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Conformally Flat Semi-Riemannian Manifolds with Commuting Curvature and Ricci Operators

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Abstract. We classify the conformally flat, semi-Riemannian manifolds satisfying $R(X, Y) \cdot Q = 0$, where R and Q are the curvature tensor and the Ricci operator, respectively. As the cases which do not occur in the Riemannian manifolds, the Ricci operator Q has pure imaginary eigenvalues or it satisfies $Q^2 = 0$.

1. Introduction

Let (M, g) be the conformally flat Riemannian manifold satisfying the condition $R(X, Y) \cdot Q = 0$ where R is the curvature tensor and Q is the Ricci operator of M. Such manifolds were studied and classified by Sekigawa and Takagi [12] under the assumption of completeness and Bishop and Goldberg [1] without such assumption. If (M, g) is the semi-Riemannian manifold, the Ricci operator Q_p of M is a symmetric linear endomorphism of an indefinite scalar product space (T_pM, g_p). According to Petrov [11], Q_p is not always diagonalizable in this case. Let (M, g) be the conformally flat Lorentzian manifold satisfying the condition $R(X, Y) \cdot Q = 0$. The case when the Ricci operator Q is diagonalizable was classified by Erdogan and Ikawa [4]. In this paper, we study and classify the conformally flat semi-Riemannian manifold satisfying the condition $R(X, Y) \cdot Q = 0$. The main result is the following.

MAIN THEOREM. Let M^n_q be an n-dimensional ($n \geq 4$), simply connected, complete, conformally flat semi-Riemannian manifold of index q satisfying $R(X, Y) \cdot Q = 0$. Then M is one of the following:

(1) M is a semi-Riemannian manifold of constant curvature.

(2) M is the product manifold of a k-dimensional semi-Riemannian manifold of constant curvature K ($\neq 0$) and an $n-k$-dimensional semi-Riemannnian manifold of constant curvature $-K$; that is, $M^k_{q_1}(K) \times M^{n-k}_{q-q_1}(-K)$, where $1 < k < n-1$.

(3) M is the product manifold of an $n-1$-dimensional semi-Riemannian manifold of index $q-1$ of constant curvature $K(\neq 0)$ and a 1-dimensional Lorentzian manifold, or the

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product manifold of an $n-1$-dimensional semi-Riemannnian manifold of index q of constant curvature $K(\neq 0)$ and a 1-dimensional Riemannian manifold; that is, $M^{n-1}_{q-1}(K) \times M^1_1$ or $M^{n-1}_q(K) \times M^1$.

(4) M is an m-dimensional complex sphere in C^{m+1} defined by

$$z_1^2 + \cdots + z_{m+1}^2 = \sqrt{-1}b \quad (b \neq 0, b \in R),$$

where $2m = n$.

(5) The Ricci operator satisfies $Q^2 = 0$ everywhere. Moreover on an open set where the Ricci operator has maximal rank, the kernel of Q is an integrable distribution and gives a totally geodesic foliation whose leaves are flat and complete with respect to the induced connection.

REMARK. The detailed definition of a complex sphere will be given in section 4.

The cases of (4) and (5) in the theorem above never occur if M is a Riamnnian manifold. The Ricci operator of the semi-Riemannian manifold in (4) has two pure imaginary eigenvalues which are mutually conjugate.

After preliminaries in section 2, in section 3 the possible Ricci operator Q under the assumption of Main Theorem are classified algebraically (Theorem 3.1). Moreover we consider the case when Q is diagonalizable and obtain the similar result to Sekigawa and Takagi (Proposition 3.3). In section 4, we study the case when Q has two pure imaginary eigenvalues which are mutually conjugate and show the classification result (Theorem 4.1). We study the case when the Ricci operator is nilpotent in section 5 and show such examples in section 6.

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2. Preliminaries

Let (M^n_q, g) be an n-dimensional semi-Riemannian manifold of index q, i.e., the signature of $g = (\overbrace{-, \cdots, -}^{q}, +, \cdots, +)$. If $q = 0$, M is a Riemannian manifold, and if $q = 1$, M is a Lorentzian manifold. We denote by ∇ the Levi-Civita connection of M^n_q and by R the curvature tensor of M. The Ricci operator Q is a field of symmetric endomorphism which corresponds to the Ricci tensor ric, that is, $ric(X, Y) = g(QX, Y)$. r denotes the scalar curvature defined by $r = tr\,Q$. The *Weyl conformal curvature tensor field C* on M^n_q is a tensor field of type (1,3) defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z + \frac{r}{(n-1)(n-2)}(X \wedge Y)Z \tag{2.1}$$

where $X \wedge Y$ denotes the endomorphism defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$.

It is known that M is conformally flat if and only if C vanishes for $n > 3$. The Weyl conformal curvature tensor field C vanishes identically for $n = 3$. We put the tensor field c of type (1,2) as follows:

$$c(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(n-1)}\{(\nabla_X r)Y - (\nabla_Y r)X\}. \tag{2.2}$$

It is well known that $C = 0$ implies $c = 0$ for $n > 3$. So if M_q^n is conformally flat with $n > 3$, from (2.1) and (2.2), we have the following equations:

$$R(X, Y)Z = \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z - \frac{r}{(n-1)(n-2)}(X \wedge Y)Z, \tag{2.3}$$

$$(\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(n-1)}\{(\nabla_X r)Y - (\nabla_Y r)X\} = 0. \tag{2.4}$$

In this paper, we consider conformally flat semi-Riemannian manifolds whose Ricci operator Q satisfies the following condition:

$$R(X, Y) \cdot Q = 0 \tag{2.5}$$

The condition (2.5) is equivalent to

$$R(QX, X) = 0. \tag{2.6}$$

From (2.3) and (2.6), we have the following lemma

LEMMA 2.1. *Let M_q^n be an n $(n > 3)$-dimensional conformally flat semi-Riemannian manifold satisfying* (2.5). *Then*

$$Q^2 - \frac{r}{n-1}Q = \rho I, \tag{2.7}$$

where ρ is a smooth function on M_q^n and I is the identity field.

We recall the form of a symmetric linear operator in an indefinite scalar product due to Petrov [11].

PROPOSITION 2.2. *A linear operator Q in an indefinite scalar product space is symmetric if and only if Q can be put into the following form:*

$$Q = \begin{pmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_k & & & \\ & & & C_1 & & \\ & & & & \ddots & \\ & & & & & C_m \end{pmatrix}$$

where B_i is $s_i \times s_i$ matrix

$$B_i = \begin{pmatrix} \lambda_i & & & & \\ 1 & \lambda_i & & & \\ & & \ddots & & \\ & & & \lambda_i & \\ & & & 1 & \lambda_i \end{pmatrix}$$

relative to a basis $v_1, \cdots, v_{s_i} (s_i \geq 1)$ with all scalar products zero except $\langle v_k, v_l \rangle = \varepsilon = \pm 1$ if $k + l = s_i + 1$, and C_j is $2t_j \times 2t_j$ matrix

$$C_j = \begin{pmatrix} a_j & b_j & & & & & & & & \\ -b_j & a_j & & & & & & & & \\ 1 & 0 & a_j & b_j & & & & & & \\ 0 & 1 & -b_j & a_j & & & & & & \\ & & 1 & 0 & a_j & b_j & & & & \\ & & 0 & 1 & -b_j & a_j & & & & \\ & & & & & & \ddots & & & \\ & & & & & & 1 & 0 & a_j & b_j \\ & & & & & & 0 & 1 & -b_j & a_j \end{pmatrix} \quad (b_j \neq 0)$$

relative to a basis $u_1, v_1, \cdots, u_{t_j}, v_{t_j}$ with all scalar products zero except $\langle u_k, u_l \rangle = -\langle v_k, v_l \rangle = 1$ if $k + l = t_j + 1$.

3. Ricci operator

At first, we classify possible Ricci operators algebraically. From Proposition 2.2 and (2.7), we obtain the following theorem:

THEOREM 3.1. *Let M_q^n be an $n(n \geq 4)$-dimensional conformally flat semi-Riemannian manifold satysfying* (2.5). *Then the Ricci operator Q_x at each point $x \in M$ is either diagonalizable relative to an orthonormal basis or has one of the following two forms*:

$$
Q_x = \begin{pmatrix} \overbrace{\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \quad \ddots \quad \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}}^{2r} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad (r \le q) \tag{3.1}
$$

relative to a basis $v_1, \cdots, v_n$ *of* $T_x M$ *with all scalar products zero except*

$$
g(v_{2i-1}, v_{2i}) = \varepsilon \quad \varepsilon = \pm 1 \quad (i = 1, \cdots, r), \quad g(v_i, v_i) = \varepsilon_i ,
$$

$$
\varepsilon_i = \begin{cases} -1 \ (i = 2r+1, \cdots, q+r) \\ \ \ 1 \ (i = q+r+1, \cdots, n) \end{cases} ,
$$

or

$$
Q_x = \begin{pmatrix} 0 & b & & & \\ -b & 0 & & & \\ & & \ddots & & \\ & & & 0 & b \\ & & & -b & 0 \end{pmatrix} \tag{3.2}
$$

relative to a basis $u_1, v_1, \cdots, u_m, v_m (n = 2m)$ *with all scalar products zero except*

$$
g(u_i, u_i) = 1 = -g(v_i, v_i) .
$$

In the last case n *is even, its index* q *is* $n/2$ *and* Q_x *has the pure imaginary eigenvalues* $\pm\sqrt{-1}b$.

PROOF. The Ricci operator Q_x has the form in Proposition 2.2. One computes that

$$
B_i^2 = \begin{pmatrix} \lambda_i^2 & & & & & \\ 2\lambda_i & \lambda_i^2 & & & & \\ 1 & 2\lambda_i & \lambda_i^2 & & & \\ & 1 & 2\lambda_i & & & \\ & & & \ddots & & \\ & & & & 2\lambda_i & \lambda_i^2 \end{pmatrix} ,
$$

$$C_j^2 = \begin{pmatrix} a_j^2 - b_j^2 & 2a_jb_j & & & & \\ -2a_jb_j & a_j^2 - b_j^2 & & & & \\ 2a_j & 2b_j & a_j^2 - b_j^2 & 2a_jb_j & & \\ -2b_j & 2a_j & -2a_jb_j & a_j^2 - b_j^2 & & \\ 1 & 0 & 2a_j & 2b_j & & \\ 0 & 1 & -2b_j & 2a_j & & \\ & & & & \ddots & \\ & & & & & a_j^2 - b_j^2 \end{pmatrix}.$$

Q_x satisfies the equation (2.7). Therefore it is clear from the form of B_i^2 and C_j^2 that $s_i \leq 2$ and $t_j \leq 1$. So Q_x has blocks of the form

$$(\mu_i) \quad \text{or} \quad \begin{pmatrix} \lambda_j & 0 \\ 1 & \lambda_j \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix}$$

with squares

$$(\mu_i^2) \quad \text{or} \quad \begin{pmatrix} \lambda_j^2 & 0 \\ 2\lambda_j & \lambda_j^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_k^2 - b_k^2 & 2a_kb_k \\ -2a_kb_k & a_k^2 - b_k^2 \end{pmatrix}.$$

The equation (2.7) yields

$$\mu_i^2 - \frac{r}{n-1}\mu_i = \rho\,, \quad \lambda_j^2 - \frac{r}{n-1}\lambda_j = \rho\,, \quad a_k^2 - b_k^2 - \frac{r}{n-1}a_k = \rho\,,$$
$$2\lambda_j - \frac{r}{n-1} = 0\,, \quad 2a_kb_k - \frac{r}{n-1}b_k = 0\,.$$

If Q_x is diagonalizable,

$$\mu_i = \frac{r \pm \sqrt{r^2 + 4(n-1)^2\rho}}{2(n-1)}\,.$$

Then Q_x has at most two real eigenvalues.

Next we consider the case Q_x is not diagonalizable. If there are any blocks with a's and b's, we have $\lambda_j = a_k = r/2(n-1)$ for each j and k since $b_k \neq 0$. Thus all λ_j's and a_k's are equal. It is clear that all b_k's are equal. The equations became

$$\mu_i^2 - \frac{r}{n-1}\mu_i = \rho\,, \quad \lambda^2 = -\rho\,, \quad a^2 + b^2 = -\rho\,, \quad \lambda = a = \frac{r}{2(n-1)}\,.$$

Since $\lambda = a$ and $b \neq 0$, there can be blocks with a's or blocks with λ's but not both. In either case we have

$$\mu_i = \frac{r}{2(n-1)} \pm \sqrt{\left(\frac{r}{2(n-1)}\right)^2 + \rho}\,.$$

If $\lambda^2 = -\rho$, then $\mu_i = \lambda$. If $a^2 + b^2 = -\rho$, then $(r/2(n-1))^2 + \rho < 0$ and there are no μ_i's. If there is a block with a λ, then $\lambda = \mu_i = r/2(n-1)$ for each i. If p is the number of

μ's which appear in Q_x and $2p'$ the number of λ's,

$$r = p\mu + 2p'\lambda = p\left(\frac{r}{2(n-1)}\right) + 2p'\left(\frac{r}{2(n-1)}\right).$$

Thus $r\left(1 - p/2(n-1) - p'/2(n-1)\right) = 0$. But $p + 2p' = n$ and $n \geq 4$, so $r = 0$. Then $\lambda = \mu = 0$ and Q_x is of the form

$$Q_x = \begin{pmatrix} 0 & 0 & & & & & & \\ 1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 0 & & & \\ & & & 1 & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}.$$

From Proposition 2.2, for a basis $v_1, \cdots, v_n$ of T_xM, we have

$$g(v_{2i-1}, v_{2i}) = \varepsilon, \quad \varepsilon = \pm 1, \quad g(v_{2i-1}, v_{2i-1}) = g(v_{2i}, v_{2i}) = 0 \quad (i = 1, \cdots, r).$$

Then the vectors v_{2i-1} and v_{2i} are lightlike, and a plane Π_i spanned by v_{2i-1} and v_{2i} is non-degenerate. The index of g on Π_i is 1. Then $r \leq q$ and

$$g(v_i, v_i) = -1 \quad (i = 2r+1, \cdots, q+r), \quad g(v_i, v_i) = 1 \quad (i = q+r+1, \cdots, n).$$

If there is a block with a b, there are no other types of blocks. Since $a = r/2(n-1) = na/2(n-1)$ and $n \geq 4$, we see that $a = 0$. Then Q_x is of the form

$$Q_x = \begin{pmatrix} 0 & b & & & \\ -b & 0 & & & \\ & & \ddots & & \\ & & & 0 & b \\ & & & -b & 0 \end{pmatrix}.$$

In this case, for a basis $u_1, v_1, \cdots, u_m, v_m$ $(n = 2m)$ of T_xM,

$$g(u_i, u_i) = 1, \quad g(v_i, v_i) = -1 \quad (1 \leq i \leq m).$$

Then n is even and the index $q = m = n/2$. Q_x has the pure imaginary eigenvalues $\pm\sqrt{-1}b$. □

Suppose that Q_x is diagonalizable relative to an orthonormal basis. Then Q_x has at most two eigenvalues. Suppose that Q_x has distinct eigenvalues λ and μ with multiplicities k and $n-k$, respectively. Then (2.7) implies

$$(\lambda - \mu)\{(n-k-1)\lambda + (k-1)\mu\} = 0.$$

Since $\lambda \neq \mu$, we have $(n-k-1)\lambda + (k-1)\mu = 0$. If $k = 1$, then $\lambda = 0$. If $k = n-1$, then $\mu = 0$. Otherwise we have

$$\lambda\mu < 0\,. \tag{3.3}$$

Now we define 6 types of subsets U_i $(i = 1, 2, \cdots, 6)$ associated with the types of the Ricci operator Q_x:

$U_1 = \{x \in M \mid Q_x$ has two non-zero real eigenvalues.$\}$

$U_2 = \{x \in M \mid Q_x$ has only one non-zero real eigenvalue.$\}$

$U_3 = \{x \in M \mid Q_x$ has two pure imaginary eigenvalues which are mutually conjugate.$\}$

$U_4 = \{x \in M \mid Q_x$ has two real eigenvalues one of which is zero.$\}$

$U_5 = \{x \in M \mid {Q_x}^2 = 0$ and $Q_x \neq 0.\}$

$U_6 = \{x \in M \mid Q_x = 0.\}$

Q_x is diagonalizable relative to an orthonormal basis at $x \in U_i$ $(i = 1, 2, 4, 6)$. On U_1, two eigenvalues have the opposite signs by (3.3) and their multiplicities are not less than 2. On U_4, the multiplicity of the eigenvalue 0 is equal to 1.

PROPOSITION 3.2. *M is the disjoint union of* U_i $(i = 1, \cdots, 6)$. *For each i* $(i = 1, \cdots, 6)$, *the rank of* Q_x *at* $x \in U_i$ *and the openness of* U_i *are the following*:

	The rank of Q_x	*Openness*
U_1	n	*open*
U_2	n	*open*
U_3	n	*open*
U_4	$n-1$	*open*
U_5	$1 \leq$ *The rank of* Q_x $\leq min\{q, n-q\}$	*?*
U_6	0	*?*

Here n and q denote the dimension of M and its index, respectively. The symbol ? in the table means that we can not determine whether it is open or not.

PROOF. The former part of this proposition and the rank of Q_x are easily seen by Theorem 3.1 and the argument before this proposition. We will prove the openness of U_i $(1 = 1, 2, 3, 4)$.

For each $x \in U_1$, by the continuity of eigenvalues of the Ricci operator Q, there exists a neighbourhood U of x on which Q has at least two eigenvalues and hence has exactly two eigenvalues. Moreover on a neighbourhood U' $(\subset U)$ of x, the real parts of two eigenvalues are not zeros. Any points of U' do not belong to U_i $(i = 2, 3, 4, 5, 6)$. Therefore $U' \subset U_1$. This implies that U_1 is open.

For each $x \in U_2$, by the continuity of eigenvalues of the Ricci operator Q, on some neighbourhood U of x, the real parts of eigenvalues of Q has the same sign (the plus sign or the minus sign). Therefore $U \subset U_2$ and hence U_2 is open.

For each point $x \in U_3$, by the similar reason above there exists a neighbourhood U of x on which the imaginary parts of eigenvalues of the Ricci operator Q are not zeros. Hence U is contained in U_3.

For each $x \in U_4$, there exists a neighbourhood U of x on which Q has exactly two eigenvalues. Moreover on some neighbourhood U' ($\subset U$) of x, the real part of one eigenvalue is not zero and the multiplicity of the other eigenvalue is equal to 1. Therefore $U' \subset U_4$. This implies that U_4 is open. □

Let M_q^n be an n ($n \geq 4$)-dimensional, simply connected, complete, conformally flat semi-Riemannian manifold of index q satisfying (2.5). Now we consider the case when the Ricci operator is diagonalizable. We can prove the following similarly to Sekigawa and Takagi [12].

PROPOSITION 3.3. (1) *If* U_1 *of Proposition* 3.2 *is not empty, then* $U_1 = M$ *and the semi-Riemannian manifold of* M *is isometric to the product manifold of a* k*-dimensional semi-Riemannian manifold of constant positive curvature* K *and an* $(n-k)$*-dimensional semi-Riemannaian manifold of constant negative curvature* $-K$*, that is,* $M_{q_1}^k(K) \times M_{q-q_1}^{n-k}(-K)$*, where* $1 < k < n-1$.

(2) *If* U_2 *of Proposition* 3.2 *is not empty, then* $U_2 = M$ *and* M *is a semi-Riemannain manifold of a non-zero constant curvature.*

(3) *If* U_4 *of Proposition* 3.2 *is not empty, then* $U_4 = M$ *and the semi-Riemannian manifold of* M *is isometric to the product manifold of an* $(n-1)$*-dimensional semi-Riemannian manifold of index* $q-1$ *of constant curvature* K *and a* 1*-dimensional Lorentzian manifold or the product manifold of an* $(n-1)$*-dimensional semi-Riemannian manifold of index* q *of constant curvature* K *and a* 1*-dimensional Riemannian manifold, that is,* $M_{q-1}^{n-1}(K) \times M_1^1$ *or* $M_q^{n-1}(K) \times M^1$*, where* $K \neq 0$.

We study the remaining cases in sections 4 and 5 and show Theorem 4.1 and 5.3, which together with Proposition 3.3, yield our Main Theorem in the Introduction.

4. The case when the Ricci operator has pure imaginary eigenvalues

In this section, we discuss the case when the Ricci operator has pure imaginary eigenvalues.

At first, we show an example— a complex sphere $CS^n(\sqrt{-1}b)$ with a real b. We define a semi-Riemannian metric g on an $(n+1)$-dimensional complex vector space C^{n+1} by

$$g = 2 \text{ the real part of } \sum_{i=1}^{n+1} dz_i \otimes dz_i$$

$$= 2\left(\sum_{i=1}^{n+1} dx_i \otimes dx_i - \sum_{i=1}^{n+1} dy_i \otimes dy_i\right),$$

where $z_i = x_i + \sqrt{-1}y_i$. It has signature $(n+1, n+1)$. The complexification of the semi-Riemannian metric g coinsides with $\sum_{i=1}^{n+1} dz_i \otimes dz_i$, for which we use the same notaion g. We consider a complex hypersurface M in C^{n+1} defined as follows:

$$M = \{(z_1, \cdots, z_{n+1}) \in C^{n+1} |\, z_1^2 + \cdots + z_{n+1}^2 = c\}$$

for $c \in C, c \neq 0$. It is called a complex sphere. A complex sphere is diffeomorphic to the tangent bundle TS^n of the sphere S^n. In fact, put $\sqrt{c} = h$ and define the linear transformation $F : C^{n+1} \Rightarrow C^{n+1}$ by $F(z_1, \cdots, z_{n+1}) = 1/h(z_1, \cdots, z_{n+1})$. Then we have

$$M' = F(M) = \{(z_1, \cdots, z_{n+1}) \in C^{n+1} |\, z_1^2 + \cdots + z_{n+1}^2 = 1\}.$$

We identify $(x_1 + \sqrt{-1}y_1, \cdots, x_{n+1} + \sqrt{-1}y_{n+1}) \in C^{n+1}$ with $(x_1, y_1, \cdots, x_{n+1}, y_{n+1}) \in R^{2(n+1)}$. Then M' is a submanifold of $R^{2(n+1)}$ with codimension 2 which is defined by $\sum_{j=1}^{n+1} x_j y_j = 0$ and $\sum_{j=1}^{n+1}(x_j^2 - y_j^2) = 1$. It is easily seen that M' is diffeomorphic to the tangent bundle TS^n of the sphere S^n. In particular it is simply connected for $n > 1$. We will calculate the curvature tensor of M applying the formulas in Chapter 4 § 9 in Nomizu and Sasaki [10]. By the defining equation of M, we have $\sum_{i=1}^{n+1} z_i dz_i = 0$ on M. Let $\zeta = \sum_{i=1}^{n+1} z_i \partial/\partial z_i$ be a holomorphic vector field on C^{n+1}. Then $(\sum_{i=1}^{n+1} z_i dz_i)(\zeta) = \sum_{i=1}^{n+1} z_i^2 = c \neq 0$ on M. Therefore ζ is a transversal vector field along M. Furthermore ζ is a normal vector field along M. Indeed,

$$g(W, \zeta) = \sum_{i=1}^{n+1} dz_i(W) dz_i(\zeta) = \left(\sum_{i=1}^{n+1} z_i dz_i\right)(W) = 0$$

for any $W \in T_z M^{1,0}$. Moreover, $g(\zeta, \zeta) = \sum_{i=1}^{n+1} z_i^2 = c \neq 0$ on M. Therefore the induced metric g on M is non-degenerate and it satisfies $g(JX, Y) = g(X, JY)$ for $X, Y \in T_z M$, where J denotes complex structure on M, that is, g is a so-called complex Riemannian metric on M [8]. This induced semi-Riemannian metric has the signature (n, n). M can be viewed as a semi-Riemannian symmetric space $SO(n+1, C)/SO(n, C)$. In particular M is a complete semi-Riemannian manifold.

We calculate the curvature tensor and the Ricci operator of M, applying the equation of Gauss. We denote by D the usual flat affine connection on C^{n+1}. It is also a Levi-Civita connection with respect to g. Since $D_W \zeta = W$ and $D_{\bar{W}} \zeta = 0$ for $W \in \Gamma(TM^{1,0})$, the shape operator S of M is given by

(4.1) $$S = -I \quad \text{on } TM^{1,0}$$

and

(4.2) $$S = 0 \quad \text{on } TM^{0,1}.$$

We have the Gauss formula:

$$D_X Y = \nabla_X Y + h(X, Y)\zeta \quad \text{for } X, Y \in \Gamma(TM^{1,0}),$$

where ∇ is the Levi-Civita connection with respect to the induced metric and h is the second fundamental form. Then h is given by

$$h(X, Y) = -\frac{1}{c} g(X, Y) \quad \text{for } X, Y \in TM^{1,0}. \tag{4.3}$$

Indeed, for $X, Y \in \Gamma(TM^{1,0})$,

$$ch(X, Y) = g(D_X Y, \zeta) = -g(Y, D_X \zeta) = -g(X, Y).$$

We denote by R the curvature tensor of the Levi-Civita connection on M. By the equation of Gauss, we have

$$\begin{aligned} R(Z, W)U &= h(W, U)SZ - h(Z, U)SW \\ R(Z, W)\overline{U} &= 0 \\ R(Z, \overline{W})U &= -h(Z, U)S\overline{W} \end{aligned}$$

for $Z, W, U \in TM^{1,0}$ (Nomizu and Sasaki [10], p. 191). By (4.1), (4.2) and (4.3), we have

$$\begin{aligned} R(Z, W)U &= \frac{1}{c}\{g(W, U)Z - g(Z, U)W\} \\ R(Z, \overline{W})U &= 0 \\ R(\overline{Z}, W)\overline{U} &= \overline{R(Z, \overline{W})U} = 0 \\ R(\overline{Z}, \overline{W})\overline{U} &= \overline{R(Z, W)U} \\ &= \frac{1}{\bar{c}}\{g(\overline{W}, \overline{U})\overline{Z} - g(\overline{Z}, \overline{U})\overline{W}\} \end{aligned} \tag{4.4}$$

Next we calculate the Ricci tensor ric. Let $\{e_1, \cdots, e_n\}$ be a basis of $TM^{1,0}$ satisfying $g(e_i, e_j) = \delta_{ij}$. Then we have

$$\begin{aligned} ric(W, U) &= \sum_{i=1}^{n} g(R(e_i, W)U, e_i) + \sum_{i=1}^{n} g(R(\bar{e_i}, W)U, \bar{e_i}) \\ &= \frac{n-1}{c} g(W, U) \\ ric(W, \overline{U}) &= 0 \\ ric(\overline{W}, \overline{U}) &= \overline{ric(W, U)} = \frac{n-1}{\bar{c}} g(\overline{W}, \overline{U}). \end{aligned} \tag{4.5}$$

For the Ricci operator Q, using (4.5) we have

$$g(QW, U) = ric(W, U) = \frac{n-1}{c} g(W, U)$$

$$g(QW,\overline{U}) = ric(W,\overline{U}) = 0\,.$$

Then

$$QW = \frac{n-1}{c}W\,. \tag{4.6}$$

While

$$g(Q\overline{W},U) = ric(\overline{W},U) = 0$$

$$g(Q\overline{W},\overline{U}) = ric(\overline{W},\overline{U}) = \frac{n-1}{\bar{c}}g(\overline{W},\overline{U})\,,$$

then we obtain

$$Q\overline{W} = \frac{n-1}{\bar{c}}\overline{W}\,. \tag{4.7}$$

The scalar curvature $r = tr\,Q$ becomes

$$r = n(n-1)\left(\frac{1}{c} + \frac{1}{\bar{c}}\right)\,.$$

Now we assume that c is pure imaginary, that is, $c = \sqrt{-1}b$ $(b \in R, b \neq 0)$. Then by (4.6) and (4.7), the Ricci operator has the pure imaginary eigenvalues $\pm\sqrt{-1}(n-1)/b$ and its scalar curvature r vanishies. The Ricci operator Q is parallel with respect to ∇ and hence it satisfies $R(X,Y)\cdot Q = 0$ for $X, Y \in TM$. Moreover we have

$$R(X,Y)Z = \frac{1}{2n-2}(QX \wedge Y + X \wedge QY)Z \tag{4.8}$$

for $X, Y, Z \in TM^C$. Comparing with the equation (2.3), we see that a complex sphere with the pure imaginary c is conformally flat. From now on, we denote by $CS^n(c)$ the complex sphere defined by $z_1^2 + \cdots + z_{n+1}^2 = c$.

For the case when the Ricci operator has pure imaginary eigenvalues, we will show the following:

THEOREM 4.1. *Let M be an n (≥ 4)-dimensional, simply connected, complete, conformally flat semi-Riemannian manifold satisfying the condition* (2.5). *If the Ricci operator Q_x has pure imaginary eigenvalues at some point $x \in M$, then M is isometric to a complex sphere $CS^{n/2}(\sqrt{-1}b)$ with some real b.*

PROOF. We define a subset U in M by $U = \{x \in M \mid Q_x$ has pure imaginary eigenvalues.$\}$ Then by Proposition 3.2, U is open in M. Because of the assumption, U is not empty. We denote by W a connected component of U. On W there exists a pure imaginary valued function λ such that the Ricci operator Q_x has the eigenvalues $\lambda(x)$ and $\overline{\lambda(x)} = -\lambda(x)$ at $x \in W$. We define two complex subbundles T_1 and T_2 of TM^C on W as follows at $x \in W$:

$$T_1(x) = \{X \in T_xM^C \mid QX = \lambda(x)X\}\,,$$

$$T_2(x) = \{X \in T_x M^C \mid QX = -\lambda(x)X\}\,.$$

Then we have an orthogonal direct sum decomposition:

$$TM^C = T_1 + T_2$$

and the complex conjugation is a real linear isomorphism between T_1 and T_2. For $X \in TM^C$, we denote by $(X)_{T_1}$ and $(X)_{T_2}$ the components of X which belong to T_1 and T_2, respectively. The scalar curvature r vanishes on W. Therefore by (2.4), we obtain

$$(\nabla_X Q)(Y) - (\nabla_Y Q)(X) = 0 \tag{4.9}$$

for $X, Y \in \Gamma(TM)$. For $X, Y \in \Gamma(T_1)$, by (4.9)

$$\begin{aligned} 0 &= (\nabla_X Q)(Y) - (\nabla_Y Q)(X) \\ &= (X\lambda)Y - (Y\lambda)X + 2\lambda\{(\nabla_X Y)_{T_2} - (\nabla_Y X)_{T_2}\}\,. \end{aligned}$$

If X and Y are linearly independent, $X\lambda = 0$. Similarly we have $X\lambda = 0$ for $X \in \Gamma(T_2)$. Therefore λ is constant on W.

For $X \in \Gamma(T_1)$ and $Y \in \Gamma(T_2)$, by (4.9)

$$\begin{aligned} 0 &= (\nabla_X Q)(Y) - (\nabla_Y Q)(X) \\ &= -2\lambda(\nabla_X Y)_{T_1} - 2\lambda(\nabla_Y X)_{T_2}\,. \end{aligned}$$

Therefore we have $(\nabla_X Y)_{T_1} = 0$, $(\nabla_Y X)_{T_2} = 0$. This means that T_1 and T_2 are parallel subbundles of TM^C. In particular the Ricci operator Q is parallel. Since the curvature tensor R has the form

$$R(X, Y) = \frac{1}{n-2}(QX \wedge Y + X \wedge QY)\,,$$

we have $\nabla R = 0$ on W.

We take a constant pure imaginary number λ which is an eigenvalue of the Ricci operator Q_x, $x \in W$ and define a subset V of M as follows: $V = \{x \in M \mid Q_x^2 - \lambda^2 I = 0\}$. Then V is not empty and evidently it is closed. By the argument above, we see that V is open and hence $V = M$. Consequently M is a simply connected, complete, locally symmetric semi-Riemannian manifold. We put $c = (n-2)/2\lambda$ and a complex sphere $CS^{n/2}(\sqrt{-1}b)$. By the form of curvature tensor, it follows that there exists a linear isometry $F : T_x M \to T_y CS^{n/2}(\sqrt{-1}b)$ which preserves the curvature tensor. Applying Theorem 7.8, in Kobayashi and Nomizu [7], Chapter VI, we see that M is isometric to a complex sphere $CS^{n/2}(\sqrt{-1}b)$.

5. The case when the Ricci operator is nilpotent

In this section we assume that there exists a point $x \in M$ at which $Q_x^2 = 0$. Then by Proposition 3.3 and Theorem 4.1, $Q^2 \equiv 0$ on whole M. Then scalar curvature $r = trQ$ vanishes and hence we have

$$(\nabla_X Q)(Y) = (\nabla_Y Q)(X) \tag{5.1}$$

for $X, Y \in \Gamma(TM)$. We put

$$k = max\{\text{The rank of } Q_x \,|\, x \in M\}\,.$$

If $k = 0$, that is, $Q \equiv 0$ on M, then M is flat. From now on, we assume that $k > 0$. We put

$$U = \{x \in M \,|\, \text{The rank of } Q_x = k\}\,.$$

Then U is open in M. We denote by W a connected component of U. From now on, we discuss on W.

For each point $x \in W$, we define

$$T_0(x) = ker\, Q_x\,, \quad L(x) = Im\, Q_x\,.$$

$L(x)$ is included in $T_0(x)$. By Lemma 3.1, the semi-Riemannian metric g restricted to $T_0(x)$ is degenerate and its nullity subspace

$$\{X \in T_0(x) |\, g(X, Y) = 0 \quad \text{for any } Y \in T_0(x)\}$$

coinsides with $L(x)$. T_0 and L are subbundles of TM on W of dimensions $n - k$ and k, respectively.

PROPOSITION 5.1. *The subbundle L is parallel along T_0-direction.*

PROOF. For $X \in \Gamma(T_0)$ and $Z \in \Gamma(TM)$, it follows from (5.1) that

$$\nabla_X(QZ) = Q([X, Z])\,.$$

□

PROPOSITION 5.2. *T_0 is a totally geodesic foliation of W and the leaves are flat with respect to the induced connection.*

PROOF. For $X, Y \in \Gamma(T_0)$ and $Z \in \Gamma(TM)$, by (5.1) we have

$$\begin{aligned} g(Q(\nabla_X Y), Z) &= -g((\nabla_X Q)Y, Z) \\ &= -g((\nabla_X Q)Z, Y) \\ &= -g((\nabla_Z Q)X, Y) \\ &= -g(\nabla_Z(Q(X)) - Q(\nabla_Z X), Y) \\ &= g(\nabla_Z X, QY) = 0 \end{aligned}$$

Therefore $Q(\nabla_X Y) = 0$, that is, $\nabla_X Y \in \Gamma(T_0)$. Hence $[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(T_0)$. This implies T_0 is completely integrable. Moreover for $X, Y \in \Gamma(T_0)$, it follows by (2.3) that $R(X, Y) = 0$. □

THEOREM 5.3. *The leaf $M_0(x_0)$ of the distribution T_0 through $x_0 \in W$ is complete with respect to the induced connection.*

PROOF. By the similar argument to Graves [6], we prove this theorem. For an arbitrary point $x \in M_0(x_0)$, let $\gamma : R \to M$ be a geodesic of M such that $\gamma(0) = x$, $\gamma'(0) \in T_0(x)$.

Since $M_0(x_0)$ is a totally geodesic submanifold of M, there exists a positive number $\varepsilon > 0$ such that $\gamma(t) \in M_0(x_0)$ for $-\varepsilon < t < \varepsilon$. We will show that $\gamma(t) \in M_0(x_0)$ for all $t \in R$. In fact we have the following:

LEMMA 5.4. *Let γ be a geodesic as above. If $\gamma(t) \in M_0(x_0)$ for $0 \le t < b$, then $\gamma(b) \in W$.*

PROOF OF LEMMA. For a basis $\{Z_1, \cdots, Z_k\}$ of $L(x)$, we take k tangent vectors $\xi_1, \cdots, \xi_k$ of T_xM which satisfy $g(\xi_i, Z_j) = -\delta_{ij}$. Let $\xi_i(t)$ and $Z_i(t)$ be parallel vector fields along γ with $\xi_i(0) = \xi_i$, $Z_i(0) = Z_i$ $(i = 1, \cdots, k)$. By Proposition 5.1, it follows that $\{Z_1(t), \cdots, Z_k(t)\}$ is a basis of $L(\gamma(t))$ and the subspace spanned by $\xi_1(t), \cdots, \xi_k(t)$ is a complementary subspace of $T_0(\gamma(t))$ in $T_{\gamma(t)}M$ for $0 \le t < b$. We define a $k \times k$ matrix $\Phi(t) = (\Phi_{ij}(t))$ by

$$\Phi_{ij}(t) = -g(Q\xi_j(t), \xi_i(t))$$

for $t \in R$. Then $\Phi(t)$ is a non-singular matrix for $0 \le t < b$. If $\Phi(b)$ is non-singular, $Q_{\gamma(b)}$ has rank k and hence $\gamma(b) \in W$. So we will prove that $\Phi(b)$ is non-singular.

For $v \in T_0(\gamma(t))$, $0 \le t < b$, we extend v to a T_0-vector field V in a neighbourhood of $\gamma(t)$. Then $g((\nabla_{\xi_j(t)}V)_{\gamma(t)}, Z_i(t))$ does not depend on an extension V of v. In particular we extend $\gamma'(t)$ to a T_0-vector field X and put

$$C_{ij}(t) = g((\nabla_{\xi_j(t)}X)_{\gamma(t)}, Z_i(t))$$

and define a $k \times k$ matrix $C(t)$ by $C(t) = (C_{ij}(t))$ for $0 \le t < b$. $\Phi(t)$ and $C(t)$ satisfy the following differential equations:

$$\Phi'(t) = \Phi(t)C(t)\,, \tag{5.2}$$

$$C'(t) = C(t)^2 \tag{5.3}$$

for $0 \le t < b$. We will show the equations above. For a fixed t $(0 \le t < b)$, we extend $\gamma'(t)$ to a T_0-vector field X and $\xi_j(t)$ to a vector field Ξ_j in a neighbourhood of $\gamma(t)$. By (5.1), we have

$$\nabla_X(Q\Xi_j) = Q([X, \Xi_j])\,.$$

At $\gamma(t)$,

$$\begin{aligned} g(\nabla_X(Q\Xi_j), \Xi_i) &= Xg(Q\Xi_j, \Xi_i) - g(Q\Xi_j, \nabla_X\Xi_i) \\ &= -\Phi'_{ij}(t) \end{aligned}$$

On the other hand, at $\gamma(t)$,

$$\begin{aligned} [X, \Xi_j] &= \nabla_X\Xi_j - \nabla_{\xi_j(t)}X \\ &= -(\nabla_{\xi_j(t)}X)_{T_0} + \sum_{a=1}^{k} g((\nabla_{\xi_j(t)}X)_{\gamma(t)}, Z_a(t))\xi_a(t) \\ &= -(\nabla_{\xi_j(t)}X)_{T_0} + \sum_{a=1}^{k} C_{aj}(t)\xi_a(t)\,. \end{aligned}$$

Therefore

$$g(Q([X, \Xi_j]), \xi_i(t)) = \sum_{a=1}^{k} C_{aj}(t) g(Q\xi_a(t), \xi_i(t))$$
$$= -\sum_{a=1}^{k} \Phi_{ia}(t) C_{aj}(t) .$$

Thus we obtain (5.2). Next we will obtain the equation (5.3).

$$\begin{aligned} C'_{ij}(t) &= Xg(\nabla_{\Xi_j} X, Z_i(t)) \\ &= g(\nabla_X(\nabla_{\Xi_j} X), Z_i(t)) + g(\nabla_{\Xi_j} X, \nabla_X Z_i(t)) \\ &= g(R(X, \Xi_j)X, Z_i(t)) + g(\nabla_{\Xi_j}(\nabla_X X), Z_i(t)) + g(\nabla_{[X,\Xi_j]} X, Z_i(t)) . \end{aligned}$$

At $\gamma(t)$, by Proposition 5.2,

$$g(R(X, \Xi_j)X, Z_i(t)) = g(R(\gamma'(t), Z_i(t))\gamma'(t), \xi_j(t)) = 0 .$$

$\nabla_X X$ is a T_0-vector field and $(\nabla_X X)_{\gamma(t)} = 0$. Since $g(\nabla_{\Xi_j}(\nabla_X X), Z_i(t))$ does not depend on an extension of $(\nabla_X X)_{\gamma(t)}$, $g(\nabla_{\Xi_j}(\nabla_X X), Z_i(t)) = 0$ at $\gamma(t)$.

For the last term, we have

$$g(\nabla_{[X,\Xi_j]} X, Z_i(t)) = -g(\nabla_{(\nabla_{\xi_j(t)} X)_{T_0}} X, Z_i(t)) + \sum_{a=1}^{k} C_{aj}(t) g(\nabla_{\xi_a(t)} X, Z_i(t))$$
$$= \sum_{a=1}^{k} C_{ia}(t) C_{aj}(t) .$$

Then (5.3) yields.

We put $d(t) = det\Phi(t)$ for $t \in R$. Because of the assumption, $d(t) \neq 0$ for $0 \leq t < b$ and we have

$$d'(t) = d(t) tr(\Phi^{-1}(t)\Phi'(t)) = d(t) trC(t) .$$

By (5.3), we obtain

$$d(t) = d(0) \prod_{i=1}^{k} \frac{1}{1 - \mu_i t} \quad \text{for } 0 \leq t < b ,$$

where $\mu_1, \cdots, \mu_k$ are the eigenvalues of $C(0)$. Since $d(t)$ is bounded on $0 \leq t \leq b$, $1 - \mu_i b \neq 0$ $(i = 1, \cdots, k)$. Then

$$d(b) = \lim_{t \to b} d(t) = \lim_{t \to b} d(0) \prod_{i=1}^{k} \frac{1}{1 - \mu_i t} \neq 0 .$$

Consequently, $\Phi(b)$ is non-singular. □

We continue our proof of Theorem 5.3. As in the proof of Theorem 5.14 in Graves [6], the fact $\gamma(b) \in W$ in Lemma 5.4 implies that

$$\sup\{s \in R | \gamma(t) \in M_0(x_0) \text{ for } 0 \leq t < s\}$$

is infinite and gives our theorem. □

Suppose that the maximal rank of the Ricci operator is 1. In this case the distributions T_0 and L on W are of dimension $n-1$ and 1, respectively. We can obtain more detailed description of T_0 and L.

COROLLARY 5.5. *Suppose that tha maximal rank of the Ricci operator Q is* 1. *Let U be the set of $x \in M$ at which the rank of Q_x is* 1. *Then the kernel distribution T_0 and the image distribution L of the Ricci operator are parallel subbundles of TM on U.*

PROOF. In the proof of Lemma 5.4, $\Phi(t)$ and $C(t)$ are real valued functions globally defined on R and we have $\Phi(t) = \Phi(0)/(1 - C(0)t)$. This implies $C(0) = 0$. Then we see that T_0 is parallel. L is also parallel. In fact, for $X \in \Gamma(T_0)$ and $Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} g(\nabla_Y(QZ), X) &= Yg(QZ, X) - g(QZ, \nabla_Y X) \\ &= Yg(Z, QX) - g(Z, Q(\nabla_Y X)) \\ &= 0\,. \end{aligned}$$

Therefore $\nabla_Y(QZ) \in \Gamma(L)$. □

REMARK. When M is a Lorentzian manifold, the maximal rank of the Ricci operator is 1 unless Q vanishes identically.

6. Examples with nilpotent Ricci operators

In this section, we will give examples of Main Theorem (5), which are *symmetric domains of projective quadrics* constructed and classified by Cahen and Kerbrat [2]. Here we give a slightly different description and compute their Ricci operators.

Let $(R^{n+2}_{p+1}, \langle\ ,\ \rangle)$ be the semi-Euclidean space with an inner product $\langle\ ,\ \rangle$ of signature $(p+1, n-p+1)$ and Γ be the lightcone which is a hypersurface of $R^{n+2}_{p+1} - \{0\}$ defined by

$$\Gamma = \{x \in R^{n+2}_{p+1} - \{0\} | \langle x, x\rangle = 0\}\,.$$

If a linear endomorphism A of R^{n+2}_{p+1} satisfies the following conditions, it is said to be of type N (Cahen and Parker [3] Definition 1.7.3, 45p.):

(1) A is self-adjoint with respect to $\langle\ ,\ \rangle$, i.e., $\langle Ax, y\rangle = \langle x, Ay\rangle$.

(2) $A^2 = 0$.

(3) There exists a point $x \in \Gamma$ such that $\langle x, Ax\rangle > 0$.

We consider the following subset M of R^{n+2}_{p+1}:

$$M = \{x \in R^{n+2}_{p+1} | \langle x, x\rangle = 0,\ \langle x, Ax\rangle = 1\}$$
$$= \Gamma \cap \{x \in R^{n+2}_{p+1} | \langle x, Ax\rangle = 1\}\,.$$

If M is not connected, we take a connected component and use the same notation M. Then M is a submanifold of R^{n+2}_{p+1} with codimension 2.

At each point $x \in M$, x and Ax are linearly independent. We denote by $V(x)$ a 2-dimensional subspace of R^{n+2}_{p+1} spanned by x and Ax. Then $\langle\ ,\ \rangle|_{V(x)}$ is non-degenerate. Indeed, we have the following:

$$\langle x, x\rangle = \langle Ax, Ax\rangle = 0\,, \quad \langle x, Ax\rangle = \langle Ax, x\rangle = 1\,.$$

Next we will show that $T_xM = V(x)^{\perp}$. Let f and g be the functions on R^{n+2}_{p+1} given by $f(x) = \langle x, x\rangle$ and $g(x) = \langle x, Ax\rangle$, respectively. For $X \in T_xM$, we have $df(X) = 0$ and $dg(X) = 0$, and on the other hand we have

$$df(X) = 2\,\langle Xx, x\rangle = 2\,\langle X, x\rangle$$
$$dg(X) = \langle Xx, Ax\rangle + \langle x, XAx\rangle$$
$$= \langle X, Ax\rangle + \langle x, AXx\rangle$$
$$= 2\,\langle X, Ax\rangle\ .$$

Since $V(x)$ is non-degenerate, T_xM is non-degenerate with respect to $\langle\ ,\ \rangle$. In particular M endowed with an induced metric is an n-dimensional semi-Riemannian manifold of index p.

At each point $x \in M$, we define a linear endomorphism ϕ_x of R^{n+2}_{p+1} by

$$\phi_x = I|_{V(x)} \oplus -I|_{V(x)^{\perp}}\,.$$

Then ϕ_x becomes a linear isometry of $(R^{n+2}_{p+1}, \langle\ ,\ \rangle)$ and satisfies $\phi_x A = A\phi_x$. Indeed, for any $\xi, \eta \in R^{n+2}_{p+1}$, we set

$$\xi = \xi' + \xi'', \quad \eta = \eta' + \eta''\,,$$

where $\xi', \eta' \in V(x)$ and $\xi'', \eta'' \in V(x)^{\perp}$. Then we have

$$\langle \phi_x\xi, \phi_x\eta\rangle = \left\langle \xi' - \xi'', \eta' - \eta''\right\rangle$$
$$= \left\langle \xi', \eta'\right\rangle + \left\langle \xi'', \eta''\right\rangle$$
$$= \left\langle \xi' + \xi'', \eta' + \eta''\right\rangle$$
$$= \langle \xi, \eta\rangle\ .$$

So ϕ_x is a linear isometry with respect to $\langle\ ,\ \rangle$. Since $V(x)$ is an A-invariant subspace and A is self-adjoint with respect to $\langle\ ,\ \rangle$, $V(x)^{\perp}$ is also A-invariant. Then we obtain $\phi_x A = A\phi_x$.

We will show that M is an extrinsic symmetric submanifold of R^{n+2}_{p+1}, and then M is a semi-Riemannian symmetric space. See Ferus [5] or Naitoh [9] for the basic facts of extrinsic symmetric submanifolds.

At each point $x \in M$ we take a linear isometry ϕ_x defined as above. By the difinition of ϕ_x, clearly we have

(1) $\phi_x(x) = x$.

Also we have

(2) $\phi_x(M) = M$.

In fact, since ϕ_x is a linear isometry,

$$\langle \phi_x(y), \phi_x(y) \rangle = \langle y, y \rangle = 0$$

for $y \in M$. Since ϕ_x commutes with A, we have

$$\begin{aligned} \langle \phi_x(y), A\phi_x(y) \rangle &= \langle \phi_x(y), \phi_x(Ay) \rangle \\ &= \langle y, Ay \rangle = 1 . \end{aligned}$$

Since $T_x M = V(x)^\perp$ and $T_x^\perp M = (V(x)^\perp)^\perp = V(x)$, we have

(3) $\phi_x = \begin{cases} -Id & \text{on } T_x M \\ \ \ Id & \text{on } T_x^\perp M . \end{cases}$

Let σ and S be the second fundamental form and the shape operator of the submanifold M, respectively. Then at each point $x \in M$, we have

$$S_x X = -X , \quad S_{Ax} X = -AX ,$$

$$\sigma(X, Y) = -\langle AX, Y \rangle x - \langle X, Y \rangle Ax .$$

By the equation of Gauss, the curvature tensor R of M is given by

$$\begin{aligned} R(X, Y)Z &= \{AX \wedge Y + X \wedge AY\}(Z) \\ &= \langle Y, Z \rangle AX - \langle AX, Z \rangle Y + \langle AY, Z \rangle X - \langle X, Z \rangle AY . \end{aligned}$$

Since A is nilpotent, we have $tr A|_{T_x M} = 0$. Then the Ricci tensor ric is given by

$$ric(X, Y) = (n-2) \langle AX, Y \rangle .$$

So the Ricci operator Q becomes $Q = (n-2)A|_{T_x M}$ and consequently $Q^2 = 0$. Hence the scalar curvature vanishes.

From these, the curvature tensor R of M satisfies

$$R(X, Y)Z = \frac{1}{n-2}\{QX \wedge Y + X \wedge QX\}$$

and we see that M is conformally flat. Thus we obtain examples of Main Theorem (5).

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