq \{1\}$.

LEMMA 2.13. *If T is of type $\langle G/H\rangle$, then*

(i) $T(G\times_U D(\sigma))=0$ *for any slice type* $\sigma\in St[U]$ *with* $U\not\supseteq H$.

(ii) $T(G/K)=|G/K|$ *for any* $K\supseteq H$. *In particular,* $\beta=T(G/H)=|G/H|$.

PROOF. For subgroups P, Q of G and $\sigma\in St[P]$, we have

$$T(G\times_P D(\sigma))\cdot T(G/Q)=a(P,Q)\cdot T(D^b)\cdot T(G\times_{P\cap Q} D(\sigma_{(P\cap Q)})) \tag{2.13.1}$$

by Lemma 2.3 (i), where $b=|\sigma|-|\sigma_{(P\cap Q)}|$. First, set $P=Q=U$ for a proper subgroup U of H, then

$$T(G\times_U D(\sigma))\cdot T(G/U)=|G/U|\cdot T(D^0)\cdot T(G\times_U D(\sigma))\,.$$

Since $T(G/U)=0$ by definition and $T(D^0)=1$, we have $T(G\times_U D(\sigma))=0$. Next, set $P=U$ for $U\not\supseteq H$ and $Q=H$ in (2.13.1). The right-hand side vanishes because $U\cap H$ is a proper subgroup of H, while $T(G/H)=\beta\neq 0$ in the left-hand side. Thus $T(G\times_U D(\sigma))=0$ and (i) is obtained. To show (ii), set $P=K$, $\sigma=\sigma^K(\mathbf{0})$ for $K\supseteq H$ and $Q=H$ in (2.13.1), then $T(G/K)\cdot\beta=a(K,H)\cdot\beta$. Hence $T(G/K)=a(K,H)=|G/K|$. q.e.d.

LEMMA 2.14. *For subgroups H and K with $H\subset K\subseteq G$, let $J(K)=\bigcup_i J(i)$ and $J(i)=J(H,K;i)$ be the partition in Remark* 1.2 (iii). *Let $\sigma_j=\sigma^K(\mathbf{e}_j)$, where $\mathbf{e}_j$ has components zero except for its j-th component, which is equal to* 1 $(1\le j\le t_K)$. *Then we have*

(i)

$$(\sigma_j)_{(H)}=\begin{cases}\sigma^H(\mathbf{0}) & \text{if } j\in J(0)\,,\\ \sigma^H(\mathbf{e}_i) & \text{if } j\in J(i)\ (1\le i\le p_H, 1\le j\le p_K \text{ or } i>p_H, j>p_K)\,,\\ \sigma^H(2\mathbf{e}_i) & \text{if } j\in J(i)\ (1\le i\le p_H, j>p_K)\,.\end{cases}$$

Further, if T is an invariant of type $\langle G/H\rangle$, then

(ii)

$$T(G\times_K D(\sigma_j))=\begin{cases}|G/K|\cdot a & \text{if } j\in J(0)\ (1\le j\le p_K)\,,\\ |G/K|\cdot a^2 & \text{if } j\in J(0)\ (j>p_K)\,,\\ |G/K|\cdot\gamma_i & \text{if } j\in J(i)\ (1\le i\le p_H, 1\le j\le p_K \text{ or } i>p_H, j>p_K)\,,\\ |G/K|\cdot\gamma_i^2 & \text{if } j\in J(i)\ (1\le i\le p_H, j>p_K)\,,\end{cases}$$

where $a=T(D^1)$ and γ_i is the integer given by $\gamma_i=|G/H|^{-1}\cdot T(G\times_H D(\sigma^H(\mathbf{e}_i)))$ $(1\le i\le t_H)$.

Note that $G\times_K D(\sigma_j)=G\times_K D(W_j)$, where $\{W_j\}$ is the set of non-trivial irreducible K-modules.

PROOF OF THE LEMMA. Write $(\sigma_j)_{(H)}$ as $(\sigma_j)_{(H)}=\sigma^H(a(1),\cdots,a(p_H);a(p_H+1),\cdots,a(t_H))$. Suppose that $j\in J(i)=J'(i)\cup J''(i)$ for some i $(1\le i\le p_H)$. Then

$a(i) = 1$ (resp. $a(i) = 2$) if $j \in J'(i)$ (resp. $j \in J''(i)$) and $a(k) = 0$ if $k \neq i$ by the first equality in (1.2.1). Hence $(\sigma_j)_{(H)} = \sigma^H(\mathbf{e}_i)$ if $1 \leq j \leq p_K$ or $\sigma^H(2\mathbf{e}_i)$ if $j > p_K$. Similarly $(\sigma_j)_{(H)} = \sigma^H(\mathbf{e}_i)$ if $i > p_H$, $j > p_K$. Finally $(\sigma_j)_{(H)} = \sigma^H(\mathbf{0})$ if and only if $j \in J(0)$. Thus (i) follows. Next we prove (ii). Put $\lambda_j = T(G \times_K D(\sigma_j))$ and $\xi_i = T(G \times_H D(\sigma^H(\mathbf{e}_i)))$ for convenience. We have $\lambda_j \cdot T(G/H) = |G/K|a^b \cdot T(G \times_H D((\sigma_j)_{(H)}))$ by (2.13.1), where $b = |\sigma_j| - |(\sigma_j)_{(H)}|$. This implies that

$$\lambda_j = |G/K||G/H|^{-1}a^b \cdot T(G \times_H D((\sigma_j)_{(H)})) \tag{2.14.1}$$

because $T(G/H) = |G/H|$ by Lemma 2.13 (ii). Suppose first that $j \in J(i)$ with $i \geq 1$. In this case, we see that $|\sigma_j| = |(\sigma_j)_{(H)}|$ and $b = 0$ by (i). Since T is non-trivial, we note that $a^0 = T(D^0) = 1$ even if $a = 0$. Hence

$$\lambda_j = \begin{cases} |G/K||G/H|^{-1} \cdot \xi_i \text{ if } j \in J(i)\ (1 \leq i \leq p_H, 1 \leq j \leq p_K \text{ or } i > p_H, j > p_K)\,, \\ |G/K||G/H|^{-1} \cdot T(G \times_H D(\sigma^H(2\mathbf{e}_i))) \text{ if } j \in J(i)\ (1 \leq i \leq p_H, j > p_K)\,. \end{cases} \tag{2.14.2}$$

In the second case, since $\xi_i^2 = |G/H| \cdot T(G \times_H D(\sigma^H(2\mathbf{e}_i)))$ by Lemma 2.3 (i), we have $\lambda_j = |G/K||G/H|^{-2} \cdot \xi_i^2$. Now consider the case where $K = G$ for the first equality in (2.14.2) and denote by γ_i the integer $T(D(\sigma^G(\mathbf{e}_j)))$ if $1 \leq i \leq p_H$ (and some $j \in J(H, G; i)$ with $1 \leq j \leq p_G$) or $i > p_H$ (and some $j \in J(H, G; i)$ with $j > p_G$). Then $\gamma_i = |G/H|^{-1} \cdot \xi_i$ $(1 \leq i \leq t_H)$. Taking this in (2.14.2), we have $\lambda_j = |G/K| \cdot \gamma_i$ in the first case or $|G/K| \cdot \gamma_i^2$ in the second one. Next, in case $j \in J(0)$, it is seen that $\lambda_j = |G/K|a^b$ by (2.14.1) because $(\sigma_j)_{(H)} = \sigma^H(\mathbf{0})$ and $T(G \times_H D(\sigma^H(\mathbf{0}))) = |G/H|$ by Lemma 2.13 (ii). Since $b = |\sigma_j| = 1$ if $1 \leq j \leq p_K$ or 2 if $j > p_K$, the result follows. q.e.d.

DEFINITION 2.15. The above T is said to take a class of integers $\mathcal{V} = \{a\} \cup \{\gamma_i \mid 1 \leq i \leq t_H\}$. The integer a or γ_i is given by $a = T(D^1)$ or $\gamma_i = |G/H|^{-1} \cdot T(G \times_H D(V_i))$ respectively, where $\{V_i \mid 1 \leq i \leq t_H\}$ is the set of non-trivial irreducible H-modules.

LEMMA 2.16. *Let T be the invariant in the above lemma. Then we have*

$$T(G \times_K D(\sigma)) = |G/K| \cdot a^{|\sigma| - |\sigma_{(H)}|}\gamma_{\sigma_{(H)}} \tag{2.16.1}$$

for any slice type $\sigma \in St[K]$ with $K \supseteq H$, where $\gamma_{\sigma_{(H)}} = \prod_i \gamma_i^{a(i)}$ if $\sigma_{(H)} = \sigma^H(a(1), \cdots, a(p_H); a(p_H+1), \cdots, a(t_H))$. We regard a^0 (or γ_i^0) as 1 if $a = 0$ (or $\gamma_i = 0$) respectively.

PROOF. Write σ as $\sigma = \sigma^K(b(1), \cdots, b(p_K); b(p_K+1), \cdots, b(t_K))$ and set $\sigma(j) = \sigma^K(0, \cdots, b(j), 0, \cdots, 0)$ $(1 \leq j \leq t_K)$. Since $\sigma = \prod_j \sigma(j)$ and $\sigma(j) = \sigma^K(\mathbf{e}_j)^{b(j)}$, we have $|G/K|^{t_K-1}[G \times_K D(\sigma)] = \prod_j [G \times_K D(\sigma(j))]$ and $|G/K|^{b(j)-1}[G \times_K D(\sigma(j))] = [G \times_K D(\sigma^K(\mathbf{e}_j))]^{b(j)}$ by using Lemma 2.3 (i) inductively. Then it follows that

$$T(G \times_K D(\sigma(j))) = |G/K|^{1-b(j)} \cdot \lambda_j^{b(j)} \quad (1 \leq j \leq t_K)\,,$$

where $\lambda_j = T(G \times_K D(\sigma^K(\mathbf{e}_j)))$. Further, we have

$$\begin{aligned} T(G \times_K D(\sigma)) &= |G/K|^{1-t_K} \prod_j T(G \times_K D(\sigma(j))) \\ &= |G/K|^{1-t_K} |G/K|^{t_K - l(\sigma)} \prod_j \lambda_j^{b(j)} \\ &= |G/K|^{1-l(\sigma)} \prod_i L_i \,, \end{aligned} \tag{2.16.2}$$

where $l(\sigma) = \sum_j b(j)$ and $L_i = \prod_{j \in J(i)} \lambda_j^{b(j)}$, $J(i) = J(H, K; i)$ $(0 \le i \le t_H)$ as in Lemma 2.14. It follows from Remark 1.2 (iii) and Lemma 2.14 (ii) that $L_i = |G/K|^{l_0} a^{s_0}$ if $i = 0$ or $|G/K|^{l_i} \gamma^{s_i}$ if $1 \le i \le t_H$, where $l_i = \sum_{j \in J(i)} b(j)$ and

$$s_i = \begin{cases} \displaystyle\sum_{j \in J'(0)} b(j) + 2 \sum_{j \in J''(0)} b(j) = |\sigma| - |\sigma_{(H)}| \text{ if } i = 0\,, \\ \displaystyle\sum_{j \in J'(i)} b(j) + 2 \sum_{j \in J''(i)} b(j) = a(i) \text{ if } 1 \le i \le p_H\,, \\ \displaystyle\sum_{j \in J(i)} b(j) = a(i) \text{ if } p_H < i \le t_H\,. \end{cases}$$

Note that $\sum_{0 \le i \le t_H} l_i = l(\sigma)$. Hence we obtain the desired formula by substituting L_i in (2.16.2). Let $\sigma = \sigma^H(\mathbf{0})$ in (2.16.1). Then $T(G/H) = |G/H| \cdot a^0 \gamma_{\sigma^H(\mathbf{0})}$ which is equal to $|G/H|$ by Lemma 2.13 (ii). Hence $a^0 \gamma_1^0 \cdots \gamma_{t_H}^0 = 1$ and this means that a^0 or each γ_i^0 must be regarded as 1 even if a or $\gamma_i = 0$ respectively. q.e.d.

THEOREM 2.17. *Let T be a (non-trivial) multiplicative invariant of type $\langle G/H \rangle$ with $H \ne \{1\}$. Then it is uniquely determined by a class of integers $\mathcal{V} = \{a\} \cup \{\gamma_i \mid 1 \le i \le t_H\}$ in Definition 2.15 and has a form*

$$T(M) = \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)} \gamma_\sigma \cdot \chi(M_\sigma) \tag{2.17.1}$$

for any G-manifold M. If a or $\gamma_i = 0$ for some i, then we regard a^0 or γ_i^0 as 1 respectively.

PROOF. Since $\gamma_{\sigma'} \gamma_{\sigma''} = \gamma_\sigma$ for any σ' and $\sigma'' \in \mathrm{St}[H]$ such that $\sigma' \times \sigma'' = \sigma$, the above T is multiplicative by making use of the product formula (2.6.1). We see that T takes integers $\mathcal{V} = \{a, \gamma_i\}$. In fact, let $\sigma \in \mathrm{St}[H]$, then $\chi_\sigma(D^1) = 1$ if $\sigma = \sigma^H(\mathbf{0})$ or zero otherwise. Hence $T(D^1) = a^1 \gamma_{\sigma^s(\mathbf{0})} \cdot 1 = a$. Further $\chi_\sigma(G \times_H D(\sigma^H(\mathbf{e}_i))) = |G/H|$ if $\sigma = \sigma^H(\mathbf{e}_i)$ or zero otherwise $(1 \le i \le t_H)$, which implies that $T(G \times_H D(\sigma^H(\mathbf{e}_i))) = a^0 \gamma_{\sigma^H(\mathbf{e}_i)} \cdot |G/H| = \gamma_i \cdot |G/H|$. Therefore, T takes integers in $\mathcal{V}$ and is determined by the class $\mathcal{V}$. On the other hand, $T(G/H) = a^0 \gamma_{\sigma^H(\mathbf{0})} \cdot |G/H| = |G/H|$ and $T(G/U) = 0$ for any proper subgroup U of H. This verifies that T is of type $\langle G/H \rangle$ (cf. Definition 2.11).

Let I be an invariant which is not necessarily multiplicative. To proceed with our proof, for an integer j (≥ 0), consider an invariant $I_{(j)}$ defined by $I_{(j)}(M) = I(M)$ if $j = \dim(M)$ or zero if $j \neq \dim(M)$. Now let T be any multiplicative invariant of type $\langle G/H \rangle$, which takes integers $\mathcal{V} = \{a, \gamma_i\}$. We show that T has a form in (2.17.1), or equivalently has a form

$$T = \sum_{k,\,\sigma} a^k \gamma_\sigma \chi_{\sigma,(k+|\sigma|)}, \tag{2.17.2}$$

where $\chi_{\sigma,(j)} = (\chi_\sigma)_{(j)}$ in the sense mentioned above and the sum is taken over all slice types $\sigma \in \mathrm{St}[H]$ and the integer k (≥ 0) (Remark. Note that $\chi_{\sigma,(j)}$ may be defined for $j \geq |\sigma|$ because $M_\sigma = \emptyset$ if $\dim(M) < |\sigma|$. Thus, j is written as $j = k + |\sigma|$ for some $k \geq 0$. If $\dim(M) = k + |\sigma|$, then $k = \dim(M) - |\sigma| = \dim(M_\sigma)$ as in (2.17.1) (cf. Remark 2.6).). Now let us consider an invariant $\theta_{\sigma,(j)} = (\theta_\sigma)_{(j)}$ for θ_σ in Proposition 2.8. Since $T \in \mathcal{T}_*^G = \sum_m \mathcal{T}_m^G$, we can write T as $T = \sum_{k,\sigma} a_{\sigma,(k+|\sigma|)} \theta_{\sigma,(k+|\sigma|)}$ summing over all $\sigma \in \mathrm{St}(G)$ and $k \geq 0$. To begin with, we show that $a_{\sigma,(k+|\sigma|)} = 0$ for each $\sigma \in St[U]$ with $U \not\supseteq H$ and $k \geq 0$. Recall the total ordering $<$ on $\mathrm{St}(G)$ and rename the slice types $\sigma \in \bigcup_{U \not\supseteq H} St[U]$ as $\rho_1 = \sigma_0 < \rho_2 < \rho_3 < \cdots$. First it follows that $T(D^k \times G) = a_{\sigma_0,(k)} \cdot \theta_{\sigma_0}(D^k \times G) = a_{\sigma_0,(k)} \cdot |G|^{-1} \chi(D^k \times G) = a_{\sigma_0,(k)}$ because $\theta_{\sigma_0} = |G|^{-1}(\chi + \eta)$ by definition, where η is a sum of χ_τ with $\tau \neq \sigma_0$. Since T is of type $\langle G/H \rangle$ with $H \neq \{1\}$, we have $T(G) = 0$ and $T(D^k \times G) = T(D^k) \cdot T(G) = 0$. Thus $a_{\sigma_0,(k)} = 0$ for each $k \geq 0$. Next suppose that $a_{\rho_j,(k+|\rho_j|)} = 0$ for any ρ_j with $1 \leq j < t$ and $k \geq 0$. Let take $M = D^k \times G \times_U D(\rho_t)$. Then $T(M) = T(D^k) \cdot T(G \times_U D(\rho_t)) = 0$ by Lemma 2.13 (i), while

$$T(M) = \sum_{L \subseteq U} a_{(\rho_t)_{(L)},(k+|\rho_t|)} \theta_{(\rho_t)_{(L)},(k+|\rho_t|)}(M) = \sum_{L \subseteq U} a_{(\rho_t)_{(L)},(k+|\rho_t|)}$$

because $\theta_{(\rho_t)_{(L)},(k+|\rho_t|)}(M) = \theta_{(\rho_t)_{(L)}}(G \times_U D(\rho_t)) = 1$ by Proposition 2.8 (i). If L is a proper subgroup of U, then $(\rho_t)_{(L)} = \rho_{j_L}$ for some $j_L < t$. Hence $a_{(\rho_t)_{(L)},(k+|\rho_t|)}$ vanishes by the inductive assumption and so does $a_{\rho_t,(k+|\rho_t|)}$. Therefore T is written as

$$T = \sum_{\sigma \in St[H],\ k \geq 0} P_{\sigma,(k+|\sigma|)} \chi_{\sigma,(k+|\sigma|)} + \sum_{H \subset K \subseteq G} \ \sum_{\tau \in St[K],\ k \geq 0} Q_{\tau,(k+|\tau|)} \chi_{\tau,(k+|\tau|)} \tag{2.17.3}$$

for some rational numbers $P_{\sigma,(l)}$ and $Q_{\tau,(l)}$ (See the expression of θ_σ in Proposition 2.8.). Next we prove that

$$P_{\sigma,(k+|\sigma|)} = a^k \gamma_\sigma \tag{2.17.4}$$

for any $\sigma \in St[H]$ and k (≥ 0). To show this, consider the value $T(M)$ for $M = D^k \times G \times_H D(\sigma)$. Then we have that $T(M) = P_{\sigma,(k+|\sigma|)} \cdot \chi_\sigma(M) = P_{\sigma,(k+|\sigma|)} \cdot |G/H|$ by (2.17.3), while $T(M) = T(D^k) \cdot T(G \times_H D(\sigma)) = a^k \cdot |G/H| a^0 \gamma_\sigma$ by (2.16.1). Therefore we obtain (2.17.4). We recall that a^0 (or γ_i^0) is regarded as 1 even if $a = 0$ (or $\gamma_i = 0$) respectively as remarked in the proof of Lemma 2.16. To complete the proof, we must show that $Q_{\tau,(k+|\tau|)} = 0$. Let us fix an integer k (≥ 0). Then, for $M = D^k \times G \times_K D(\tau)$ ($\tau \in St[K], H \subset K \subseteq G$), we have that $T(M) = x + y$ by (2.17.3) and Remark 2.6, where

$$\begin{cases} x = P_{\tau(H),(k+|\tau|)} \cdot \chi_{\tau(H),(k+|\tau|)}(M) \\ y = \sum_{H \subset U \subseteq K} Q_{\tau(U),(k+|\tau|)} \cdot \chi_{\tau(U),(k+|\tau|)}(M) . \end{cases} \tag{2.17.5}$$

It follows that

$$x = a^{k+|\tau|-|\tau(H)|}\gamma_{\tau(H)} \cdot |G/K| = a^k \cdot |G/K| a^{|\tau|-|\tau(H)|}\gamma_{\tau(H)}$$

by Example 2.7 and (2.17.4), which means that $x = T(D^k) \cdot T(G \times_K D(\tau)) = T(M)$ by (2.16.1). Hence $y = 0$ and

$$\sum_{H \subset U \subseteq K} Q_{\tau(U),(k+|\tau|)} = 0$$

because $\chi_{\tau(U)}(M) = |G/K|$ ($H \subset U \subseteq K$). We can order these $U : H < U_1 < \cdots < U_p = K$ by using the total ordering in Section 1. Then the inductive argument gives that $Q_{\tau,(k+|\tau|)} = 0$. Hence T has the desired form (2.17.2). q.e.d.

REMARK 2.18. In case where $G = \{1\}$, we have that $\mathrm{St}[\{1\}] = \{\sigma_0\}$ and $\chi_{\sigma_0} = \chi$. Then the invariant in (2.17.1) has the form $T(M) = a^{\dim(M)}\chi(M)$ because $M_{\sigma_0} = M$, where γ_{σ_0} is regarded as 1 formally (cf. Remark 2.6 and Example 2.7). Such a T coincides with T_0 in Proposition 2.10.

We have shown that $SK_*(\partial) \cong \mathbf{Z}[[D^1]]$ as a polynomial ring over $\mathbf{Z}$. Further, an element $x \in SK_*(\partial)$ is determined by the value $\chi(x)$ and $[M] = \chi(M) \cdot [D^1]^{\dim(M)}$ for any manifold M (cf. [8; Theorem 1.2]).

COROLLARY 2.19. *Let $R : SK_*^G(\partial) \to SK_*(\partial)$ be a (non-trivial) ring homomorphism, then it has a form*

$$R([M]) = \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)}\gamma_\sigma \cdot [M_\sigma] = \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)}\gamma_\sigma \chi(M_\sigma) \cdot [D^1]^{\dim(M_\sigma)}$$

for an $H \subseteq G$ and a class of integers $\mathcal{V}_H = \{\gamma_i \mid 1 \leq i \leq t_H\}$, where $a = \chi(R(D^1))$ and $\gamma_i = |G/H|^{-1} \cdot \chi(R(G \times_H D(V_i)))$ as in Definition 2.15.

PROOF. Consider a map $T = \chi \circ R : SK_*^G(\partial) \to \mathbf{Z}$, then it is a multiplicative invariant and has a form $T = \chi \circ R_0$ as in (2.17.1), where $R_0 : SK_*^G(\partial) \to SK_*(\partial)$ is given by $R_0([M]) = \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)}\gamma_\sigma \cdot [M_\sigma]$. Since $\chi(R([M])) = \chi(R_0([M]))$, we have $R([M]) = R_0([M])$ for any $[M]$ as mentioned above. Thus, $R = R_0$. q.e.d.

REMARK 2.20. Let a multiplicative invariant T of type $\langle G/H \rangle$ be determined by integers $\mathcal{V} = \{a, \gamma_1, \cdots, \gamma_{t_H}\}$ and have the form as in (2.17.1). Then an invariant T' defined by $T'(M) = (-1)^{\dim(M)}T(M)$ is also multiplicative and is of type $\langle G/H \rangle$. In fact, T' coincides with the one T'' which takes integers $\mathcal{V}'' = \{-a, -\gamma_1, \cdots, -\gamma_{p_H}, \gamma_{p_H+1}, \cdots, \gamma_{t_H}\}$. To show this, write T'' as

$$T''(M) = \sum_{\sigma \in St[H]} (-a)^{\dim(M_\sigma)} (-\gamma_1)^{a(1)} \cdots (-\gamma_{p_H})^{a(p_H)} \gamma_{\sigma_1} \cdot \chi(M_\sigma)$$

for a G-manifold M, where $\sigma_1 = \sigma^H(0, \cdots, 0, a(p_H+1), \cdots, a(t_H))$ if $\sigma = \sigma^H(a(1), \cdots, a(t_H))$. Since $|\sigma| \equiv \sum_{1 \le i \le p_H} a(i)$ (mod 2) and $\dim(M_\sigma) = \dim(M) - |\sigma|$, we have

$$T''(M) = (-1)^{\dim(M)} \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)} \gamma_\sigma \cdot \chi(M_\sigma) = (-1)^{\dim(M)} T(M) = T'(M) .$$

EXAMPLE 2.21. Finally we consider an invariant T of type $\langle G/H \rangle$ which takes integers $\mathcal{V}_H = \{a\} \cup \{\gamma_i \mid 1 \le i \le t_H\}$ with a or each $\gamma_i \in \{-1, 0, 1\}$ and give some typical example by using the formula (2.17.1). Let $l(\sigma) = \sum_{1 \le i \le t_H} a(i)$ be the length of $\sigma = \sigma^H(a(1), \cdots, a(p_H); a(p_H + 1), \cdots, p(t_H)) \in St[H]$ and $l_2(\sigma) = \sum_{p_H < i \le t_H} a(i)$ the length of the two-dimensional irreducible H-modules in σ. Note that $l_2(\sigma) = |\sigma| - l(\sigma)$.

(i) Suppose that $\gamma_i = 1$ $(1 \le i \le t_H)$, then $\gamma_\sigma = 1$ for any $\sigma \in St[H]$. If $a = 1$, then

$$T(M) = \sum_{|\sigma| \le \dim(M)} \chi(M_\sigma) = \chi(M^H)$$

(cf. Remark 2.6). If $a = 0$, then

$$T(M) = \sum_{|\sigma| \le \dim(M)} 0^{\dim(M_\sigma)} \chi(M_\sigma) = \sum_{|\sigma| = \dim(M)} 0^0 \chi(M_\sigma) = \chi(M^{H,\,0}) ,$$

where $M^{H,\,0} = \sum_{|\sigma| = \dim(M)} M_\sigma$ is the isolated points of M^H (Note that $0^0 = 1$). If $a = -1$, then

$$T(M) = \sum_{|\sigma| \le \dim(M)} (-1)^{\dim(M_\sigma)} \chi(M_\sigma) = \chi(M^{H,\,\mathrm{ev}}) - \chi(M^{H,\,\mathrm{od}}) ,$$

where $M^{H,\,\mathrm{ev}}$ (or $M^{H,\,\mathrm{od}}$) is the even-dimensional (or odd-dimensional) components of M^H respectively.

(ii) Suppose that $\gamma_i = -1$ $(1 \le i \le t_H)$, then $\gamma_\sigma = (-1)^{l(\sigma)}$ for any $\sigma \in St[H]$. If $a = 1$, then

$$T(M) = \sum_{|\sigma| \le \dim(M)} (-1)^{l(\sigma)} \chi(M_\sigma) = \chi(M^H_+) - \chi(M^H_-) ,$$

where M^H_+ (or M^H_-) is the subset of M^H consisting of those points x having slice types σ_x with $l(\sigma_x)$ even (or odd) respectively. If $a = 0$, then

$$T(M) = \sum_{|\sigma| = \dim(M)} 0^0 (-1)^{l(\sigma)} \chi(M_\sigma) = \chi(M^{H,\,0}_+) - \chi(M^{H,\,0}_-) ,$$

where $M_{\varepsilon}^{H,\,0} = M_{\varepsilon}^{H} \cap M^{H,\,0}$ ($\varepsilon = +$ or $-$). If $a = -1$, then

$$T(M) = \sum_{|\sigma| \le \dim(M)} (-1)^{\dim(M_\sigma)+l(\sigma)} \chi(M_\sigma) = (-1)^{\dim(M)}\{\chi(M_{2,\,+}^{H}) - \chi(M_{2,\,-}^{H})\}$$

because $\dim(M_\sigma) + l(\sigma) = \dim(M) - l_2(\sigma)$, where $M_{2,\,+}^{H}$ (or $M_{2,\,-}^{H}$) is the subset of M^H consisting of those points x having slice types σ_x with $l_2(\sigma_x)$ even (or odd) respectively.

(iii) Finally, suppose that $\gamma_i = 0$ ($1 \le i \le t_H$), then $\gamma_\sigma = 0^0 \cdots 0^0 = 1$ if $\sigma = \sigma^H(\mathbf{0})$ or zero otherwise. If $a = 1$, then

$$T(M) = \gamma_{\sigma^H(\mathbf{0})} \chi(M_{\sigma^H(\mathbf{0})}) = \chi(M_{\sigma^H(\mathbf{0})}),$$

where $M_{\sigma^H(\mathbf{0})}$ is the components of M^H with $\dim(M_{\sigma^H(\mathbf{0})}) = \dim(M) - |\sigma^H(\mathbf{0})| = \dim(M)$. If $a = 0$, then

$$T(M) = 0^{\dim(M)} \gamma_{\sigma^H(\mathbf{0})} \chi(M_{\sigma^H(\mathbf{0})}) = \begin{cases} \chi(M^H) & \text{if } \dim(M) = 0 \\ 0 & \text{if } \dim(M) > 0\,. \end{cases}$$

If $a = -1$, then

$$T(M) = (-1)^{\dim(M_{\sigma^H(\mathbf{0})})} \gamma_{\sigma^H(\mathbf{0})} \chi(M_{\sigma^H(\mathbf{0})}) = (-1)^{\dim(M)} \chi(M_{\sigma^H(\mathbf{0})})\,.$$

In a similar way, we have another examples.

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